Chapter 4

Parity results for broken 5-, 7- and 11-diamond partitions

4.1 Introduction

Several mathematician studied the congruence properties for broken k-diamond partitions. Very recently, Lin, Malik and Wang [44] studied extensively the congruence properties for broken 5-diamond partitions modulo 2. In the next section, we give some preliminary lemmas which will be used in finding the parity results for $k \in \{5, 7, 11\}$ in the subsequent sections.

The results of this chapter appeared in [3].

4.2 Preliminary Lemmas

Lemma 4.2.1. [19, p. 48, Entry 31] *Let* $U_n = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$ *and* $V_n = a^{\frac{n(n-1)}{2}} b^{\frac{n(n+1)}{2}}$ *for an integer* n*. Then*

$$f(U_1,\ V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r},\ \frac{V_{n-r}}{U_r}\right). \tag{4.2.1}$$

Lemma 4.2.2. [19, p. 69, Eq. (36.8)] *For an even integer* μ *and an integer* ν *with*

$\mu > \nu \geq 0$, *we have*

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \varphi(q^{\mu(\mu^2-\nu^2)})\psi(q^{2\mu}) + \sum_{m=1}^{\mu/2-1} q^{\mu m^2-\nu m} f(q^{(\mu+2m)(\mu^2-\nu^2)},\ q^{(\mu-2m)(\mu^2-\nu^2)}) f(q^{2\nu m},\ q^{2\mu-2\nu m}) + q^{\mu^3/4-\mu\nu/2}\psi(q^{2\mu(\mu^2-\nu^2)}) f(q^{\mu\nu},\ q^{2\mu-\mu\nu}). \tag{4.2.2}$$

By (1.8.2), we also have [19, p. 51, Example (v); p. 350, Eq. (2.3)]

$$f(q, q^5) = \psi(-q^3)\chi(q) \tag{4.2.3}$$

and

$$f(q, q^2) = \frac{\varphi(-q^3)}{\chi(-q)}. \tag{4.2.4}$$

4.3 Parity results for broken 5-diamond partitions

Theorem 4.3.1. *For any non-negative integer* α, *we have*

$$\sum_{n=0}^{\infty} \Delta_5\left(44 \cdot 3^{2\alpha} \cdot n + \frac{44 \cdot 9^{\alpha} + 4}{8}\right) q^n \equiv \psi(q) \pmod 2. \tag{4.3.1}$$

Proof. Setting $k = 5$ in (1.5.2), we have

$$\sum_{n=0}^{\infty} \Delta_5(n) q^n = \frac{(q^2; q^2)_\infty}{(q; q)_\infty^3 (-q^{11}; q^{11})_\infty}.$$

Since

$$(q; q)_\infty^2 \equiv (q^2; q^2)_\infty (\text{mod } 2)$$

and

$$(-q^{11}; q^{11})_\infty \equiv (q^{11}; q^{11})_\infty (\text{mod } 2),$$

we find that

$$\sum_{n=0}^{\infty} \Delta_5(n) q^n \equiv \frac{1}{(q : q)_\infty (q^{11}; q^{11})_\infty} \pmod 2. \tag{4.3.2}$$

Now, setting $\mu = 6$ and $\nu = 5$ in (4.2.2), we have

$$\psi(q)\psi(q^{11}) = \varphi(q^{66})\psi(q^{12}) + qf(q^{88}, q^{44})f(q^2,\ q^{10}) + q^{14}f(q^{20}, q^{-8})f(q^{110}, q^{22}) + q^{15}\psi(q^{132})\varphi(q^6).$$

Employing the trivial identity $f(a, b) = af(a^2b, a^{-1})$, (4.2.3), and (4.2.4) in the above, we obtain

$$\psi(q)\psi(q^{11}) = \varphi(q^{66})\psi(q^{12}) + q^6\frac{\psi(-q^{66})\chi(q^{22})\varphi(-q^{12})}{\chi(-q^4)} + q\frac{\psi(-q^6)\chi(q^2)\varphi(-q^{132})}{\chi(-q^{44})} + q^{15}\psi(q^{132})\varphi(q^6),$$

which, by (1.8.3), is equivalent to

$$\frac{1}{(q;q)_\infty(q^{11};q^{11})_\infty} = \frac{1}{(q^2;q^2)^2_\infty(q^{22};q^{22})^2_\infty}\Big\{\varphi(q^{66})\psi(q^{12}) + q^6\frac{\psi(-q^{66})\chi(q^{22})\varphi(-q^{12})}{\chi(-q^4)} + q\frac{\psi(-q^6)\chi(q^2)\varphi(-q^{132})}{\chi(-q^{44})} + q^{15}\psi(q^{132})\varphi(q^6)\Big\}.$$

Since $\varphi(q) \equiv 1 \pmod 2$, we arrive at

$$\frac{1}{(q;q)_\infty(q^{11};q^{11})_\infty} \equiv \frac{1}{(q^2;q^2)^2_\infty(q^{22};q^{22})^2_\infty}\Big\{\psi(q^{12}) + q^6\frac{\psi(-q^{66})\chi(q^{22})}{\chi(-q^4)} + q\frac{\psi(-q^6)\chi(q^2)}{\chi(-q^{44})} + q^{15}\psi(q^{132})\Big\} \pmod 2. \tag{4.3.3}$$

From (4.3.3) and (4.3.2), we have

$$\sum_{n=0}^{\infty}\Delta_5(n)q^n \equiv \frac{1}{(q^2;q^2)^2_\infty(q^{22};q^{22})^2_\infty}\Big\{\psi(q^{12}) + q^6\frac{\psi(-q^{66})\chi(q^{22})}{\chi(-q^4)} + q\frac{\psi(-q^6)\chi(q^2)}{\chi(-q^{44})} + q^{15}\psi(q^{132})\Big\} \pmod 2.$$

Extracting the terms involving q^{2n} from both sides of the above congruence, and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty}\Delta_5(2n)q^n \equiv \frac{1}{(q;q)^2_\infty(q^{11};q^{11})^2_\infty}\left\{\psi(q^6) + q^3\frac{\psi(-q^{33})\chi(q^{11})}{\chi(-q^2)}\right\} \equiv \frac{1}{(q^2;q^2)_\infty(q^{22};q^{22})_\infty}\left\{\psi(q^6) + q^3\frac{1}{\chi(-q^2)}\frac{(q^{33};q^{33})^3_\infty}{(q^{11};q^{11})_\infty}\right\} \pmod 2. \tag{4.3.4}$$

Now, from Hirschhorn and Roselin's paper [38], we recall that

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}=\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}+q\frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty}.$$

Employing the above in (4.3.4), we have

$$\sum_{n=0}^{\infty}\Delta_5(2n)q^n\equiv\frac{1}{(q^2;q^2)_\infty(q^{22};q^{22})_\infty}\Big\{\psi(q^6)+q^3\frac{1}{\chi(-q^2)}\Big\{\frac{(q^{44};q^{44})_\infty^3(q^{66};q^{66})_\infty^2}{(q^{22};q^{22})_\infty^2(q^{132};q^{132})_\infty}$$
$$+q^{11}\frac{(q^{132};q^{132})_\infty^3}{(q^{44};q^{44})_\infty}\Big\}\Big\}\pmod 2.$$

Extracting the terms involving q^{2n+1} from both sides of the above congruence and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\Delta_5(2(2n+1))q^n\equiv q\frac{(q^{22};q^{22})_\infty^2}{(q;q)_\infty(q^{11};q^{11})_\infty\chi(-q)}\pmod 2.$$

By simplifying the above and using (1.8.3), we have

$$\sum_{n=0}^{\infty}\Delta_5(4n+2)q^n\equiv q(q^{11};q^{11})_\infty^3\pmod 2.$$

Extracting the terms involving q^{11n+1} from both sides of the above and then replacing q^{11} by q, we find that

$$\sum_{n=0}^{\infty}\Delta_5(44n+6)q^n\equiv(q;q)_\infty^3\pmod 2.$$

But

$$(q;q)_\infty^3\equiv(q^2;q^2)_\infty(q;q)_\infty\equiv(-q;q)_\infty(q;q)_\infty^2\equiv\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\equiv\psi(q)\pmod 2.$$

From the above two identities, we arrive at

$$\sum_{n=0}^{\infty}\Delta_5(44n+6)q^n\equiv\psi(q)\pmod 2. \tag{4.3.5}$$

Thus, (4.3.1) holds for $\alpha=0$.

Now, from [19, p. 49, Corollary(ii)]

$$\psi(q)=f(q^3,q^6)+q\psi(q^9). \tag{4.3.6}$$

Employing (4.3.6) in (4.3.5), and then extracting the terms involving q^{3n}, q^{3n+1}, and q^{3n+2}, respectively, from both sides, we find that

$$\sum_{n=0}^{\infty} \Delta_5(44(3n)+6)q^n \equiv f(q,q^2) \pmod{2}, \tag{4.3.7}$$

$$\sum_{n=0}^{\infty} \Delta_5(44(3n+1)+6)q^n \equiv \psi(q^3) \pmod{2}, \tag{4.3.8}$$

and

$$\Delta_5(44(3n+2)+6) \equiv 0 \pmod{2}.$$

Now, extracting the terms involving q^{3n} from both sides of (4.3.8) and then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} \Delta_5(44\cdot 3^2\cdot n+50)q^n \equiv \psi(q) \pmod{2}. \tag{4.3.9}$$

Hence, (4.3.1) is true for $\alpha = 1$.

Now, let (4.3.1) be true for some integer $\alpha \geq 1$, i.e.,

$$\sum_{n=0}^{\infty} \Delta_5\left(44\cdot 3^{2\alpha}\cdot n+\frac{44\cdot 9^{\alpha}+4}{8}\right) q^n \equiv \psi(q) \pmod{2}. \tag{4.3.10}$$

Employing (4.3.6) in (4.3.10), we have

$$\sum_{n=0}^{\infty} \Delta_5\left(44\cdot 3^{2\alpha}\cdot n+\frac{44\cdot 9^{\alpha}+4}{8}\right) q^n \equiv f(q^3,q^6)+q\psi(q^9) \pmod{2}.$$

Extracting the terms involving q^{9n+1} from both sides of the above congruence, we find that

$$\sum_{n=0}^{\infty} \Delta_5\left(44\cdot 3^{2(\alpha+1)}\cdot n+\frac{44\cdot 9^{\alpha+1}+4}{8}\right) q^n \equiv \psi(q) \pmod{2}.$$

Thus, (4.3.1) is also true for $\alpha+1$ when it is true for α. Hence, by mathematical induction the congruence (4.3.1) holds for all $\alpha \geq 1$. $\square$

Remark 4.3.2. *Employing* (1.8.2) *in* (4.3.7), *we find that*

$$\sum_{n=0}^{\infty}\Delta_5(44(3n)+6)q^n \equiv (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty \equiv \frac{(-q;q)_\infty(q^3;q^3)_\infty}{(-q^3;q^3)_\infty}$$
$$\equiv \frac{(-q;q)_\infty(q^3;q^3)_\infty^2}{(q^6;q^6)_\infty} \equiv (-q;q)_\infty \pmod 2, \tag{4.3.11}$$

which can be rewritten as

$$\sum_{n=0}^{\infty}\Delta_5(44(3n)+6)q^n \equiv \sum_{n=0}^{\infty}p_d(n)q^n \pmod 2,$$

where $p_d(n)$ *is the number of partitions of* n *into distinct parts. Thus, we arrive at the following interesting result:*

$$\Delta_5(132n+6) \equiv p_d(n) \pmod 2.$$

Since $(-q;q)_\infty \equiv (q;q)_\infty \pmod 2$, *it is clear from* (4.3.11) *and pentagonal number theorem that*

$$\sum_{n=0}^{\infty}\Delta_5(132n+6)q^n \equiv \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2} \pmod 2.$$

Hence, if n *is not a pentagonal number, then*

$$\Delta_5(132n+6) \equiv 0 \pmod 2.$$

Corollary 4.3.3. *If* n *is not a triangular number then for any non-negative integer* α, *we have*

$$\Delta_5\left(44\cdot 3^{2\alpha}\cdot n + \frac{44\cdot 9^\alpha + 4}{8}\right) \equiv 0 \pmod 2. \tag{4.3.12}$$

Proof. From the definition of $\psi(q)$ in (1.8.1), we observe that the coefficients of q^r is zero if r is not a triangular number. Thus from (4.3.1), we readily arrive at (4.3.12). □

Theorem 4.3.4. *For any odd prime* p *and for any non-negative integers* α *and* n, *we have*

$$\sum_{n=0}^{\infty}\Delta_5\left(396\cdot p^{2\alpha}\cdot n + \frac{99\cdot p^{2\alpha}+1}{2}\right)q^n \equiv \psi(q) \pmod 2. \tag{4.3.13}$$

Proof. From (4.3.9), we have

$$\sum_{n=0}^{\infty} \Delta_5(396 \cdot n + 50) q^n \equiv \psi(q) \pmod 2.$$

Therefore, the congruence (4.3.13) holds for $\alpha = 0$. Now, let (4.3.13) be true for some $\alpha > 0$, i.e.,

$$\sum_{n=0}^{\infty} \Delta_5 \left(396 \cdot p^{2\alpha} \cdot n + \frac{99 \cdot p^{2\alpha} + 1}{2} \right) q^n \equiv \psi(q) \pmod 2. \tag{4.3.14}$$

From Cui and Gu's paper [32], a p-dissection of $\psi(q)$, where p is an odd prime, is

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}} \right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}).$$

Furthermore, for $0 \leq k \leq \dfrac{p-3}{2}$,

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod p.$$

By using the above p-dissection for $\psi(q)$ in (4.3.14) and extracting the terms involving $q^{p^2 n + \frac{p^2-1}{8}}$ from both sides of the congruence and then replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty} \Delta_5 \left(396 \cdot p^{2(\alpha+1)} \cdot n + 396 \cdot p^{2\alpha} \cdot \frac{p^2-1}{8} + \frac{99 \cdot p^{2\alpha} + 1}{2} \right) q^n \equiv \psi(q),$$

which is equivalent to

$$\sum_{n=0}^{\infty} \Delta_5 \left(396 \cdot p^{2(\alpha+1)} \cdot n + \frac{99 \cdot p^{2(\alpha+1)} + 1}{2} \right) q^n \equiv \psi(q) \pmod 2.$$

Thus, the congruence (4.3.13) is true for $\alpha + 1$ if it is true for α. So the proof of (4.3.13) is complete by mathematical induction. □

Corollary 4.3.5. *For any odd prime p, $\alpha \geq 0$ and if n is not a triangular number, then*

$$\Delta_5 \left(396 \cdot p^{2\alpha} \cdot n + \frac{99 \cdot p^{2\alpha} + 1}{2} \right) \equiv 0 \pmod 2. \tag{4.3.15}$$

Proof. From the definition of $\psi(q)$ in (1.8.1), we observe that the coefficients of q^r is zero if r is not a triangular number. Therefore, from (4.3.13), we readily arrive at (4.3.15). □

4.4 Parity results for broken 7-diamond partitions

Theorem 4.4.1. *For all non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty}\Delta_7\left(8\cdot 5^{2\alpha}\cdot n+\frac{16\cdot 5^{2\alpha}+2}{3}\right)q^n\equiv (q;q)_\infty(q^{15};q^{15})_\infty \pmod 2. \tag{4.4.1}$$

Proof. Setting $k=7$ in (1.5.2), we have

$$\sum_{n=0}^{\infty}\Delta_7(n)q^n=\frac{(q^2;q^2)_\infty}{(q;q)_\infty^3(-q^{15};q^{15})_\infty}.$$

Taking congruence modulo 2, we find that

$$\sum_{n=0}^{\infty}\Delta_7(n)q^n\equiv\frac{1}{(q;q)_\infty(q^{15};q^{15})_\infty} \pmod 2. \tag{4.4.2}$$

Now, setting $\mu=8$ and $\nu=7$ in (4.2.2), we obtain

$$\begin{aligned}\psi(q)\psi(q^{15})=\psi(q^{16})\varphi(q^{120})+q^6f(q^{60},q^{180})f(q^4,q^{12})+q^{28}\psi(q^{240})\varphi(q^8)\\ +qf(q^{90},q^{150})f(q^2,q^{14})+q^{15}f(q^{30},q^{210})f(q^6,q^{10}).\end{aligned} \tag{4.4.3}$$

Replacing q by $-q$ in (4.4.3) and adding the resulting identity and (4.4.3), we find that

$$\begin{aligned}\psi(q)\psi(q^{15})+\psi(-q)\psi(-q^{15})=2\{\psi(q^{16})\varphi(q^{120})+q^6f(q^{60},q^{180})f(q^4,q^{12})\\ +q^{28}\psi(q^{240})\varphi(q^8)\}.\end{aligned}$$

But, from [19, p. 377, Entry 9(iv)], we recall that

$$\psi(q)\psi(q^{15})+\psi(-q)\psi(-q^{15})=2\psi(q^6)\psi(q^{10}).$$

From the above two identities, we have

$$\psi(q^6)\psi(q^{10}) = \psi(q^{16})\varphi(q^{120}) + q^6 f(q^{60}, q^{180}) f(q^4, q^{12}) + q^{28}\psi(q^{240})\varphi(q^8). \quad (4.4.4)$$

Employing (4.4.4) in (4.4.3), we obtain

$$\psi(q)\psi(q^{15}) = \psi(q^6)\psi(q^{10}) + q f(q^{90}, q^{150}) f(q^2, q^{14}) + q^{15} f(q^{30}, q^{210}) f(q^6, q^{10}).$$

Thus,

$$\frac{1}{(q;q)_\infty (q^{15};q^{15})_\infty} = \frac{1}{(q^2;q^2)_\infty^2 (q^{30};q^{30})_\infty^2}\{\psi(q^6)\psi(q^{10}) + q f(q^{90}, q^{150}) f(q^2, q^{14}) + q^{15} f(q^{30}, q^{210}) f(q^6, q^{10})\}. \quad (4.4.5)$$

Employing (4.4.5) in (4.4.2) and then extracting the terms involving q^{2n} from both sides of (4.4.2), replacing q^2 by q and then taking congruence modulo 2, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_7(2n) q^n &\equiv \frac{1}{(q^2;q^2)_\infty (q^{30};q^{30})_\infty}\psi(q^3)\psi(q^5) \pmod 2,\\ &\equiv \frac{(q^6;q^6)_\infty^2 (q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty (q^{30};q^{30})_\infty (q^3;q^3)_\infty (q^5;q^5)_\infty} \pmod 2 \end{aligned} \quad (4.4.6)$$

Now, from Baruah and Ojah's paper [16, Eq. (4.11)], we recall that

$$\sum_{n=0}^{\infty} p_{[3^1 5^1]}(2n+1) q^n = q\frac{(q^2;q^2)_\infty^2 (q^{30};q^{30})_\infty^2}{(q^3;q^3)_\infty^2 (q^5;q^5)_\infty^2 (q;q)_\infty (q^{15};q^{15})_\infty}, \quad (4.4.7)$$

where $p_{[3^1 5^1]}(n)$ is the number of partitions of n into parts that are multiples of either 3 or 5 or equivalently,

$$\sum_{n=0}^{\infty} p_{[3^1 5^1]}(n) q^n := \frac{1}{(q^3;q^3)_\infty (q^5;q^5)_\infty}.$$

Extracting the terms involving q^{2n+1} from both sides of (4.4.6), replacing q^2 by q and then employing (4.4.7), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \Delta_7(4n+2) q^n &\equiv q\frac{(q^3;q^3)_\infty^2 (q^5;q^5)_\infty^2 (q^2;q^2)_\infty^2 (q^{30};q^{30})_\infty^2}{(q;q)_\infty (q^{15};q^{15})_\infty (q^3;q^3)_\infty^2 (q^5;q^5)_\infty^2 (q;q)_\infty (q^{15};q^{15})_\infty} \pmod 2\\ &\equiv q(q^2;q^2)_\infty (q^{30};q^{30})_\infty \pmod 2. \end{aligned} \quad (4.4.8)$$

Thus, extracting the terms involving q^{2n+1} from both sides of the above congruence, we obtain

$$\sum_{n=0}^{\infty} \Delta_7(8n+6)q^n \equiv (q;q)_\infty(q^{15};q^{15})_\infty \pmod 2, \tag{4.4.9}$$

which is the case for $\alpha = 0$ in (4.4.1).

Now, let the congruence (4.4.1) be true for some integer $\alpha > 0$, that is,

$$\sum_{n=0}^{\infty} \Delta_7\left(8 \cdot 5^{2\alpha} \cdot n + \frac{16 \cdot 5^{2\alpha}+2}{3}\right) q^n \equiv (q;q)_\infty(q^{15};q^{15})_\infty \pmod 2. \tag{4.4.10}$$

Recall the following 5-dissection of $(q;q)_\infty$ from [19, p. 82],

$$(q;q)_\infty = (q^{25};q^{25})_\infty \left\{ \frac{f(-q^{15}, \ -q^{10})}{f(-q^{20}, \ -q^{5})} - q - q^2 \frac{f(-q^{20}, \ -q^{5})}{f(-q^{15}, \ -q^{10})} \right\}. \tag{4.4.11}$$

Employing (4.4.11) in (4.4.10) and then extracting the terms involving q^{5n+1} from both sides of the resulting congruence, we find that

$$\sum_{n=0}^{\infty} \Delta_7\left(8 \cdot 5^{2\alpha} \cdot (5n+1) + \frac{16 \cdot 5^{2\alpha}+2}{3}\right) q^n \equiv (q^3;q^3)_\infty(q^5;q^5)_\infty \pmod 2.$$

Again, employing (4.4.11), with q replaced by q^3, in the above and then extracting the terms involving q^{5n+3}, we obtain

$$\sum_{n=0}^{\infty} \Delta_7\left(8 \cdot 5^{2\alpha+1} \cdot (5n+3) + 8 \cdot 5^{2\alpha} + \frac{16 \cdot 5^{2\alpha}+2}{3}\right) q^n \equiv (q;q)_\infty(q^{15};q^{15})_\infty \pmod 2,$$

which can be rewritten as

$$\sum_{n=0}^{\infty} \Delta_7\left(8 \cdot 5^{2(\alpha+1)} \cdot n + \frac{16 \cdot 5^{2(\alpha+1)}+2}{3}\right) q^n \equiv (q;q)_\infty(q^{15};q^{15})_\infty \pmod 2.$$

Thus, (4.4.1) is also true for $\alpha+1$ when it is true for α. Therefore, by mathematical induction the congruence (4.4.1) is true for all non negative integer α. □

Corollary 4.4.2. *For any non-negative integers n and α,*

$$\Delta_7\left(8 \cdot 5^{2\alpha+1} \cdot n + 8 \cdot r \cdot 5^{2\alpha} + \frac{16 \cdot 5^{2\alpha}+2}{3}\right) \equiv 0 \pmod 2, \tag{4.4.12}$$

for $r = 3,\ 4,\ 8,\ 9,\ 13,$ *and* 14.

Proof. From (2.2.2), we recall that

$$f(-q,-q^2)=(q;q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^k q^{\frac{k(3k-1)}{2}}.$$

If $\dfrac{k(3k-1)}{2}$ is of the form $15m+r$, then $\dfrac{k(3k-1)}{2}\equiv r \pmod{15}$, which is true for $r=0,\ 1,\ 2,\ 5,\ 6,\ 7,\ \ 10,\ 11,$ and 12 only. Hence by comparing the coefficients of q^{15m+r} on both sides of (4.4.1) for $r=3,\ 4,\ 8,\ 9,\ 13,$ and 14, we easily arrive at (4.4.12). □

Corollary 4.4.3. *We have*

$$\Delta_7(8n+2)\equiv 0 \pmod{2} \tag{4.4.13}$$

and

$$\Delta_7(64n+54)\equiv 0 \pmod{2}. \tag{4.4.14}$$

Proof. Comparing q^{2n} from both sides of (4.4.8) and then replacing q^2 by q, we readily obtain (4.4.13).

Now, from (4.4.9), we have

$$\sum_{n=0}^{\infty}\Delta_7(8n+6)q^n\equiv\frac{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}{(q;q)_\infty(q^{15};q^{15})_\infty} \pmod{2}.$$

Employing (4.4.5) and extracting the terms involving q^{2n} from both sides of the above congruence and then replacing q^2 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_7(16n+6)q^n&\equiv(q;q)_\infty(q^{15};q^{15})_\infty\frac{\psi(q^3)\psi(q^5)}{(q;q)_\infty^2(q^{15};q^{15})_\infty^2}\\&\equiv(q;q)_\infty(q^{15};q^{15})_\infty\frac{(q^3;q^3)_\infty^3(q^5;q^5)_\infty^3}{(q;q)_\infty^2(q^{15};q^{15})_\infty^2}\\&\equiv\frac{(q^3;q^3)_\infty^4(q^5;q^5)_\infty^4}{(q;q)_\infty(q^3;q^3)_\infty(q^5;q^5)_\infty(q^{15};q^{15})_\infty} \pmod{2}.\end{aligned} \tag{4.4.15}$$

From Baruah and Ojah's paper [16, Eq. 4.1], we recall that

$$\sum_{n=0}^{\infty} p_{[1^1 3^1 5^1 15^1]}(2n+1)q^n = \frac{(q^2;q^2)_\infty (q^6;q^6)_\infty (q^{10};q^{10})_\infty (q^{30};q^{30})_\infty}{(q;q)_\infty^2 (q^3;q^3)_\infty^2 (q^5;q^5)_\infty^2 (q^{15};q^{15})_\infty^2} + 2q\frac{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty^2 (q^{10};q^{10})_\infty^2 (q^{30};q^{30})_\infty^2}{(q;q)_\infty^3 (q^3;q^3)_\infty^3 (q^5;q^5)_\infty^3 (q^{15};q^{15})_\infty^3}, \quad (4.4.16)$$

where $p_{[1^1 3^1 5^1 15^1]}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{[1^1 3^1 5^1 15^1]}(n)q^n := \frac{1}{(q;q)_\infty (q^3;q^3)_\infty (q^5;q^5)_\infty (q^{15};q^{15})_\infty}.$$

Extracting the terms involving q^{2n+1} from both sides of (4.4.15), replacing q^2 by q and then employing (4.4.16), we find that

$$\sum_{n=0}^{\infty} \Delta_7(16(2n+1)+6)q^n \equiv (q^6;q^6)_\infty (q^{10};q^{10})_\infty \pmod 2. \quad (4.4.17)$$

Comparing the coefficients of q^{2n+1} from both sides of the above congruence, we easily arrive at (4.4.14). □

Theorem 4.4.4. *For all non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} \Delta_7\left(64\cdot 5^{2\alpha}\cdot n + \frac{64\cdot 5^{2\alpha}+2}{3}\right)q^n \equiv (q^3;q^3)_\infty (q^5;q^5)_\infty \pmod 2. \quad (4.4.18)$$

Proof. From (4.4.17), we have

$$\sum_{n=0}^{\infty} \Delta_7(32n+22)q^n \equiv (q^6;q^6)_\infty (q^{10};q^{10})_\infty \pmod 2.$$

Extracting the terms involving q^{2n} from both sides of the above congruence and replacing q^2 by q , we obtain

$$\sum_{n=0}^{\infty} \Delta_7(64n+22)q^n \equiv (q^3;q^3)_\infty (q^5;q^5)_\infty \pmod 2.$$

From the above congruence we see that (4.4.18) is proved for $\alpha = 0$.

The rest of the proof by mathematical induction is similar to that of (4.4.1). So we omit the details. □

Corollary 4.4.5. *For all non-negative integers α and n, we have*

$$\sum_{n=0}^{\infty} \Delta_7 \left(64 \cdot 5^{2\alpha+1} \cdot n + \frac{448 \cdot 5^{2\alpha} + 2}{3} \right) q^n \equiv 0 \pmod{2}$$
$$\sum_{n=0}^{\infty} \Delta_7 \left(64 \cdot 5^{2\alpha+1} \cdot n + \frac{832 \cdot 5^{2\alpha} + 2}{3} \right) q^n \equiv 0 \pmod{2}.$$

Proof. The above congruences easily follow from (4.4.18) and (4.4.11). □

4.5 Parity results for broken 11-diamond partitions

Theorem 4.5.1. *For any non negative integer α, we have*

$$\sum_{n=0}^{\infty} \Delta_{11}(2 \cdot 23^{\alpha} \cdot n + 1) q^n \equiv 1 + q(q;q)_\infty (q^{23};q^{23})_\infty \pmod{2}. \tag{4.5.1}$$

Proof. From (1.5.2), we noticed that

$$\sum_{n=0}^{\infty} \Delta_{11}(n) q^n = \frac{(q^2;q^2)_\infty (q^{23};q^{23})_\infty}{(q;q)_\infty^3 (q^{46};q^{46})_\infty}.$$

Taking modulo 2, we find that

$$\sum_{n=0}^{\infty} \Delta_{11}(n) q^n \equiv \frac{1}{(q;q)_\infty (q^{23};q^{23})_\infty} \pmod{2}. \tag{4.5.2}$$

From Baruah and Ojah's paper [16, Eq. 1.9], we recall that

$$\sum_{n=0}^{\infty} p_{[1^1 23^1]}(2n+1) q^n = \frac{(q^2;q^2)_\infty (q^{46};q^{46})_\infty}{(q;q)_\infty^2 (q^{23};q^{23})_\infty^2} + q \frac{(q^2;q^2)_\infty^2 (q^{46};q^{46})_\infty^2}{(q;q)_\infty^3 (q^{23};q^{23})_\infty^3}, \tag{4.5.3}$$

where $p_{[1^1 23^1]}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{[1^1 3^1 5^1 15^1]}(n) q^n := \frac{1}{(q;q)_\infty (q^{23};q^{23})_\infty}.$$

Extracting the terms involving q^{2n+1} from both sides of (4.5.2), replacing q^2 by q and employing (4.5.3), we obtain

$$\sum_{n=0}^{\infty} \Delta_{11}(2n+1)q^n \equiv 1 + q(q;q)_\infty (q^{23};q^{23})_\infty \pmod 2 \tag{4.5.4}$$

which is the case for $\alpha = 0$ in (4.5.1).

Now, let (4.5.1) be true for some integer $\alpha > 0$, i.e.

$$\sum_{n=0}^{\infty} \Delta_{11}(2 \cdot 23^\alpha \cdot n+1)q^n \equiv 1 + q(q;q)_\infty (q^{23};q^{23})_\infty \pmod 2. \tag{4.5.5}$$

Setting $U_1 = a = -q$, $V_1 = b = -q^2$ and $n = 23$ in (4.2.1), we find the following 23-dissection of $(q;q)_\infty$:

$$\begin{aligned} f(-q,-q^2) = (q;q)_\infty \\ &= f(-q^{782},-q^{805}) - qf(-q^{851},-q^{736}) + q^5 f(-q^{920},-q^{667}) \\ &\quad - q^{12} f(-q^{989},-q^{598}) + q^{22} f(-q^{1058},-q^{529}) - q^{35} f(-q^{1127},-q^{460}) \\ &\quad + q^{51} f(-q^{1196},-q^{391}) - q^{70} f(-q^{1265},-q^{322}) + q^{92} f(-q^{1334},-q^{253}) \\ &\quad - q^{117} f(-q^{1403},-q^{184}) + q^{145} f(-q^{1472},-q^{115}) - q^{176} f(-q^{1541},-q^{46}) \\ &\quad - q^{187} f(-q^{23},-q^{1564}) + q^{155} f(-q^{92},-q^{1495}) - q^{126} f(-q^{161},-q^{1426}) \\ &\quad + q^{100} f(-q^{230},-q^{1357}) - q^{77} f(-q^{299},-q^{1288}) + q^{57} f(-q^{368},-q^{1219}) \\ &\quad - q^{40} f(-q^{437},-q^{1150}) + q^{26} f(-q^{506},-q^{1081}) - q^{15} f(-q^{575},-q^{1012}) \\ &\quad + q^7 f(-q^{644},-q^{943}) - q^2 f(-q^{713},-q^{874}). \end{aligned} \tag{4.5.6}$$

From (4.5.4) and (4.5.6), it is clear that,

$$\Delta_{11}(2(23n+r)+1) \equiv 0 \pmod 2, \tag{4.5.7}$$

where $r = 5,\ 7,\ 10,\ 11,\ 14,\ 15,\ 17,\ 19,\ 20,\ 21$ and 22.

Again, employing (4.5.6) in (4.5.5), extracting the terms involving q^{23n} from both sides of the resulting congruence and then replacing q^{23} by q, we find that

$$\sum_{n=0}^{\infty} \Delta_{11}(2 \cdot 23^{\alpha+1} \cdot n+1)q^n \equiv 1 + q(q;q)_\infty (q^{23};q^{23})_\infty \pmod 2.$$

Thus, (4.5.1) is also true for $\alpha+1$ when it is true for α. Hence by mathematical induction, (4.5.1) is true for all $\alpha \geq 0$. □

Remark 4.5.2. *Result* (4.5.7) *was earlier proved by Radu and Sellers [53] by using the theory of modular forms.*

Corollary 4.5.3. *For any non-negative integers n and α,*

$$\Delta_{11}(2\cdot 23^{\alpha+1}\cdot n+2\cdot r\cdot 23^{\alpha}+1)\equiv 0 \pmod 2, \tag{4.5.8}$$

for $r=5,\ 7,\ 10,\ 11,\ 14,\ 15,\ 17,\ 19,\ 20,\ 21,\ 22.$

Proof. Comparing the coefficients of q^{23n+r} from both sides of (4.5.1) where $r=5,\ 7,\ 10,\ 11,\ 14,\ 15, 17,\ 19,\ 20,\ 21,\ 22$ from (4.5.1) by employing (4.5.6), we easily arrive at (4.5.8). □

Remark 4.5.4. *For more congruences modulo* 2 *for* 11*-diamond partitions, we refer to a recent paper by Yao [61].*