Chapter 5

Congruences modulo p^2 and p^3 for k dots bracelet partition functions

5.1 Introduction

This chapter deals with the congruences for k dots bracelet partition functions and t-cores. To find the congruences for those partition functions, we use Ramanujan's theta functions and the binomial expansion of q-products for modulo p^2 and p^3.

This chapter is organized as follows.

In the next section, we find two useful congruences modulo p^n for $(q;q)_\infty^{p^n}$ when $n = 2$ and $n = 3$. Section 5.3 contains congruences modulo p^2 and p^3 for k dots bracelet partitions whereas Section 5.4 is on new parity results for 7 and 11 dots bracelet partitions. Congruences modulo 9 and 25 for 3- and 5-cores, respectively, are presented in the final section.

The contents of this chapter appeared in [11] excluding the results for 7 dots bracelet partitions.

5.2 Preliminary Lemmas

Lemma 5.2.1. *For any prime p,*

$$(q;q)_\infty^{p^2} \equiv (q^p;q^p)_\infty^p \pmod{p^2}. \tag{5.2.1}$$

Proof. It is sufficient to prove that

$$(1-q)^{p^2} \equiv (1-q^p)^p \pmod{p^2}.$$

In fact, by the binomial theorem, it is enough to show that

$$\binom{p^2}{np} \equiv \binom{p}{n} \pmod{p^2}. \tag{5.2.2}$$

But from Bailey's paper [10], we recall that, for any positive integer k, r and any prime p,

$$\binom{kp}{rp} \equiv \binom{k}{r} \pmod{p^2},$$

which immediately implies (5.2.2) by setting $k = p$ and $r = n$. □

By taking (np^{r-2})-th power of the congruence (5.2.1), we also note that

$$(q;q)_\infty^{np^r} \equiv (q^p;q^p)_\infty^{np^{r-1}} \pmod{p^2}, \tag{5.2.3}$$

for any $n \in \mathbb{N}$ and $r \geq 2$.

Lemma 5.2.2. *For any prime $p > 3$,*

$$(q;q)_\infty^{p^3} \equiv (q^p;q^p)_\infty^{p^2} \pmod{p^3}. \tag{5.2.4}$$

Proof. The proof is similar to the above lemma. Here we use the following congruence from Bailey's paper [9]:
For any positive integer k, r and any prime $p > 3$

$$\binom{kp}{rp} \equiv \binom{k}{r} \pmod{p^3}.$$

Setting $k = p^2$ and $r = n$, we can easily arrive at (5.2.4). □

By taking (np^{s-3})-th power of the congruence (5.2.1), we also note that

$$(q;q)_\infty^{np^s} \equiv (q^p;q^p)_\infty^{np^{s-1}} \pmod{p^3}, \tag{5.2.5}$$

for any $n \in \mathbb{N}$ and $s \geq 3$.

5.3 Congruences modulo p^2 and p^3 for k dots bracelet partitions

Theorem 5.3.1. *Let $k = mp^r$, where $m \in \mathbb{N}$, $p \geq 5$ and $r \geq 2$. Then for any positive integer n, we have*

$$\mathfrak{B}_k(pn + l) \equiv 0 \pmod{p^2}, \tag{5.3.1}$$

where $1 \leq l \leq p - 1$ and $12l + 1$ is quadratic nonresidue modulo p, i.e., in Legendre symbol $\left(\frac{12l+1}{p}\right) = -1$.

Proof. We note that

$$\sum_{n=0}^{\infty} \mathfrak{B}_k(n)q^n = \frac{(q^2; q^2)_\infty (q^{mp^r}; q^{mp^r})_\infty}{(q; q)_\infty^{mp^r} (q^{2mp^r}; q^{2mp^r})_\infty}. \tag{5.3.2}$$

Employing (5.2.3) in (5.3.2), we obtain

$$\sum_{n=0}^{\infty} \mathfrak{B}_k(n)q^n \equiv \frac{(q^2; q^2)_\infty (q^{mp^r}; q^{mp^r})_\infty}{(q^p; q^p)_\infty^{mp^{r-1}} (q^{2mp^r}; q^{2mp^r})_\infty} \pmod{p^2}. \tag{5.3.3}$$

It is sufficient to find the coefficients of q^{pn+l}, where $1 \leq l \leq p - 1$ and $n \in \mathbb{N}$, in $(q^2; q^2)_\infty$ only, because the coefficients of q^{pn+l} in the other products in (5.3.3) are zero.

From (2.2.2), we have

$$(q^2; q^2)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)}.$$

If the coefficients of q^{pn+l} in the above equation is nonzero, then for some k, we have

$$pn + l = k(3k - 1),$$

i.e.,

$$l \equiv k(3k - 1) \pmod{p}.$$

Thus,

$$12l+1 \equiv (6k-1)^2 \pmod{p}, \tag{5.3.4}$$

which contradict the fact that $12l+1$ is a non quadratic residue modulo p. Thus, we complete the proof of (5.3.1). □

Theorem 5.3.2. *Let $k = mp^s$, where $m \in \mathbb{N}$, $p \geq 5$ and $s \geq 3$. Then for any positive integer n, we have*

$$\mathfrak{B}_k(pn+i) \equiv 0 \pmod{p^3},$$

where $1 \leq i \leq p-1$ and $12i+1$ is quadratic nonresidue modulo p, i.e., in Legendre symbol $\left(\frac{12i+1}{p}\right) = -1$.

Proof. The proof is similar to the above one. Here we use (5.2.5) in place of (5.2.3). □

Theorem 5.3.3. *Let $k = mp^s$, where $m \in \mathbb{N}$, $p \geq 5$ and $s \geq 3$. Then for any positive integer n, we have*

$$\mathfrak{B}_k\left(p(pn+j) + \frac{p^2-1}{12}\right) \equiv 0 \pmod{p^2}, \tag{5.3.5}$$

for $j = 1,\ 2,\ \ldots,\ p-1$.

Proof. We note that

$$\sum_{n=0}^{\infty} \mathfrak{B}_k(n) q^n = \frac{(q^2;q^2)_\infty (q^{mp^s};q^{mp^s})_\infty}{(q;q)_\infty^{mp^s} (q^{2mp^s};q^{2mp^s})_\infty}.$$

Employing (5.2.5) in the above, we obtain

$$\sum_{n=0}^{\infty} \mathfrak{B}_k(n) q^n \equiv \frac{(q^2;q^2)_\infty (q^{mp^s};q^{mp^s})_\infty}{(q^p;q^p)_\infty^{mp^{s-1}} (q^{2mp^s};q^{2mp^s})_\infty} \pmod{p^3}. \tag{5.3.6}$$

Now, replacing q by q^2 in (2.2.2), employing the result in the above and then extracting the terms containing $q^{pn+\frac{p^2-1}{12}}$, dividing both sides by $q^{\frac{p^2-1}{12}}$ and then replacing q^p by q in the above congruence, we arrive at

$$\sum_{n=0}^{\infty}\mathfrak{B}_k\left(pn+\frac{p^2-1}{12}\right)q^n \equiv (-1)^{\frac{\pm p-1}{6}}\frac{(q^{2p};q^{2p})_\infty(q^{mp^{s-1}};q^{mp^{s-1}})_\infty}{(q;q)_\infty^{mp^{s-1}}(q^{2mp^{s-1}};q^{2mp^{s-1}})_\infty} \pmod{p^3},$$

which is also true for modulo p^2, i.e.,

$$\sum_{n=0}^{\infty}\mathfrak{B}_k\left(pn+\frac{p^2-1}{12}\right)q^n \equiv (-1)^{\frac{\pm p-1}{6}}\frac{(q^{2p};q^{2p})_\infty(q^{mp^{s-1}};q^{mp^{s-1}})_\infty}{(q;q)_\infty^{mp^{s-1}}(q^{2mp^{s-1}};q^{2mp^{s-1}})_\infty} \pmod{p^2}.$$

Employing (5.2.3) in the above congruence, we find that

$$\sum_{n=0}^{\infty}\mathfrak{B}_k\left(pn+\frac{p^2-1}{12}\right)q^n \equiv (-1)^{\frac{\pm p-1}{6}}\frac{(q^{2p};q^{2p})_\infty(q^{mp^{s-1}};q^{mp^{s-1}})_\infty}{(q^p;q^p)_\infty^{mp^{s-2}}(q^{2mp^{s-1}};q^{2mp^{s-1}})_\infty} \pmod{p^2} \tag{5.3.7}$$

Comparing the coefficients of q^{pn+j}, where $j = 1,\ 2,\ \ldots,\ p-1$, we complete the proof of (5.3.5). □

Corollary 5.3.4. *Let $k = mp^s$, where $m \in \mathbb{N}$, $p \geq 5$ and $s \geq 3$. Then for any positive integer n, we have*

$$\mathfrak{B}_k\left(p^2n+\frac{p^2-1}{12}\right) \equiv (-1)^{\frac{\pm p-1}{6}}\mathfrak{B}_{k'}(n) \pmod{p^2}, \tag{5.3.8}$$

where $k' = np^{s-2}$.

Proof. Extracting the terms containing q^{pn} from both sides of (5.3.7), and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\mathfrak{B}_k\left(p^2n+\frac{p^2-1}{12}\right)q^n \equiv (-1)^{\frac{\pm p-1}{6}}\frac{(q^2;q^2)_\infty(q^{mp^{s-2}};q^{mp^{s-2}})_\infty}{(q;q)_\infty^{mp^{s-2}}(q^{2mp^{s-2}};q^{2mp^{s-2}})_\infty},$$

i.e.,

$$\sum_{n=0}^{\infty}\mathfrak{B}_k\left(p^2n+\frac{p^2-1}{12}\right)q^n \equiv (-1)^{\frac{\pm p-1}{6}}\sum_{n=0}^{\infty}\mathfrak{B}_{k'}(n)q^n \pmod{p^2}. \tag{5.3.9}$$

Comparing the coefficients of q^n from both side of the above congruence, we readily arrive at (5.3.8). □

5.4 Parity results for 7 and 11 dots bracelet partitions

Theorem 5.4.1. *For any non-negative integer n, we have*

$$\mathfrak{B}_7(2n+1) \equiv b_7(n) \pmod{2}. \tag{5.4.1}$$

Proof. We note from (1.6.1) that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{B}_7(n)q^n &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^4(-q^4;q^4)_\infty} \\ &= \frac{(q^2;q^2)_\infty(q^7;q^7)_\infty}{(q;q)_\infty^7(q^{14};q^{14})_\infty}.\end{aligned}$$

Taking congruent modulo 2 on both sides of the above equation, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{B}_7(n)q^n &\equiv \frac{1}{(q^4;q^4)_\infty(q;q)_\infty(q^7;q^7)_\infty} \\ &\equiv \frac{1}{(q^4;q^4)_\infty}\sum_{n=0}^{\infty} p_{[1^1 7^1]}(n)q^n,\end{aligned} \tag{5.4.2}$$

where $p_{[1^1 7^1]}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{[1^1 7^1]}(n)q^n := \frac{1}{(q;q)_\infty(q^7;q^7)_\infty}.$$

From Baruah and Ojah's paper [16, Eq. 1.7], we recall that

$$\sum_{n=0}^{\infty} p_{[1^1 7^1]}(2n+1)q^n = \frac{(q^2;q^2)_\infty^2(q^{14};q^{14})_\infty^2}{(q;q)_\infty^3(q^7;q^7)_\infty^3},$$

which is equivalent to,

$$\sum_{n=0}^{\infty} p_{[1^1 7^1]}(2n+1)q^n \equiv (q;q)_\infty(q^7;q^7)_\infty \pmod{2}.$$

Now, extracting the terms containing q^{2n+1} from both sides of (5.4.2) and using the above congruence, we easily arrive at (5.4.1). □

Theorem 5.4.2. *If $r \in \{3,4,6\}$ and $s \in \{1,5,6\}$, then for any non-negative integers n and k, we have*

$$\mathfrak{B}_7\left(4\cdot 7^{2k+1}n + 4r\cdot 7^{2k} + \frac{5(7^{2k}-1)}{2} + 3\right) \equiv 0 \pmod 2 \tag{5.4.3}$$

and

$$\mathfrak{B}_7\left(4\cdot 7^{2(k+1)}n + 4s\cdot 7^{2k+1} + \frac{21\cdot 7^{2k}-1}{2} + 1\right) \equiv 0 \pmod 2. \tag{5.4.4}$$

Proof. From (5.4.1), we have

$$\mathfrak{B}_7(2n+1) \equiv b_7(n) \pmod 2. \tag{5.4.5}$$

From Theorem 1.3 of [13], we recall that if $r \in \{3,4,6\}$ and $s \in \{1,5,6\}$, then for all $n, k \geq 0$,

$$b_7\left(2\cdot 7^{2k+1}n + 2r\cdot 7^{2k} + \frac{5(7^{2k}-1)}{4} + 1\right) \equiv 0 \pmod 2$$

and

$$b_7\left(2\cdot 7^{2(k+1)}n + 2s\cdot 7^{2k+1} + \frac{21\cdot 7^{2k}-1}{4}\right) \equiv 0 \pmod 2.$$

Using the above two congruences in (5.4.5), we easily arrive at (5.4.3) and (5.4.4).
□

It has also been proved in [13] that for any prime $p \geq 5$ with $\left(\frac{-14}{p}\right) = -1$, $1 \leq j \leq p-1$, and for all non-negative integers n and k,

$$b_7\left(2\cdot p^{2k+2}n + \frac{(8j+5p)p^{2k+1}-1}{4}\right) \equiv 0 \pmod 2$$

and

$$b_7\left(14\cdot p^{2k+2}n + \frac{(56j+21p)p^{2k+1}-1}{4}\right) \equiv 0 \pmod 2.$$

Employing the above two congruences in (5.4.1), it is now easy to find the following congruences modulo 2 for 7 dots bracelet partitions.

Theorem 5.4.3. *If $p \geq 5$ is a prime with $\left(\frac{-14}{p}\right) = -1$ and $1 \leq j \leq p-1$, then for any non-negative integers n and k,*

$$\mathfrak{B}_7\left(4 \cdot p^{2k+2}n + \frac{(8j+5p)p^{2k+1}-1}{2} + 1\right) \equiv 0 \pmod{2}$$

and

$$\mathfrak{B}_7\left(28 \cdot p^{2k+2}n + \frac{(56j+21p)p^{2k+1}-1}{2} + 1\right) \equiv 0 \pmod{2}.$$

In the remaining part of this section, we find congruences modulo 2 for 11 dots bracelet partitions.

Theorem 5.4.4. *If $p \geq 5$ is a prime, then for any non-negative integer α, we have*

$$\sum_{n=0}^{\infty} \mathfrak{B}_{11}\left(4 \cdot p^{2\alpha} \cdot n + \frac{p^{2\alpha}+5}{6}\right) q^n \equiv (q;q)_\infty \pmod{2}. \tag{5.4.6}$$

Proof. We note from (1.6.1) that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{B}_{11}(n)q^n &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^{11}(-q^{11};q^{11})_\infty} \\ &= \frac{(q^2;q^2)_\infty(q^{11};q^{11})_\infty}{(q;q)_\infty^{11}(q^{22};q^{22})_\infty}.\end{aligned} \tag{5.4.7}$$

Since

$$(q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod{2},$$

we find that

$$\sum_{n=0}^{\infty} \mathfrak{B}_{11}(n)q^n \equiv \frac{1}{(q^8;q^8)_\infty(q;q)_\infty(q^{11};q^{11})_\infty} \pmod{2}. \tag{5.4.8}$$

Employing (4.3.3) in the above, we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{B}_{11}(n)q^n \equiv \frac{1}{(q^8;q^8)_\infty}\frac{1}{(q^2;q^2)_\infty^2(q^{22};q^{22})_\infty^2}\Big\{\psi(q^{12}) + q^6\frac{\psi(-q^{66})\chi(q^{22})}{\chi(-q^4)} \\ + q\frac{\psi(-q^6)\chi(q^2)}{\chi(-q^{44})} + q^{15}\psi(q^{132})\Big\} \pmod{2}.\end{aligned}$$

Extracting the terms involving q^{2n+1} from both sides of the above congruence, dividing both sides by q and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty}\mathfrak{B}_{11}(2n+1)q^n \equiv \frac{1}{(q^4;q^4)_\infty}\frac{1}{(q^2;q^2)_\infty(q^{22};q^{22})_\infty}\Big\{\frac{\psi(-q^3)\chi(q)}{\chi(-q^{22})}+q^7\psi(q^{66})\Big\}$$
$$\equiv \frac{1}{(q^2;q^2)^3_\infty(q^{22};q^{22})_\infty}\Big\{\frac{(q^6;q^6)^2_\infty(q^{44};q^{44})_\infty}{(q^2;q^2)_\infty(q^{22};q^{22})_\infty}\frac{(q;q)_\infty}{(q^3;q^3)_\infty}$$
$$+q^7\frac{(q^{132};q^{132})_\infty}{(q^{66};q^{66})_\infty}\Big\}$$
$$\equiv \frac{1}{(q^2;q^2)^3_\infty(q^{22};q^{22})_\infty}\Big\{\frac{(q^6;q^6)^2_\infty(q^{22};q^{22})_\infty}{(q^2;q^2)_\infty}\frac{(q;q)_\infty}{(q^3;q^3)_\infty}$$
$$+q^7(q^{66};q^{66})_\infty\Big\} \pmod 2. \tag{5.4.9}$$

Now, from Hirschhon and Roselin [38], we recall that

$$\frac{1}{(q;q)_\infty(q^3;q^3)_\infty}=\frac{(q^8;q^8)^2_\infty(q^{12};q^{12})^5_\infty}{(q^2;q^2)^2_\infty(q^4;q^4)_\infty(q^6;q^6)^4_\infty(q^{24};q^{24})^2_\infty}$$
$$+\frac{(q^4;q^4)^5_\infty(q^{24};q^{24})^2_\infty}{(q^2;q^2)^4_\infty(q^6;q^6)^2_\infty(q^8;q^8)^2_\infty(q^{12};q^{12})_\infty}.$$

Multiplying both sides of the above by $(q;q)^2_\infty$ and taking congruent modulo 2, we find that

$$\frac{(q;q)_\infty}{(q^3;q^3)_\infty}\equiv (q^2;q^2)_\infty\Big\{\frac{(q^8;q^8)^2_\infty(q^{12};q^{12})^5_\infty}{(q^2;q^2)^2_\infty(q^4;q^4)_\infty(q^6;q^6)^4_\infty(q^{24};q^{24})^2_\infty}$$
$$+q\frac{(q^4;q^4)^5_\infty(q^{24};q^{24})^2_\infty}{(q^2;q^2)^4_\infty(q^6;q^6)^2_\infty(q^8;q^8)^2_\infty(q^{12};q^{12})_\infty}\Big\}$$
$$\equiv \frac{(q^2;q^2)^5_\infty}{(q^{12};q^{12})_\infty}+q\frac{(q^{12};q^{12})^2_\infty}{(q^2;q^2)_\infty} \pmod 2.$$

Employing the above in (5.4.9), we obtain

$$\sum_{n=0}^{\infty}\mathfrak{B}_{11}(2n+1)q^n \equiv \frac{1}{(q^2;q^2)^3_\infty(q^{22};q^{22})_\infty}\Big\{\frac{(q^6;q^6)^2_\infty(q^{22};q^{22})_\infty}{(q^2;q^2)_\infty}\Big(\frac{(q^2;q^2)^5_\infty}{(q^{12};q^{12})_\infty}$$
$$+q\frac{(q^{12};q^{12})^2_\infty}{(q^2;q^2)_\infty}\Big)+q^7\frac{(q^{132};q^{132})_\infty}{(q^{66};q^{66})_\infty}\Big\}$$
$$\equiv (q^2;q^2)_\infty+q\frac{(q^{12};q^{12})^3_\infty}{(q^2;q^2)^5_\infty}+q^7\frac{(q^{66};q^{66})_\infty}{(q^2;q^2)^3_\infty(q^{22};q^{22})_\infty} \pmod 2.$$

Extracting the terms involving q^{2n} from both sides of the above congruence, and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty}\mathfrak{B}_{11}(4n+1)q^n \equiv (q;q)_\infty \pmod 2. \tag{5.4.10}$$

Hence, (5.4.6) is proved for $\alpha = 0$. Now, let (5.4.6) be true for some $\alpha > 0$, i.e.,

$$\sum_{n=0}^{\infty} \mathfrak{B}_{11}\left(4 \cdot p^{2\alpha} \cdot n + \frac{p^{2\alpha}+5}{6}\right) q^n \equiv (q;q)_\infty \pmod{2}. \tag{5.4.11}$$

Using the above p-dissection for $(q;q)_\infty$ from (2.2.2) in (5.4.11), then extracting the terms involving $q^{pn+\frac{p^2-1}{24}}$ from both sides of the resulting congruence, dividing both sides by $q^{\frac{p^2-1}{24}}$ and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \mathfrak{B}_{11}\left(4 \cdot p^{2\alpha}(p \cdot n + \frac{p^2-1}{24}) + \frac{p^{2\alpha}+5}{6}\right) q^n \equiv (q^p;q^p)_\infty \pmod{2},$$

i.e.,

$$\sum_{n=0}^{\infty} \mathfrak{B}_{11}\left(4 \cdot p^{2\alpha+1} \cdot n + 4 \cdot p^{2\alpha}\frac{p^2-1}{24} + \frac{p^{2\alpha}+5}{6}\right) q^n \equiv (q^p;q^p)_\infty \pmod{2}.$$

Again, extracting the terms involving q^{pn} from both sides of the above and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \mathfrak{B}_{11}\left(4 \cdot p^{2(\alpha+1)} \cdot n + \frac{p^{2(\alpha+1)}+5}{6}\right) q^n \equiv (q;q)_\infty \pmod{2}. \tag{5.4.12}$$

Hence, (5.4.6) is true for $\alpha+1$ if it is true for α. So, by mathematical induction, we complete the proof of (5.4.6). □

Corollary 5.4.5. *If $p \geq 5$ is a prime, then for any non-negative integers α and n, we have*

$$\mathfrak{B}_{11}\left(4 \cdot p^{2\alpha} \cdot n + \frac{p^{2\alpha}+5}{6}\right) \equiv 0 \pmod{2}, \tag{5.4.13}$$

if $n \neq \dfrac{k(3k-1)}{2}$.

Proof. Readily follows from (5.4.6) and (2.2.2). □

5.5 Congruences for 3- and 5-cores

In this section, we find some new congruences for 3-cores modulo 9 and a congruence modulo 25 between 5-cores and $\tau(n)$ by using (5.2.1).

Theorem 5.5.1. *For any integer $\alpha \geq 1$, we have*

$$\sum_{n=0}^{\infty} a_3\left(2^{2\alpha} \cdot n + \frac{4^{\alpha}-1}{3}\right) q^n \equiv (q;q)_{\infty}^8 \pmod 9. \tag{5.5.1}$$

Proof. We note from (1.4.1) that

$$\sum_{n=0}^{\infty} a_3(n) q^n = \frac{(q^3;q^3)_{\infty}^3}{(q;q)_{\infty}},$$

which, by (5.2.1), implies

$$\begin{aligned}\sum_{n=0}^{\infty} a_3(n) q^n &\equiv \frac{(q;q)_{\infty}^9}{(q;q)_{\infty}} \\ &\equiv (q;q)_{\infty}^8 \pmod 9.\end{aligned} \tag{5.5.2}$$

Now, from [19, p. 40, Entry 25], we have

$$\varphi^2(-q) = \varphi^2(q^2) - 4q\psi^2(q^4),$$

which is equivalent to

$$(q;q)_{\infty}^4 = \frac{(q^4;q^4)_{\infty}^{10}}{(q^2;q^2)_{\infty}^2 (q^8;q^8)_{\infty}^4} - 4q \frac{(q^2;q^2)_{\infty}^2 (q^8;q^8)_{\infty}^4}{(q^4;q^4)_{\infty}^2}. \tag{5.5.3}$$

From (5.5.2) and (5.5.3), we have

$$\sum_{n=0}^{\infty} a_3(n) q^n \equiv \left\{ \frac{(q^4;q^4)_{\infty}^{10}}{(q^2;q^2)_{\infty}^2 (q^8;q^8)_{\infty}^4} - 4q \frac{(q^2;q^2)_{\infty}^2 (q^8;q^8)_{\infty}^4}{(q^4;q^4)_{\infty}^2} \right\}^2 \pmod 9.$$

Extracting the term containing q^{4n+1} from both sides of the above congruence, dividing both sides by q and then replacing q^4 by q, we arrive at

$$\sum_{n=0}^{\infty} a_3(4n+1) q^n \equiv (q;q)_{\infty}^8 \pmod 9,$$

which is the case for $\alpha = 1$ in (5.5.1). Now, let (5.5.1) be true for some $\alpha > 1$, i.e.,

$$\sum_{n=0}^{\infty} a_3(2^{2\alpha} \cdot n + \frac{4^{\alpha}-1}{3}) q^n \equiv (q;q)_{\infty}^8 \pmod 9. \tag{5.5.4}$$

Now employing (5.5.3) in the above congruence and extracting the terms containing q^{4n+1} from both sides of the resulting congruence, dividing both sides by q and then replacing q^4 by q, we find that

$$\sum_{n=0}^{\infty} a_3(2^{2\alpha+1}\cdot n + \frac{4^{\alpha+1}-1}{3})q^n \equiv (q;q)_\infty^8 \pmod 9.$$

Hence, (5.5.1) is true for $\alpha+1$ if it is true for α. Thus, by mathematical induction, we complete the proof of (5.5.1). □

Corollary 5.5.2. *For any integer $k \geq 2$ and any non-negative integer n, we have*

$$a_3\left(2^{2k}\cdot n + \frac{3\cdot 2^{2k-1}+4^k-1}{3}\right) \equiv 0 \pmod 9. \tag{5.5.5}$$

Proof. From (5.5.1) we have

$$\sum_{n=0}^{\infty} a_3\left(2^{2\alpha}\cdot n + \frac{4^{\alpha}-1}{3}\right) q^n \equiv (q;q)_\infty^8 \pmod 9. \tag{5.5.6}$$

Employing (5.5.3) in the above congruence and extracting the term containing q^{2n+1} from both sides of the resulting congruence, dividing both sides by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} a_3\left(2^{2\alpha}(2n+1) + \frac{4^{\alpha}-1}{3}\right) q^n \equiv (q^2;q^2)_\infty^8 \pmod 9,$$

i.e.,

$$\sum_{n=0}^{\infty} a_3\left(2^{2\alpha+1}\cdot n + 2^{2\alpha} + \frac{4^{\alpha}-1}{3}\right) q^n \equiv (q^2;q^2)_\infty^8 \pmod 9.$$

Comparing the coefficients of q^{2n+1} from both sides of the above congruence, we arrive at

$$a_3\left(2^{2\alpha+1}\cdot(2n+1) + 2^{2\alpha} + \frac{4^{\alpha}-1}{3}\right) q^n \equiv 0 \pmod 9,$$

i.e.,

$$a_3\left(2^{2(\alpha+1)}\cdot n + \frac{3\cdot 2^{2\alpha+1}+4^{\alpha+1}-1}{3}\right) \equiv 0 \pmod 9,$$

which is equivalent to (5.5.5). □

Theorem 5.5.3. *For any positive integer k and any non-negative integer n, we have*

$$a_3\left(5^{2k}\cdot n+\frac{5^{2k}-1}{3}\right)\equiv(-5)^{3k}a_3(n) \pmod{9}, \tag{5.5.7}$$

$$a_3\left(2^{2k}\cdot n+\frac{2^{2k}-1}{3}\right)\equiv(-8)^{k}a_3(n) \pmod{9}. \tag{5.5.8}$$

Proof. From (5.5.2), we have

$$\sum_{n=0}^{\infty}a_3(n)q^n\equiv(q;q)_\infty^8=\sum_{n=0}^{\infty}p_8(n)q^n. \tag{5.5.9}$$

We note from Baruah and Sarmah's paper [18] that

$$p_8\left(5^{2k}\cdot n+\frac{5^{2k}-1}{3}\right)=(-5)^{3k}p_8(n),$$
$$p_8\left(2^{2k}\cdot n+\frac{2^{2k}-1}{3}\right)=(-8)^{k}p_8(n).$$

Employing the above two identities in (5.5.9), we easily obtain (5.5.7) and (5.5.8). □

Theorem 5.5.4. *For a positive integer n, we have*

$$a_5(n-1)\equiv\tau(n) \pmod{25}. \tag{5.5.10}$$

Proof. We note from (1.4.1) that

$$\sum_{n=0}^{\infty}a_5(n)q^n=\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}.$$

Using (5.2.1) in the above expression, we find that

$$\sum_{n=0}^{\infty}a_5(n)q^n\equiv(q;q)_\infty^{24} \pmod{25}.$$

Multiplying both sides by q in the above congruence and using (1.4.2), we obtain

$$\sum_{n=0}^{\infty}a_5(n)q^{n+1}\equiv\sum_{n=1}^{\infty}\tau(n)q^n \pmod{25}.$$

By comparing the coefficients of q^n in the above congruence, we complete the proof of (5.5.10). □

There are several identities known for Ramamnujan's tau function $\tau(n)$ for modulo 5, 25 and 125. for example:

$$\tau(n) \equiv n\sigma_9(n) \pmod{5},$$
$$\tau(n) \equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{125}, \text{ if } n \text{ is not a multiple of } 5,$$

where $\sigma_s(n)$ denote the (sum of the) sth power of the divisor of n, i.e.,

$$\sigma_s(n) = \sum_{d|n} d^s.$$

From the above identities for $\tau(n)$, we obtain the following congruences for 5-core partitions by using (5.5.10):

$$a_5(5n-1) \equiv n\sigma_9(n) \pmod{5},$$
$$a_5(n-1) \equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{25}.$$

For more identities of tau function, we refer to Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary by Berndt and Ono [21].

Theorem 5.5.5. *For any non-negative integers k and n, we ahve*

$$a_5\left(2^{k+2}\cdot n + 2^{k+2} - 1\right) \equiv r_k a_5(2n+1) + s_k a_5(n) \pmod{25}, \tag{5.5.11}$$

where $r_k = -24r_{k-1} + s_{k-1},\ s_k = -2048r_{k-1}$ *with* $r_0 = -24$ *and* $b_0 = -2048$.

Proof. We have

$$\sum_{n=0}^{\infty} a_5(n)q^n = \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}.$$

By using (5.2.3), taking modulo 25 on both sides of the above equation, we obtain

$$\sum_{n=0}^{\infty} a_5(n)q^n \equiv (q;q)_\infty^{24} = \sum_{n=0}^{\infty} p_{24}(n)q^n.$$

Again, from Baruah and Sarmah's paper [18], we have

$$p_{24}(2^{k+2} \cdot n + 2^{k+2} - 1) = r_k p_{24}(2n+1) + s_k p_{24}(n),$$

where $r_k = -24r_{k-1} + s_{k-1}$, $s_k = -2048r_{k-1}$ with $r_0 = -24$ and $b_0 = -2048$. By using the above identities, we complete the proof of (5.5.11). $\square$