Chapter 3

New congruences modulo 5 for the number of 2-color partitions

3.1 Introduction

In this chapter, we prove Theorem 1.3.1 for $k \in \{2, 3, 4\}$, i.e.,

$$p_2(25n + 22) \equiv 0 \pmod{5}, \tag{3.1.1}$$

$$p_3(25n + 21) \equiv 0 \pmod{5}, \tag{3.1.2}$$

and

$$p_4(25n + 20) \equiv 0 \pmod{5}, \tag{3.1.3}$$

In the next section, we present some useful lemmas and in Section 3.3, we prove (3.1.1)–(3.1.3).

The contents of this chapter published in [2].

3.2 Preliminaries

We recall the following elementary properties of Ramanujan's theta function $f(a, b)$ from Entries 29 and 30 of (Berndt [19, pp. 45–46]).

Lemma 3.2.1. *We have*

$$f(-a,-b) = f(a^2,b^2)\varphi(ab) - 2af\left(\frac{b}{a}, a^3b\right)\psi(a^2b^2). \tag{3.2.1}$$

If $ac = bd$, then

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc) \tag{3.2.2}$$

and

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c}, ac^2d\right)f\left(\frac{b}{d}, acd^2\right). \tag{3.2.3}$$

We require the following Jacobi's identity.

Lemma 3.2.2. (Berndt [19, p. 39, Entry 24]) *We have*

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{3.2.4}$$

We also require the following 5-dissection of $\varphi(-q)$.

Lemma 3.2.3. (Berndt [19, p. 39, Entry 24]) *We have*

$$\varphi(-q) = \varphi(-q^{25}) - 2qf(-q^{15},-q^{35}) + 2q^4f(-q^5,-q^{45}). \tag{3.2.5}$$

In the next four lemmas we state some well-known results on the Rogers- Ramanujan continued fraction $R(q)$.

Lemma 3.2.4. (Berndt [20, p. 161 and p. 164]) *If* $T(q) := \dfrac{q^{1/5}}{R(q)} = \dfrac{f(-q^2,-q^3)}{f(-q,-q^4)}$, *then*

$$T^5(q) - 11q - \frac{q^2}{T^5(q)} = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6} \tag{3.2.6}$$

and

$$T(q^5) - q - \frac{q^2}{T(q^5)} = \frac{(q;q)_\infty}{(q^{25};q^{25})_\infty}. \tag{3.2.7}$$

The following result gives a 5-dissection of $1/(q;q)_\infty$ in terms of $T(q)$ which also readily implies (1.3.1).

Lemma 3.2.5. (Berndt [20, p. 165, eq. (7.4.14)]) *We have*

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\Big\{T^4(q^5) + qT^3(q^5) + 2q^2T^2(q^5) + 3q^3T(q^5) + 5q^4 - \frac{3q^5}{T(q^5)} + \frac{2q^6}{T^2(q^5)} - \frac{q^7}{T^3(q^5)} + \frac{q^8}{T^4(q^5)}\Big\}. \tag{3.2.8}$$

Lemma 3.2.6. (Andrews and Berndt[6, p. 35, Entry 1.8.2]) *If $k = R(q)R^2(q^2)$ and $k \le \sqrt{5} - 2$, then*

$$\frac{\psi^2(q)}{q\psi^2(q^5)} = \frac{1 + k - k^2}{k}. \tag{3.2.9}$$

We end this section by noting the following two beautiful results found by Gugg [37].

Lemma 3.2.7. *If $u = R(q)$ and $v = R(q^3)$, then*

$$\frac{v}{u^3} + \frac{u^3}{v} = 9q^2 \frac{(q^{15};q^{15})_\infty^3 (q;q)_\infty}{(q^5;q^5)_\infty^5 (q^3;q^3)_\infty} + 2 \tag{3.2.10}$$

and

$$\frac{1}{uv^3} + uv^3 = \frac{(q^5;q^5)_\infty^5 (q^3;q^3)_\infty}{q^2(q^{15};q^{15})_\infty^5 (q;q)_\infty} - 2. \tag{3.2.11}$$

3.3 Proof of Theorem 1.3.1 for $k \in \{2, 3, 4\}$

Proof of Theorem 1.3.1 for $k = 2$. We note that

$$\begin{aligned}\sum_{n=0}^{\infty} p_2(n)q^n &= \frac{1}{(q;q)_\infty (q^2;q^2)_\infty} \\ &= \frac{(q;q)_\infty}{(q;q)_\infty^3 (-q;q)_\infty} \\ &= \frac{(q;q)_\infty^2}{(q;q)_\infty^5} \frac{(q;q)_\infty}{(-q;q)_\infty}.\end{aligned} \tag{3.3.1}$$

By the binomial theorem, it is easy to see that

$$(q;q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod 5. \tag{3.3.2}$$

Using the above and (1.8.3) in (3.3.1), we have

$$\sum_{n=0}^{\infty} p_2(n)q^n \equiv \frac{(q;q)_\infty^2}{(q^5;q^5)_\infty}\varphi(-q) \pmod 5. \tag{3.3.3}$$

Employing (3.2.7) and (3.2.5) in the above, and then extracting the terms involving q^{5n+2} from both sides, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_2(5n+2)q^n &\equiv \frac{(q^5;q^5)_\infty^2}{(q;q)_\infty}\Big\{4\,\frac{f(-q^2,-q^3)}{f(-q,-q^4)}f(-q^3,-q^7)\\
&\quad + 4q\frac{f(-q,-q^4)}{f(-q^2,-q^3)}f(-q,-q^9) - \varphi(-q^5)\Big\}\\
&\equiv \frac{(q^5;q^5)_\infty^2}{(q;q)_\infty}\Big\{4\,\frac{f^2(-q^2,-q^3)f(-q^3,-q^7)+qf^2(-q,-q^4)f(-q,-q^9)}{f(-q,-q^4)f(-q^2,-q^3)}\\
&\quad - \varphi(-q^5)\Big\} \pmod 5.\end{aligned}$$

Since by (1.8.2), $f(-q,-q^4)f(-q^2,-q^3) = (q;q)_\infty(q^5;q^5)_\infty$, the above can be written as

$$\sum_{n=0}^{\infty} p_2(5n+2)q^n \equiv A(q) + B(q) \pmod 5, \tag{3.3.4}$$

where

$$A(q) = 4\,\frac{(q^5;q^5)_\infty}{(q;q)_\infty^2}\{f^2(-q^2,-q^3)f(-q^3,-q^7) + qf^2(-q,-q^4)f(-q,-q^9)\} \tag{3.3.5}$$

and

$$B(q) = -\frac{(q^5;q^5)_\infty^2}{(q;q)_\infty}\varphi(-q^5).$$

If we can show that the coefficients of q^{5n+4} on the right side of (3.3.4) are multiples of 5, then by equating the coefficients of q^{5n+4} from both sides of (3.3.4) we shall arrive at the desired congruence (1.3.7). By (1.3.2), it is clear that the coefficients of q^{5n+4} in $B(q)$ are multiples of 5. Therefore, it is sufficient to show that the coefficients of q^{5n+4} in $A(q)$ are also multiples of 5. To that end, first we simplify $A(q)$ in terms of q-products.

Setting, in turn, $a=q$, $b=q^4$ and $a=q^2$, $b=q^3$ in (3.2.1), we have

$$f^2(-q,-q^4)=f(q^2,q^8)\varphi(q^5)-2qf(q^3,q^7)\psi(q^{10}) \tag{3.3.6}$$

and

$$f^2(-q^2,-q^3)=f(q^4,q^6)\varphi(q^5)-2q^2f(q,q^9)\psi(q^{10}). \tag{3.3.7}$$

Multiplying (3.3.6) by $qf(-q,-q^9)$ and (3.3.7) by $f(-q^3,-q^7)$, and then adding the resulting identities, we find that

$$\begin{aligned}&f^2(-q^2,-q^3)f(-q^3,-q^7)+qf^2(-q,-q^4)f(-q,-q^9)\\&=\varphi(q^5)\left\{f(q^4,q^6)f(-q^3,-q^7)+qf(q^2,q^8)f(-q,-q^9)\right\}\\&\quad-2q^2\psi(q^{10})\left\{f(-q^3,-q^7)f(q,q^9)+f(q^3,q^7)f(-q,-q^9)\right\}.\end{aligned} \tag{3.3.8}$$

Now, setting $a=q, b=-q^4, c=q^2, d=q^3$ in (3.2.2) and (3.2.3), and then adding, we have

$$f(q^4,q^6)f(-q^3,-q^7)+qf(q^2,q^8)f(-q,-q^9)=f(q,-q^4)f(q^2,-q^3)=f(q)f(q^5),$$

where (1.8.2) was also used in the last equality.

On the other hand, setting $a=q, b=q^9, c=-q^3, d=-q^7$ in (3.2.2), we have

$$\begin{aligned}f(-q^3,-q^7)f(q,q^9)+f(q^3,q^7)f(-q,-q^9)&=2f(-q^4,-q^{16})f(-q^8,-q^{12})\\&=2(q^4;q^4)_\infty(q^{20};q^{20})_\infty,\end{aligned}$$

where the last equality is obtained from (1.8.2).

Employing the last two identities in (3.3.8), we arrive at

$$\begin{aligned}&f^2(-q^2,-q^3)f(-q^3,-q^7)+qf^2(-q,-q^4)f(-q,-q^9)\\&\quad=f(q)f(q^5)\varphi(q^5)-4q^2(q^4;q^4)_\infty(q^{20};q^{20})_\infty\psi(q^{10}).\end{aligned}$$

Using the above in (3.3.5) and then employing (3.3.2), we find that

$$A(q) = 4\ (q^5;q^5)_\infty f(q^5)\varphi(q^5)\frac{f(q)}{(q;q)_\infty^2} - 16q^2(q^5;q^5)_\infty(q^{20};q^{20})_\infty\psi(q^{10})\frac{(q^4;q^4)_\infty}{(q;q)_\infty^2}$$
$$\equiv 4f(q^5)\varphi(q^5)f(q)(q;q)_\infty^3 - q^2(q^{20};q^{20})_\infty\psi(q^{10})(q^4;q^4)_\infty(q;q)_\infty^3 \pmod 5.$$

Thus, to show that the coefficients of q^{5n+4} in $A(q)$ are multiples of 5, it is sufficient to check that the coefficients of q^{5n+4} in $f(q)(q;q)_\infty^3$ as well as $q^2(q^4;q^4)_\infty(q;q)_\infty^3$ are multiples of 5. We complete the proof of (1.3.7) by verifying this fact in the following.

First, from (2.2.2) and (3.2.4), we note that

$$\begin{aligned} f(q)(q;q)_\infty^3 &= (-q;-q)_\infty(q;q)_\infty^3 \\ &= \sum_{j=-\infty}^{\infty}(-1)^{j+j(3j+1)/2}q^{j(3j+1)/2}\sum_{r=0}^{\infty}(-1)^r(2r+1)q^{r(r+1)/2} \\ &= \sum_{j=-\infty}^{\infty}\sum_{r=0}^{\infty}(-1)^{3j(j+1)/2+r}(2r+1)q^{j(3j+1)/2+r(r+1)/2}. \end{aligned}$$

As in [20, p. 32], we observe that

$$2(j+1)^2+(2r+1)^2 = 8\left\{1+\frac{j(3j+1)}{2}+\frac{r(r+1)}{2}\right\} - 10j^2 - 5.$$

Hence, $\dfrac{j(3j+1)}{2}+\dfrac{r(r+1)}{2}$ is of the form $5n+4$ if and only if

$$2(j+1)^2+(2r+1)^2 \equiv 0 \pmod 5.$$

But, $2(j+1)^2 \equiv 0, 2$ or $3 \pmod 5$ and $(2r+1)^2 \equiv 0, 1$ or $4 \pmod 5$. Therefore, the above congruence is true if and only if $2(j+1)^2 \equiv 0 \equiv (2r+1)^2 \pmod 5$; in particular, $2r+1 \equiv 0 \pmod 5$. Hence, the coefficients of q^{5n+4} in $f(q)(q;q)_\infty^3$ are multiples of 5.

Next, from (2.2.2) and (3.2.4) again, we have

$$\begin{aligned} q^2(q^4;q^4)_\infty(q;q)_\infty^3 &= q^2\sum_{j=-\infty}^{\infty}(-1)^j q^{2j(3j+1)}\sum_{r=0}^{\infty}(-1)^r(2r+1)q^{r(r+1)/2} \\ &= \sum_{j=-\infty}^{\infty}\sum_{r=0}^{\infty}(-1)^{j+r}(2r+1)q^{2+2j(3j+1)+r(r+1)/2}. \end{aligned} \tag{3.3.9}$$

Observe that

$$8(j+1)^2+(2r+1)^2=8\left\{1+2+2j(3j+1)+\frac{r(r+1)}{2}\right\}-40j^2-15.$$

Thus, the exponents $2+2j(3j+1)+\dfrac{r(r+1)}{2}$ on the right side of (3.3.9) are of the form $5n+4$ if and only if

$$8(j+1)^2+(2r+1)^2\equiv 0 \pmod 5.$$

But, $8(j+1)^2\equiv 0,2$ or $3 \pmod 5$ and $(2r+1)^2\equiv 0,1$ or $4 \pmod 5$. Therefore, the only values for which the above congruence is satisfied are $8(j+1)^2\equiv(2r+1)^2\equiv 0 \pmod 5$. Hence, $2r+1\equiv 0 \pmod 5$, and consequently from (3.3.9), the coefficients of q^{5n+4} in $q^2(q^4;q^4)_\infty(q;q)_\infty^3$ are multiples of 5. □

Proof of Theorem 1.3.1 for $k=3$. We have

$$\sum_{n=0}^{\infty}p_3(n)q^n=\frac{1}{(q;q)_\infty(q^3;q^3)_\infty}.$$

Using (3.2.8) in the above and then extracting the terms involving q^{5n+1} from both sides and also with the aid of (3.3.2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}p_3(5n+1)q^n&=\frac{(q^5;q^5)_\infty^5(q^{15};q^{15})_\infty^5}{(q;q)_\infty^6(q^3;q^3)_\infty^6}\Big[2qx_1^4x_3^2-\frac{q^4x_1^4}{x_3^3}+x_1^3x_3^4-\frac{3q^3x_1^3}{x_3}+6q^2x_1^2x_3\\&\quad+\frac{2q^5x_1^2}{x_3^4}+3qx_1x_3^3+\frac{6q^4x_1}{x_3^2}-\frac{6q^2x_3^2}{x_1}+\frac{3q^5}{x_1x_3^3}+\frac{2qx_3^4}{x_1^2}-\frac{6q^4}{x_1^2x_3}\\&\quad-\frac{3q^3x_3}{x_1^3}-\frac{q^6}{x_1^3x_3^4}+\frac{q^2x_3^3}{x_1^4}+\frac{2q^5}{x_1^4x_3^2}\Big]\\&\equiv\frac{(q^5;q^5)_\infty^4(q^{15};q^{15})_\infty^4}{(q;q)_\infty(q^3;q^3)_\infty}\Big[2qA(q)-3qB(q)+6q^2C(q)\\&\quad+D(q)\Big] \pmod 5,\end{aligned}\tag{3.3.10}$$

where $x_1:=T(q)$ and $x_3=T(q^3)$,

$$\begin{aligned}A(q)&=x_1^4x_3^2+\frac{q^4x_1^2}{x_3^4}+\frac{x_3^4}{x_1^2}+\frac{q^4}{x_1^4x_3^2},\\B(q)&=\frac{q^2x_1^3}{x_3}-x_1x_3^3-\frac{q^4}{x_1x_3^3}+\frac{q^2x_3}{x_1^3},\\C(q)&=x_1^2x_3+\frac{q^2x_1}{x_3^2}-\frac{x_3^2}{x_1}-\frac{q^2}{x_1^2x_3},\end{aligned}$$

and

$$D(q)=x_1^3x_3^4-\frac{q^6}{x_1^3x_3^4}+\frac{q^2x_3^3}{x_1^4}-\frac{q^4x_1^4}{x_3^3}.$$

To prove Theorem 1.3.1 for $k=3$, i.e., to prove that $p_3(25n+21)\equiv 0 \pmod 5$, we need to show that the coefficients of q^{5n+4} on the right side of (3.3.10) are multiples of 5. To that end, we simplify $A(q)$, $B(q)$, $C(q)$, and $D(q)$ in terms of $(q^s;q^s)_\infty$ with $s\geq 1$.

Multiplying Gugg's identities (3.2.10) and (3.2.11), we have

$$\frac{1}{u^4v^2}+\frac{v^4}{u^2}+\frac{u^2}{v^4}+u^4v^2=5-18q^2\frac{(q^{15};q^{15})_\infty^3(q;q)_\infty}{(q^5;q^5)_\infty^5(q^3;q^3)_\infty}+2\frac{(q^5;q^5)_\infty^5(q^3;q^3)_\infty}{q^2(q^{15};q^{15})_\infty^5(q;q)_\infty}.$$

As $u=R(q)=\dfrac{q^{1/5}}{x_1}$ and $v=R(q^3)=\dfrac{q^{3/5}}{x_3}$, the above can be rewritten as

$$\frac{x_1^4x_3^2}{q^2}+\frac{q^2x_1^2}{x_3^4}+\frac{x_3^4}{q^2x_1^2}+\frac{q^2}{x_1^4x_3^2}=Z(q),$$

where

$$Z(q)=5-18q^2\frac{(q^{15};q^{15})_\infty^3(q;q)_\infty}{(q^5;q^5)_\infty^5(q^3;q^3)_\infty}+2\frac{(q^5;q^5)_\infty^5(q^3;q^3)_\infty}{q^2(q^{15};q^{15})_\infty^5(q;q)_\infty}.$$

Thus,

$$A(q)=q^2Z(q). \tag{3.3.11}$$

Again from (3.2.10) and (3.2.11), we have

$$\left\{\frac{v}{u^3}+\frac{u^3}{v}-2\right\}\left\{\frac{1}{uv^3}+uv^3+2\right\}=9,$$

which is equivalent to

$$2\left\{-\frac{x_1x_3^3}{q^2}-\frac{q^2}{x_1x_3^3}+\frac{x_1^3}{x_3}+\frac{x_3}{x_1^3}\right\}=13-Z(q).$$

Therefore,

$$B(q)=\frac{13q^2-q^2Z(q)}{2}. \tag{3.3.12}$$

Now,

$$C^2(q)=\left\{x_1^4x_3^2+\frac{q^4x_1^2}{x_3^4}+\frac{x_3^4}{x_1^2}+\frac{q^4}{x_1^4x_3^2}\right\}-4q^2+2\left\{\frac{q^2x_1^3}{x_3}-x_1x_3^3-\frac{q^4}{x_1x_3^3}+\frac{q^2x_3}{x_1^3}\right\}$$

With the help of (3.3.11) and (3.3.12), we see that

$$C^2(q)=q^2Z(q)-4q^2+13q^2-q^2Z(q)=9q^2.$$

Thus,

$$C(q)=\pm 3q. \tag{3.3.13}$$

Next, (3.3.12) and (3.3.13) together give

$$\left\{\frac{q^2x_1^3}{x_3}-x_1x_3^3-\frac{q^4}{x_1x_3^3}+\frac{q^2x_3}{x_1^3}\right\}\left\{x_1^2x_3+\frac{q^2x_1}{x_3^2}-\frac{x_3^2}{x_1}-\frac{q^2}{x_1^2x_3}\right\}=\pm 3q^3\frac{13-Z(q)}{2},$$

which can be simplified as

$$-D(q)-2q^2C(q)=q^2\left\{\frac{q^2}{x_1^5}-x_1^5\right\}+\frac{q^6}{x_3^5}-x_3^5\pm 3q^3\frac{13-Z(q)}{2}.$$

Employing (3.2.6) and (3.3.13) in the above, we have

$$\begin{aligned}D(q)\pm 6q^3=q^2&\left\{\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}+11q\right\}+\frac{(q^3;q^3)_\infty^6}{(q^{15};q^{15})_\infty^6}+11q^3\\&\mp 3q^3\frac{13-Z(q)}{2},\end{aligned}$$

which is equivalent to

$$D(q)=\mp\frac{51}{2}q^3\pm\frac{3}{2}q^3Z(q)+q^2\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}+\frac{(q^3;q^3)_\infty^6}{(q^{15};q^{15})_\infty^6}+22q^3. \tag{3.3.14}$$

Employing (3.3.11) – (3.3.14) in (3.3.10), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty}a_3(5n+1)q^n&\equiv\frac{(q^5;q^5)_\infty^4(q^{15};q^{15})_\infty^4}{(q;q)_\infty(q^3;q^3)_\infty}\Big[2q^3Z(q)-\frac{3}{2}q^3(13-Z(q))\pm 18q^3\\&\quad\mp\frac{51}{2}q^3\pm\frac{3}{2}q^3Z(q)+q^2\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}+\frac{(q^3;q^3)_\infty^6}{(q^{15};q^{15})_\infty^6}+22q^3\Big]\\&\equiv\frac{(q^5;q^5)_\infty^4(q^{15};q^{15})_\infty^4}{(q;q)_\infty(q^3;q^3)_\infty}\Big[2q^3Z(q)+\frac{3}{2}q^3Z(q)(1\pm 1)+q^2\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}\\&\quad+\frac{(q^3;q^3)_\infty^6}{(q^{15};q^{15})_\infty^6}+\frac{5}{2}q^3\mp\frac{51}{2}q^3\pm 18q^3\Big]\pmod 5.\end{aligned} \tag{3.3.15}$$

Now we separately handle the two cases of ambiguity in the above congruence arising from $C(q) = \pm 3q$.

First consider $C(q) = 3q$. Then (3.3.15) becomes

$$\sum_{n=0}^{\infty} a_3(5n+1)q^n \equiv \frac{(q^5;q^5)_\infty^4(q^{15};q^{15})_\infty^4}{(q;q)_\infty(q^3;q^3)_\infty}\Big[5q^3Z(q) + q^2\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6} + \frac{(q^3;q^3)_\infty^6}{(q^{15};q^{15})_\infty^6} - 5q^3\Big]$$
$$\equiv q^2\frac{(q^{15};q^{15})_\infty^4}{(q^5;q^5)_\infty(q^3;q^3)_\infty} + \frac{(q^5;q^5)_\infty^4}{(q^{15};q^{15})_\infty(q;q)_\infty} \pmod 5.$$

Employing (3.2.8) on the right side and then comparing the coefficients of q^{5n+4} from both sides of the above, we arrive at the desired congruence $p_3(25n+21) \equiv 0 \pmod 5$.

Next, consider the other case, i.e., $C(q) = -3q$. Then (3.3.15) reduces to

$$\sum_{n=0}^{\infty} a_3(5n+1)q^n \equiv \frac{(q^5;q^5)_\infty^4(q^{15};q^{15})_\infty^4}{(q;q)_\infty(q^3;q^3)_\infty}\Big[2q^3Z(q) + q^2\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}$$
$$+ \frac{(q^3;q^3)_\infty^6}{(q^{15};q^{15})_\infty^6} + 10q^3\Big]$$
$$\equiv \frac{(q^5;q^5)_\infty^4(q^{15};q^{15})_\infty^4}{(q;q)_\infty(q^3;q^3)_\infty}\Big[2q^3\Big\{5 - 18q^2\frac{(q^{15};q^{15})_\infty^5(q;q)_\infty}{(q^5;q^5)_\infty^5(q^3;q^3)_\infty}$$
$$+ 2\frac{(q^5;q^5)_\infty^5(q^3;q^3)_\infty}{q^2(q^{15};q^{15})_\infty^5(q;q)_\infty}\Big\} + q^2\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6} + \frac{(q^3;q^3)_\infty^6}{(q^{15};q^{15})_\infty^6}\Big]$$
$$\equiv 4q^5\frac{(q^{15};q^{15})_\infty^9}{(q^5;q^5)_\infty(q^3;q^3)_\infty^2} + 4q\frac{(q^5;q^5)_\infty^9}{(q^{15};q^{15})_\infty(q;q)_\infty^2}$$
$$+ q^2\frac{(q^{15};q^{15})_\infty^4}{(q^5;q^5)_\infty(q^3;q^3)_\infty} + \frac{(q^5;q^5)_\infty^4}{(q^{15};q^{15})_\infty(q;q)_\infty}$$
$$\equiv 4q^5\frac{(q^{15};q^{15})_\infty^8}{(q^5;q^5)_\infty}(q^3;q^3)_\infty^3 + 4q\frac{(q^5;q^5)_\infty^8}{(q^{15};q^{15})_\infty}(q;q)_\infty^3$$
$$+ q^2\frac{(q^{15};q^{15})_\infty^4}{(q^5;q^5)_\infty(q^3;q^3)_\infty} + \frac{(q^5;q^5)_\infty^4}{(q^{15};q^{15})_\infty(q;q)_\infty} \pmod 5.$$

But, it has already been noted that the coefficients of q^{5n+4} in the last two terms on the right side of the above congruence are multiples of 5. Therefore, to derive $p_3(25n+21) \equiv 0 \pmod 5$ from the above congruence it is enough to show that the coefficients of q^{5n+4} in $(q^3;q^3)_\infty^3$ and $q(q;q)_\infty^3$ are also multiples of 5.

Replacing q by q^3 in (3.2.4), we have

$$(q^3;q^3)_\infty^3 = \sum_{r=0}^{\infty}(-1)^r(2r+1)q^{3r(r+1)/2}. \tag{3.3.16}$$

We observe that if the exponents of q in the above sum are of the form $5n+4$, then

$$\frac{3r(r+1)}{2} \equiv 4 \pmod{5},$$

which is equivalent to

$$(2r+1)^2 \equiv 0 \pmod{5},$$

and hence $2r+1 \equiv 0 \pmod{5}$. Therefore, by (3.3.16), the coefficients of q^{5n+4} in $(q^3;q^3)_\infty^3$ are multiples of 5.

Similarly, we note from (3.2.4) that

$$q(q;q)_\infty^3 = \sum_{r=0}^{\infty} (-1)^r (2r+1) q^{1+r(r+1)/2}. \tag{3.3.17}$$

Observe that

$$1 + \frac{r(r+1)}{2} \equiv 4 \pmod{5}$$

is equivalent to $(2r+1)^2 \equiv 0 \pmod{5}$. So, $2r+1 \equiv 0 \pmod{5}$. Therefore, we see from (3.3.17) that the coefficients of q^{5n+4} in $q(q;q)_\infty^3$ are also multiples of 5. This completes the proof. □

Proof of Theorem 1.3.1 for $k=4$. We have

$$\sum_{n=0}^{\infty} p_4(n) q^n = \frac{1}{(q;q)_\infty (q^4;q^4)_\infty}.$$

Replacing q by $-q$ in the above and then noting that $(-q;-q)_\infty = \dfrac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty}$, we see that

$$\sum_{n=0}^{\infty} (-1)^n p_4(n) q^n = \frac{(q;q)_\infty}{(q^2;q^2)_\infty^3}.$$

With the aid of (3.3.2), the above implies

$$\sum_{n=0}^{\infty} (-1)^n p_4(n) q^n \equiv \frac{(q;q)_\infty (q^2;q^2)_\infty^2}{(q^{10};q^{10})_\infty} \pmod{5}.$$

Employing (3.2.7) in the above and then extracting the terms involving q^{5n} from both sides, we find that

$$\sum_{n=0}^{\infty}(-1)^n p_4(5n)q^n \equiv \frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty}\left\{T(q)T^2(q^2)+q-\frac{q^2}{T(q)T^2(q^2)}\right\} \pmod 5.$$

Since $T(q)=\dfrac{q^{1/5}}{R(q)}$, we employ (3.2.9) in the above to arrive at

$$\sum_{n=0}^{\infty}(-1)^n p_4(5n)q^n \equiv \frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty}\frac{\psi^2(q)}{\psi^2(q^5)} \pmod 5.$$

Employing (1.8.1) in the above and then using (3.3.2), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n p_4(5n)q^n &\equiv \frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty}\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^2\psi^2(q^5)}\\ &\equiv \frac{(q^{10};q^{10})_\infty^2}{\psi^2(q^5)}(q;q)_\infty^3(q^2;q^2)_\infty^3 \pmod 5.\end{aligned}$$

To complete the proof, i.e., to prove (1.3.13), it is now sufficient to show that the coefficients of q^{5n+4} in $(q;q)_\infty^3(q^2;q^2)_\infty^3$ are multiples of 5.

From (3.2.4), we have

$$(q;q)_\infty^3(q^2;q^2)_\infty^3 = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r+s}(2r+1)(2s+1)q^{r(r+1)/2+s(s+1)}.$$

If the exponents of q in the above sum are of the form $5n+4$, then

$$\frac{r(r+1)}{2}+s(s+1)\equiv 4 \pmod 5,$$

which is clearly equivalent to

$$(2r+1)^2+2(2s+1)^2\equiv 0 \pmod 5.$$

But $(2r+1)^2\equiv 0,1$ or $4 \pmod 5$ and $2(2s+1)^2\equiv 0,2$ or $3 \pmod 5$. Therefore, the above congruence is true if and only if $(2r+1)^2\equiv 0\equiv 2(2s+1)^2 \pmod 5$, i.e., $(2r+1)\equiv 0\equiv 2(2s+1) \pmod 5$. Hence, the coefficients of q^{5n+4} in $(q;q)_\infty^3(q^2;q^2)_\infty^3$ are multiples of 5, which is what we desired to show. $\square$