Chapter 6

New congruences for Andrews' singular overpartitions

6.1 Introduction

Recall from Section 1.7 of the introductory chapter that if $\overline{C}_{k,i}(n)$ denotes the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined, then

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n) q^n = \frac{(q^k, -q^i, -q^{k-i}; q^k)_\infty}{(q;q)_\infty}, \tag{6.1.1}$$

where $(a_1, a_2, \ldots, a_k; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty$.

In this chapter, we prove several new congruences for $\overline{C}_{k,i}(n)$ for some k and i by employing Ramanujan's theta functions and p-dissections of q-products.

In Section 6.2, we prove the following congruences for $\overline{C}_{3,1}(n)$ modulo 4, 18 and 36.

Theorem 6.1.1. *If $p \geq 5$ is a prime and $1 \leq j \leq p-1$, then for any non-negative integers α and n, we have*

$$\overline{C}_{3,1}\left(24p^{(2\alpha+1)}(pn+j) + p^{2(\alpha+1)}\right) \equiv 0 \pmod{4}. \tag{6.1.2}$$

Theorem 6.1.2. *For any non-negative integer n, we have*

$$\overline{C}_{3,1}\,(48n+12) \equiv 0 \pmod{18}, \tag{6.1.3}$$

$$\overline{C}_{3,1}\,(12n+7) \equiv 0 \pmod{36}, \tag{6.1.4}$$

$$\overline{C}_{3,1}\,(12n+11) \equiv 0 \pmod{36}, \tag{6.1.5}$$

$$\overline{C}_{3,1}\,(24n+14) \equiv 0 \pmod{36} \tag{6.1.6}$$

and

$$\overline{C}_{3,1}\,(24n+22) \equiv 0 \pmod{36}. \tag{6.1.7}$$

In Section 6.3, we find the following infinite families of congruences modulo 2 and 4 for $\overline{C}_{8,2}(n)$.

Theorem 6.1.3. *If p is a prime such that $p \equiv 3 \pmod 4$ and $1 \leq j \leq p-1$, then for all non-negative integers α and n, we have*

$$\overline{C}_{8,2}\left(p^{2\alpha+1}(pn+j) + \frac{5(p^{2(\alpha+1)}-1)}{24}\right) \equiv 0 \pmod 2. \tag{6.1.8}$$

Theorem 6.1.4. *If p is a prime such that $p \equiv 13,\ 17,\ 19,$ or $23 \pmod{24}$ and $1 \leq j \leq p-1$, then for any non-negative integers α and n, we have*

$$\overline{C}_{8,2}\left(p^{2\alpha+1}(pn+j) + \frac{5(p^{2(\alpha+1)}-1)}{24}\right) \equiv 0 \pmod 4. \tag{6.1.9}$$

The following congruences modulo 2 and 3 for $\overline{C}_{12,2}(n)$ and $\overline{C}_{12,4}(n)$, respectively, are proved in Section 6.4.

Theorem 6.1.5. *If p is a prime such that $p \equiv 3 \pmod 4$ and $1 \leq j \leq p-1$, then for all non-negative integers α and n, we have*

$$\overline{C}_{12,2}\left(p^{2\alpha+1}(pn+j) + 5\cdot\frac{p^{2(\alpha+1)}-1}{8}\right) \equiv 0 \pmod 3. \tag{6.1.10}$$

Theorem 6.1.6. *If p is a prime such that $p \equiv 3 \pmod 4$ and $1 \le j \le p-1$, then for any non-negative integers α and n, we have*

$$\overline{C}_{12,4}\left(p^{2\alpha}(pn+j)+\frac{p^{2\alpha}-1}{8}\right) \equiv 0 \pmod 3. \tag{6.1.11}$$

Theorem 6.1.7. *If p is an odd prime and $1 \le j \le p-1$, then for any non-negative integers α and n, we have*

$$\overline{C}_{12,4}\left(p^{2\alpha+1}(pn+j)+\frac{p^{2(\alpha+1)}-1}{8}\right) \equiv 0 \pmod 2. \tag{6.1.12}$$

In the penultimate section of this chapter, i.e., Section 6.5, we prove the following infinite family of congruences modulo 10 for $\overline{C}_{15,5}(n)$.

Theorem 6.1.8. *For any prime $p \ge 5$ and $1 \le j \le p-1$, then for any non-negative integers α and n, we have*

$$\overline{C}_{15,5}\left(100p^{2\alpha+1}(pn+j)n+25\cdot\frac{p^{2\alpha+2}-1}{6}+4\right)q^n \equiv 0 \pmod{10}. \tag{6.1.13}$$

In the final section, we prove the following parity results for $\overline{C}_{24,8}(n)$ and $\overline{C}_{48,16}(n)$.

Theorem 6.1.9. *If p is a prime such that $p \equiv -1 \pmod 6$ and $1 \le j \le p-1$, then for any non-negative integers α and n, we have*

$$\overline{C}_{24,8}\left(p^{2\alpha+1}(pn+j)+7\cdot\frac{p^{2\alpha}-1}{24}\right) \equiv 0 \pmod 2. \tag{6.1.14}$$

Theorem 6.1.10. *If p is a prime such that $p \equiv 3 \pmod 4$ and $1 \le j \le p-1$, then for any non-negative integers α and n, we have*

$$\overline{C}_{48,16}\left(p^{2\alpha+1}(pn+j)+5\cdot\frac{p^{2\alpha}-1}{8}\right) \equiv 0 \pmod 2.$$

It is worthwhile to mention that, in view of (1.3.2) – (1.3.4) and (6.1.1), for any positive integers ℓ and m with $\ell \ge m$, we have

$$\overline{C}_{10\ell,5m}(5n+4) \equiv 0 \pmod 5,$$
$$\overline{C}_{14\ell,7m}(7n+5) \equiv 0 \pmod 7,$$

and

$$\overline{C}_{22\ell,11m}(11n+6) \equiv 0 \pmod{11},$$

respectively.

The results of this chapter, except Theorem 6.1.8, appeared in [4].

6.2 Congruences modulo 4, 18 and 36 for $\overline{C}_{3,1}(n)$

Theorem 6.2.1. *We have*

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(3n)q^n \equiv \varphi^6(-q) \pmod{9}, \tag{6.2.1}$$

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(3n+1)q^n \equiv 2(q;q)_\infty^4(q^2;q^2)_\infty^2 \pmod{9}, \tag{6.2.2}$$

and

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(3n+2)q^n \equiv 4\frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^4} \pmod{9}. \tag{6.2.3}$$

Proof. Setting $k=3$ and $i=1$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(n)q^n = \frac{(q^3,-q,-q^2;q^3)_\infty}{(q;q)_\infty},$$

which can be rewritten as

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(n)q^n = \frac{\varphi(-q^3)}{\varphi(-q)}. \tag{6.2.4}$$

From Baruah and Ojha's paper [17], we recall that

$$\frac{1}{\varphi(-q)} = \frac{\varphi^3(-q^9)}{\varphi^4(-q^3)}\{1+2qw(q^3)+4q^2w^2(q^3)\},$$

where $w(q) = \dfrac{(q;q)_\infty(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty(q^3;q^3)_\infty^3}$. Using the above in (6.2.4), we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(n)q^n = \frac{\varphi^3(-q^9)}{\varphi^3(-q^3)}\{1+2qw(q^3)+4q^2w^2(q^3)\}.$$

Extracting the terms containing q^{3n+j}, for $j = 0,\ 1,\ 2$, respectively, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(3n)q^n = \frac{\varphi^3(-q^3)}{\varphi^3(-q)}, \tag{6.2.5}$$

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(3n+1)q^n = 2\frac{\varphi^3(-q^3)}{\varphi^3(-q)}w(q), \tag{6.2.6}$$

and

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(3n+2)q^n = 4\frac{\varphi^3(-q^3)}{\varphi^3(-q)}w^2(q). \tag{6.2.7}$$

With the aid of (5.2.1) for $p = 3$, it can be shown that

$$\varphi^9(-q) \equiv \varphi^3(-q^3) \pmod 9.$$

Employing the above in (6.2.5), we arrive at (6.2.1).

Identities (6.2.2) and (6.2.3) can be proved in a similar way. □

Remark 6.2.2. *Since* $\varphi(q) \equiv \varphi(-q) \equiv 1 \pmod 2$*, it follows from* (6.2.4) *that*

$$\overline{C}_{3,1}(n) \equiv 0 \pmod 2 \quad for\ n \geq 1, \tag{6.2.8}$$

which is Theorem 2.9 in [39].

Furthermore, from (6.2.7)*, we have*

$$\overline{C}_{3,1}(3n+2) \equiv 0 \pmod 4 \quad for\ n \geq 1. \tag{6.2.9}$$

Theorem 6.2.3. *If* $p \geq 5$ *is a prime, then for any non-negative integer* α*, we have*

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(24p^{2\alpha}n + p^{2\alpha}\right) q^n \equiv 2(q;q)_\infty \pmod 4. \tag{6.2.10}$$

Proof. From (6.2.6), it follows that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(3n+1)q^n \equiv 2\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} \pmod 4. \tag{6.2.11}$$

But it is known that (for example, see Hirschhorn and Roselin [38])

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = \frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty} + q\frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty}.$$

Employing the above identity in (6.2.11) and then simplifying, we find that

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(3n+1)q^n \equiv 2\left((q^8;q^8)_\infty + q\frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty}\right) \pmod 4. \tag{6.2.12}$$

Extracting the terms containing q^{8n} from both sides of the above congruence and then replacing q^8 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(24n+1)q^n \equiv 2(q;q)_\infty \pmod 4.$$

which is the $\alpha = 0$ case of (6.2.10). Now suppose (6.2.10) holds for some $\alpha \geq 0$. Using (2.2.2) in (6.2.10) and extracting the terms containing $q^{pn+\frac{p^2-1}{24}}$ from both sides of the identity and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}\left(24p^{2\alpha}\left(pn+\frac{p^2-1}{24}\right)+p^{2\alpha}\right)q^n \equiv 2(q^p;q^p)_\infty \pmod 4. \tag{6.2.13}$$

Extracting the terms containing q^{pn} from both sides of the above congruence and replacing q^p by q again, we arrive at

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}\left(24p^{2(\alpha+1)}n+p^{2(\alpha+1)}\right)q^n \equiv 2(q;q)_\infty \pmod 4,$$

which is the $\alpha+1$ case of (6.2.10). □

We now prove Theorem 6.1.1 and Theorem 6.1.2.

Proof of Theorem 6.1.1. Comparing the coefficients of q^{pn+j}, for $1 \leq j \leq p-1$, from both sides of (6.2.13), we easily arrive (6.1.2). □

Proof of Theorem 6.1.2. Recall from [19, Entries 25(i) and (ii), p. 40] that

$$\varphi(-q) = \varphi(q^4) - 2q\psi(q^8). \tag{6.2.14}$$

Employing the above identity in (6.2.1), extracting the terms containing q^{4n} from both sides, and then replacing q^4 by q, we find that

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(12n)q^n \equiv \varphi^6(q) + 6q\varphi^2(q)\psi^4(q^2) \pmod 9. \tag{6.2.15}$$

Again, recall from [19, Entries 25(v) and (vi), p. 40] that

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4). \tag{6.2.16}$$

Employing the above identity in (6.2.15), we find that

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(12n)q^n \equiv \varphi^6(q^2) + 48q^2\varphi^2(q^2)\psi^4(q^4) + 64q^3\psi^6(q^4) + 24q^2\psi^4(q^2)\psi^2(q^4) \pmod 9,$$

where we have also used the trivial identity $\varphi(q)\psi(q^2) = \psi^2(q)$. Extracting the terms involving q^{2n+1} from both sides of the above and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(24n+12)q^n \equiv 64q\psi^6(q^2) \pmod 9,$$

which readily implies that

$$\overline{C}_{3,1}(48n+12) \equiv 0 \pmod 9.$$

Now (6.1.3) follows from the above congruence and (6.2.8).

Again, replacing q by $-q$ in (6.2.16), transforming the theta functions into q-products, and then simplifying, we find that

$$(q;q)_\infty^4 = \frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^4} - 4q\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^4}{(q^4;q^4)_\infty^2},$$

which is a 2-dissection of $(q;q)_\infty^4$. Therefore, (6.2.2) can be expressed as

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(3n+1)q^n \equiv 2(q^2;q^2)_\infty^2\Big(\frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^4} \tag{6.2.17}$$

$$- 4q\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^4}{(q^4;q^4)_\infty^2}\Big) \pmod 9. \tag{6.2.18}$$

Extracting the terms involving q^{2n} from both sides of the above and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(6n+1)q^n \equiv 2\frac{(q^2;q^2)_\infty^{10}}{(q^4;q^4)_\infty^4} \pmod{9}.$$

Comparing the coefficients of q^{2n+1} from both sides of the above congruence, we find that

$$\overline{C}_{3,1}\,(12n+7) \equiv 0 \pmod{9}. \tag{6.2.19}$$

On the other hand, comparing the coefficients of q^{4n+2} and q^{4n+3}, in turn, from both sides of (6.2.12), we also have

$$\overline{C}_{3,1}\,(12n+7) \equiv 0 \pmod{4} \tag{6.2.20}$$

and

$$\overline{C}_{3,1}\,(12n+10) \equiv 0 \pmod{4}. \tag{6.2.21}$$

From (6.2.19) and (6.2.20), we readily arrive at (6.1.4).

Next, extracting the terms containing q^{2n+1} from both sides of (6.2.17) and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(6n+4)q^n \equiv -8\frac{(q;q)_\infty^4(q^4;q^4)_\infty^4}{(q^2;q^2)_\infty^2} \equiv (q^4;q^4)_\infty^4\varphi^2(-q) \pmod{9},$$

which, by (6.2.14), is

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(6n+4)q^n \equiv (q^4;q^4)_\infty^4(\varphi(q^4)-2q\psi(q^8))^2 \pmod{9}.$$

Comparing the coefficients of q^{4n+3} from both sides of the above congruence and also using (6.2.21), we arrive at (6.1.7).

Finally, we turn to prove (6.1.5) and (6.1.6).

With the aid of (6.2.16) and the elementary identity $\varphi(-q)\varphi(q) = \varphi^2(-q^2)$, we have

$$\frac{1}{\varphi^2(-q)} = \frac{\varphi^2(q)}{\varphi^2(-q)\varphi^2(q)} = \frac{\varphi^2(q^2)+4q\psi^2(q^4)}{\varphi^2(-q^2)},$$

from which it follows that

$$\frac{1}{(q;q)^4} = \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}. \tag{6.2.22}$$

Employing the above in (6.2.3), we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(3n+2)q^n \equiv 4\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^4} + 16q(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4 \pmod{9}.$$

Another application of (6.2.22) in the above gives

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(3n+2)q^n \equiv 4\frac{(q^4;q^4)_\infty^{14}}{(q^8;q^8)_\infty^4}\left(\frac{(q^8;q^8)_\infty^{14}}{(q^4;q^4)_\infty^{14}(q^{16};q^{16})_\infty^4} + 4q^2\frac{(q^8;q^8)_\infty^2(q^{16};q^{16})_\infty^4}{(q^4;q^4)_\infty^{10}}\right)$$
$$+ 16q(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4 \pmod{9}.$$

Comparing the coefficients, in turn, of q^{4n+3} and q^{8n+4}, from both sides of the above congruence and also using (6.2.9), we arrive at (6.1.5) and (6.1.6), respectively, to finish the proof. □

Remark 6.2.4. *From* [19, Entries 30 (ii) and (iii), p. 46], *we recall that*

$$f(a,b) + f(-a,-b) = 2f(a^3b, ab^3), \tag{6.2.23}$$

and

$$f(a,b) - f(-a,-b) = 2af\left(b/a, a^5b^3\right). \tag{6.2.24}$$

Setting $a = q$ and $b = q^2$ in the above two identities and with the aid of (6.1.1), *it can be easily shown that*

$$\overline{C}_{3,1}(n) = 2\overline{C}_{12,5}(n) \ \textit{for all}\ n \geq 1,$$

and

$$\overline{C}_{3,1}(n) = 2\overline{C}_{12,11}(n-1) \ \textit{for all}\ n \geq 2.$$

Therefore, the congruences for $\overline{C}_{3,1}(n)$ *found in this section can be recast in terms of* $\overline{C}_{12,5}(n)$ *and* $\overline{C}_{12,11}(n)$*. We also note from the above that*

$$\overline{C}_{12,5}(n) = \overline{C}_{12,11}(n-1) \text{ for all } n \geq 2.$$

Furthermore, adding both (6.2.23) *and* (6.2.24) *and then setting* $a = q^i$ *and* $b = q^{k-i}$ *for* $k > 2i > 1$*, it follows from* (6.1.1) *that*

$$\overline{C}_{k,i}(n+1) = \overline{C}_{4k,2i+k}(n+1) + \overline{C}_{4k,k-2i}(n) \text{ for all } n \geq 0.$$

6.3 Congruences modulo 2 and 4 for $\overline{C}_{8,2}(n)$

Theorem 6.3.1. *If* p *is a prime such that* $p \equiv 3 \pmod 4$*, then for any non-negative integer* α*, we have*

$$\sum_{n=0}^{\infty} \overline{C}_{8,2}\left(p^{2\alpha}n + \frac{5(p^{2\alpha}-1)}{24}\right) q^n \equiv (q;q)_\infty (q^4;q^4)_\infty \pmod 2. \tag{6.3.1}$$

Proof. Setting $k = 8$ and $i = 2$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{8,2}(n) q^n = \frac{(q^8, -q^2, -q^6; q^8)_\infty}{(q;q)_\infty} = \frac{\psi(q^2)}{(q;q)_\infty} = \frac{(q^4;q^4)_\infty^2}{(q;q)_\infty (q^2;q^2)_\infty}. \tag{6.3.2}$$

Since $(q;q)_\infty^2 \equiv (q^2;q^2)_\infty \pmod 2$, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{8,2}(n) q^n \equiv \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty} \equiv (q;q)_\infty (q^4;q^4)_\infty \pmod 2,$$

which is the $\alpha = 0$ case of (6.3.1). Now suppose that (6.3.1) holds for some $\alpha \geq 0$.

Substituting (2.2.2) in (6.3.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{8,2}\left(p^{2\alpha}n + \frac{5(p^{2\alpha}-1)}{24}\right)q^n$$

$$\equiv \Big[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2})\Big]$$

$$\times \Big[\sum_{\substack{m=-\frac{p-1}{2}\\ m\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{4\cdot\frac{3m^2+m}{2}} f(-q^{4\cdot\frac{3p^2+(6m+1)p}{2}}, -q^{4\cdot\frac{3p^2-(6m+1)p}{2}})$$

$$+ (-1)^{\frac{\pm p-1}{6}} q^{4\cdot\frac{p^2-1}{24}} f(-q^{4p^2})\Big] \pmod 2. \tag{6.3.3}$$

Now consider the congruence

$$\frac{3k^2+k}{2} + 4\cdot\frac{3m^2+m}{2} \equiv \frac{5p^2-5}{24} \pmod p, \tag{6.3.4}$$

where $-(p-1)/2 \le k, m \le (p-1)/2$. Since the above congruence is equivalent to

$$(6k+1)^2 + (12m+2)^2 \equiv 0 \pmod p$$

and $\left(\frac{-1}{p}\right) = -1$ as $p \equiv 3 \pmod 4$, the only solution of (6.3.4) is $k = m = \frac{\pm p-1}{6}$. Therefore, extracting the terms containing $q^{pn+\frac{5p^2-5}{24}}$ from both sides of (6.3.3) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{8,2}\left(p^{2\alpha+1}n + \frac{5(p^{2\alpha+2}-1)}{24}\right)q^n \equiv (q^p;q^p)_\infty(q^{4p};q^{4p})_\infty \pmod 2. \tag{6.3.5}$$

Again extracting the terms containing q^{pn} from both sides of the above and replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{8,2}\left(p^{2\alpha+2}n + \frac{5(p^{2\alpha+2}-1)}{24}\right)q^n \equiv (q;q)_\infty(q^4;q^4)_\infty \pmod 2,$$

which is the $\alpha+1$ case of (6.3.1). □

Theorem 6.3.2. *If p is a prime such that $p \equiv 13,\ 17,\ 19,$ or $23 \pmod{24}$, then for any non-negative integer α, we have*

$$\sum_{n=0}^{\infty} \overline{C}_{8,2}\left(p^{2\alpha}n + \frac{5(p^{2\alpha}-1)}{24}\right)q^n \equiv (-1)^{\alpha\cdot\frac{\pm p-1}{6}}\psi(q)(q^2;q^2)_\infty \pmod 4. \tag{6.3.6}$$

Proof. Since $(q^2;q^2)_\infty^2 \equiv (q;q)_\infty^4 \pmod 4$, from (6.3.2), we have

$$\sum_{n=0}^{\infty} \overline{C}_{8,2}(n)q^n \equiv \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty} \equiv \psi(q)(q^2;q^2)_\infty \pmod 4,$$

which is the $\alpha = 0$ case of (6.3.6). Now suppose that (6.3.6) holds for some $\alpha \geq 0$. With the aid of (2.2.2) and (2.2.1), we rewrite (6.3.6) as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{C}_{8,2}\left(p^{2\alpha}n + \frac{5(p^{2\alpha}-1)}{24}\right) q^n \\
&\equiv (-1)^{\alpha\cdot\frac{\pm p-1}{6}} \Big[\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}) \Big] \\
&\quad \times \Big[\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{2\cdot\frac{3k^2+k}{2}} f(-q^{2\cdot\frac{3p^2+(6k+1)p}{2}}, -q^{2\cdot\frac{3p^2-(6k+1)p}{2}}) \\
&\quad + (-1)^{\frac{\pm p-1}{6}} q^{2\cdot\frac{p^2-1}{24}} f(-q^{2p^2}) \Big] \pmod 4. \qquad (6.3.7)
\end{aligned}$$

Now we consider the congruence

$$3k^2 + k + \frac{m^2+m}{2} \equiv \frac{5(p^2-1)}{24}, \qquad (6.3.8)$$

where $-(p-1)/2 \leq k \leq (p-1)/2$ and $0 \leq m \leq p-1$. Since the above congruence is equivalent to

$$(12k+2)^2 + 6(2m+1)^2 \equiv 0 \pmod p$$

and $\left(\frac{-6}{p}\right) = -1$ as $p \equiv 13,\ 17,\ 19,$ or $23 \pmod{24}$, the only solution of (6.3.8) is $k = \frac{\pm p-1}{6}$ and $m = \frac{p-1}{2}$. So, extracting the terms containing $q^{pn+\frac{5p^2-5}{24}}$ from both sides of (6.3.7) and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{8,2}\left(p^{2\alpha+1}n + \frac{5(p^{2\alpha+2}-1)}{24}\right) q^n \equiv (-1)^{(\alpha+1)(\frac{\pm p-1}{6})} \psi(q^p)(q^{2p};q^{2p})_\infty \pmod 4. \qquad (6.3.9)$$

Extracting the terms containing q^{pn} from both sides of the above and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{8,2}\left(p^{2\alpha+2}n + \frac{5(p^{2\alpha+2}-1)}{24}\right) q^n \equiv (-1)^{(\alpha+1)(\frac{\pm p-1}{6})} \psi(q)(q^2;q^2)_\infty \pmod 4,$$

which is the $\alpha+1$ case of (6.3.6). □

We are now in a position to prove Theorem 6.1.3 and Theorem 6.1.4.

Proofs of Theorems 6.1.3 and 6.1.4. Comparing the coefficients of q^{pn+j}, for $1 \leq j \leq p-1$, from both sides of (6.3.5), we arrive at (6.1.8). On the other hand, comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$ from both sides of (6.3.9), we obtain(6.1.9). □

6.4 Congruences modulo 2 and 3 for $\overline{C}_{12,2}(n)$ and $\overline{C}_{12,4}(n)$

Theorem 6.4.1. *If p is a prime such that $p \equiv 3 \pmod 4$, then for any non-negative integer α, we have*

$$\sum_{n=0}^{\infty} \overline{C}_{12,2}\left(p^{2\alpha}n + 5 \cdot \frac{p^{2\alpha}-1}{8}\right) q^n \equiv \psi(q)\psi(q^4) \pmod 3. \tag{6.4.1}$$

Proof. Setting $k=12$ and $i=2$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{12,2}(n) q^n = \frac{(q^{12}, -q^2, -q^{10}; q^{12})_\infty}{(q;q)_\infty}.$$

Manipulating the q-products, with some additional aid from Euler's identity

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty},$$

we have

$$\sum_{n=0}^{\infty} \overline{C}_{12,2}(n) q^n = \frac{1}{(q;q)_\infty} \cdot \frac{(q^6;q^6)_\infty (q^{24};q^{24})_\infty}{(q^{12};q^{12})_\infty} \cdot \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty (q^8;q^8)_\infty}.$$

Taking congruence modulo 3 on both sides of the above and noting that $(q;q)_\infty^3 \equiv (q^3;q^3)_\infty \pmod 3$, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{12,2}(n) q^n \equiv \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \frac{(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty} \equiv \psi(q)\psi(q^4) \pmod 3,$$

which is clearly the $\alpha=0$ case of (6.4.1). Now suppose that (6.4.1) be true for some $\alpha\geq 0$. Substituting (2.2.1) in (6.4.1), we have

$$\sum_{n=0}^{\infty}\overline{C}_{12,2}\left(p^{2\alpha}n+5\cdot\frac{p^{2\alpha}-1}{8}\right)q^n$$

$$\equiv\Big[\sum_{k=0}^{\frac{p-3}{2}}q^{\frac{k^2+k}{2}}f(q^{\frac{p^2+(2k+1)p}{2}},q^{\frac{p^2-(2k+1)p}{2}})+q^{\frac{p^2-1}{8}}\psi(q^{p^2})\Big] \tag{6.4.2}$$

$$\times\Big[\sum_{m=0}^{\frac{p-3}{2}}q^{4\cdot\frac{m^2+m}{2}}f(q^{4\cdot\frac{p^2+(2m+1)p}{2}},q^{4\cdot\frac{p^2-(2m+1)p}{2}})+q^{4\cdot\frac{p^2-1}{8}}\psi(q^{4p^2})\Big]\ (\text{mod } 3). \tag{6.4.3}$$

For $0\leq k,m\leq p-1$, we now consider the congruence

$$\frac{k^2+k}{2}+4\cdot\frac{m^2+m}{2}\equiv\frac{5p^2-5}{8}\ (\text{mod } p), \tag{6.4.4}$$

which is equivalent to

$$(2k+1)^2+(4m+2)^2\equiv 0\ (\text{mod } p).$$

Since $\left(\frac{-1}{p}\right)=-1$ for $p\equiv 3\ (\text{mod } 4)$, the only solution of (6.4.4) is $k=m=\frac{p-1}{2}$. Therefore, extracting the terms involving $q^{pn+\frac{5p^2-5}{8}}$ from both sides of (6.4.2) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty}\overline{C}_{12,2}\left(p^{2\alpha}\left(pn+\frac{5p^2-5}{8}\right)+5\cdot\frac{p^{2\alpha}-1}{8}\right)q^n\equiv\psi(q^p)\psi(q^{4p})\ (\text{mod } 3). \tag{6.4.5}$$

Again extracting the terms involving q^{pn} form both sides of the above and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{12,2}\left(p^{2(\alpha+1)}n+5\cdot\frac{p^{2(\alpha+1)}-1}{8}\right)q^n\equiv\psi(q)\psi(q^4)\ (\text{mod } 3),$$

which is the $\alpha+1$ case of (6.4.1). □

Theorem 6.4.2. *If p is a prime such that $p\equiv 3\ (\text{mod } 4)$, then for any non-negative integer α, we have*

$$\sum_{n=0}^{\infty}\overline{C}_{12,4}\left(p^{2\alpha}n+\frac{p^{2\alpha}-1}{8}\right)q^n\equiv\psi(q)\varphi(q^2)\ (\text{mod } 3). \tag{6.4.6}$$

Proof. Setting $k = 12$ and $i = 4$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{12,4}(n)q^n = \frac{(q^{12}, -q^4, -q^8; q^{12})_\infty}{(q;q)_\infty},$$

which, by manipulation of the q-products, yields

$$\sum_{n=0}^{\infty} \overline{C}_{12,4}(n)q^n = \frac{(q^{12};q^{12})_\infty^2 (q^8;q^8)_\infty}{(q;q)_\infty (q^{24};q^{24})_\infty (q^4;q^4)_\infty}. \tag{6.4.7}$$

Taking congruent modulo 3 on both sides of the above and then employing $(q;q)_\infty^3 \equiv (q^3;q^3)_\infty \pmod 3$, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}_{12,4}(n)q^n &\equiv \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty^2} \\ &\equiv \psi(q)\varphi(q^2) \pmod 3,\end{aligned}$$

which is the $\alpha = 0$ case of (6.4.6). Now suppose that (6.4.6) holds for some $\alpha \geq 0$.

From [19, p. 49]), we recall that for any prime p,

$$\varphi(q) = \varphi(q^{p^2}) + \sum_{r=0}^{p-1} q^{r^2} f(q^{p(p-2r)}, q^{p(p+2r)}).$$

Now, substituting (2.2.1) and the above p-dissection of $\varphi(q)$ in (6.4.6), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}_{12,4}\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{8}\right) q^n \equiv &\Big[\sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}) + q^{\frac{p^2-1}{8}} \psi(q^{p^2})\Big] \\ &\times \Big[\varphi(q^{2p^2}) + \sum_{r=0}^{p-1} q^{2r^2} f(q^{2p(p-2r)}, q^{2p(p+2r)})\Big]\end{aligned} \tag{6.4.8}$$

Now consider the congruence

$$\frac{k^2+k}{2} + 2r^2 \equiv \frac{p^2-1}{8} \pmod p, \tag{6.4.9}$$

where $0 \leq k, r \leq p-1$. Since the above congruence is equivalent to

$$(2k+1)^2 + (4r)^2 \equiv 0 \pmod p,$$

and $\left(\frac{-1}{p}\right) = -1$ as $p \equiv 3 \pmod 4$, the only solution of (6.4.9) is $k = \frac{p-1}{2}$ and $r = 0$. Therefore, extracting the terms involving $q^{pn+\frac{p^2-1}{8}}$ from both sides of (6.4.8) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{12,4}\left(p^{2\alpha}\left(pn + \frac{p^2-1}{8}\right) + \frac{p^{2\alpha}-1}{8}\right) q^n \equiv \psi(q^p)\varphi(q^{2p}) \pmod 3. \qquad (6.4.10)$$

Again, extracting the terms involving q^{pn} from both sides of the above congruence and replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{12,4}\left(p^{2(\alpha+1)}n + \frac{p^{2(\alpha+1)}-1}{8}\right) q^n \equiv \psi(q)\varphi(q^2) \pmod 3,$$

which is the $\alpha + 1$ case of (6.4.6). □

Theorem 6.4.3. *If p is an odd prime, then for any non-negative integer α, we have*

$$\sum_{n=0}^{\infty} \overline{C}_{12,4}\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{8}\right) q^n \equiv \psi(q) \pmod 2. \qquad (6.4.11)$$

Proof. From (6.4.7), we have

$$\sum_{n=0}^{\infty} \overline{C}_{12,4}(n)q^n \equiv \frac{(q^4;q^4)_\infty}{(q;q)_\infty} \equiv \psi(q) \pmod 2,$$

which is clearly the $\alpha = 0$ case of (6.4.11). Now suppose that (6.4.11) holds for some $\alpha \geq 0$. Now using (2.2.1) in (6.4.11), extracting the terms involving $q^{pn+\frac{p^2-1}{8}}$ from both sides of the above congruence and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{12,4}\left(p^{2\alpha}\left(pn + \frac{p^2-1}{8}\right) + \frac{p^{2\alpha}-1}{8}\right) q^n \equiv \psi(q^p) \pmod 2. \qquad (6.4.12)$$

Again extracting the terms involving q^{pn} from both sides of the above congruence and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{12,4}\left(p^{2(\alpha+1)}n + \frac{p^{2(\alpha+1)}-1}{8}\right) q^n \equiv \psi(q) \pmod 2,$$

which is the $\alpha + 1$ case of (6.4.11). □

We now prove Theorems 6.1.5 – 6.1.7.

Proofs of Theorems 6.1.5 – 6.1.7. Comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$, on both sides of (6.4.5), we arrive at (6.1.10). Next, comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$, from both sides of (6.4.10), we obtain (6.1.11). Finally, comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$, on both sides of (6.4.12), we arrive at (6.1.12). □

6.5 Congruences modulo 10 for $\overline{C}_{15,5}(n)$

Theorem 6.5.1. *If $p \geq 5$ is a prime, then for any non-negative integer α, we have*

$$\sum_{n=0}^{\infty} \overline{C}_{15,5}\left(100p^{2\alpha}n + 25 \cdot \frac{p^{2\alpha}-1}{6} + 4\right) q^n \equiv (q;q)_\infty \pmod{2}. \tag{6.5.1}$$

Proof. Setting $k = 15$ and $i = 5$ in (6.1.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}_{15,5}(n) q^n &= \frac{f(q^5, q^{10})}{(q;q)_\infty} \\ &= f(q^5, q^{10}) \sum_{n=0}^{\infty} p(n) q^n,\end{aligned}$$

where $p(n)$ is the ordinary partition function. Extracting the terms containing q^{5n+4} from the above congruence, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{15,5}(5n+4) q^n = f(q, q^2) \sum_{n=0}^{\infty} p(5n+4) q^n.$$

In view of (1.3.1) and (4.2.4), the above becomes

$$\sum_{n=0}^{\infty} \overline{C}_{15,5}(5n+4) q^n = 5 \frac{\varphi(-q^3)(q^5;q^5)_\infty^5}{\chi(-q)(q;q)_\infty^6}.$$

Under modulo 2, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}_{15,5}(5n+4) q^n &\equiv \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^5} \\ &\equiv \frac{(q^{20};q^{20})_\infty (q^5;q^5)_\infty}{(q^4;q^4)_\infty (q;q)_\infty} \pmod{2}.\end{aligned} \tag{6.5.2}$$

Now, from Hirschhorn and Sellers' paper [41], we have

$$\frac{(q^5;q^5)_\infty}{(q;q)_\infty} = \frac{(q^8;q^8)_\infty(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2(q^{40};q^{40})_\infty} + q\frac{(q^4;q^4)_\infty^3(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^3(q^8;q^8)_\infty(q^{20};q^{20})_\infty}.$$

Using this in (6.5.2), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{C}_{15,5}(5n+4)q^n &\equiv \frac{(q^{20};q^{20})_\infty}{(q^4;q^4)_\infty}\Big(\frac{(q^8;q^8)_\infty(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2(q^{40};q^{40})_\infty} \\ &\quad + q\frac{(q^4;q^4)_\infty^3(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^3(q^8;q^8)_\infty(q^{20};q^{20})_\infty}\Big) \\ &\equiv \frac{(q^{20};q^{20})_\infty}{(q^4;q^4)_\infty}\left((q^4;q^4)_\infty + q\frac{(q^{10};q^{10})_\infty^3}{(q^2;q^2)_\infty}\right) \pmod 2.\end{aligned}$$

Extracting the terms containing q^{20n} from the above congruence and replacing q^{20} by q, we find that

$$\sum_{n=0}^{\infty}\overline{C}_{15,5}(100n+4)q^n \equiv (q;q)_\infty \pmod 2.$$

Hence, (6.5.1) is true for $\alpha = 0$. Now suppose that (6.5.1) is true for some $\alpha \geq 0$, i.e.,

$$\sum_{n=0}^{\infty}\overline{C}_{15,5}\left(100p^{2\alpha}n + 25\cdot\frac{p^{2\alpha}-1}{6}+4\right)q^n \equiv (q;q)_\infty \pmod 2.$$

Using the p-dissection of $(q;q)_\infty$ from (2.2.2) in the above, extracting the terms containing $q^{p^2n+\frac{p^2-1}{24}}$ from both sides, and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{15,5}\left(100p^{2\alpha}(p^2n+\frac{p^2-1}{24}) + 25\cdot\frac{p^{2\alpha}-1}{6}+4\right)q^n \equiv (q;q)_\infty$$

or

$$\sum_{n=0}^{\infty}\overline{C}_{15,5}\left(100p^{2\alpha+2}n + 25\cdot\frac{p^{2\alpha+2}-1}{6}+4\right)q^n \equiv (q;q)_\infty \pmod 2.$$

Therefore, (6.5.1) is true for $\alpha+1$ if it is true for some $\alpha \geq 0$. So, by mathematical induction, we complete the proof of (6.5.1). □

Proof of Theorem 6.1.8. From (6.5.1) and (2.2.2) and also employing (1.3.2), we easily arrive at (6.1.13). □

6.6 Congruences modulo 2 for $\overline{C}_{24,8}(n)$ and $\overline{C}_{48,16}(n)$

Theorem 6.6.1. *If p is a prime such that $p \equiv -1 \pmod 6$, then for any non-negative integer α, we have*

$$\sum_{n=0}^{\infty} \overline{C}_{24,8}\left(p^{2\alpha}n + 7\cdot\frac{p^{2\alpha}-1}{24}\right)q^n \equiv \psi(q)(q^4;q^4)_\infty \pmod 2. \tag{6.6.1}$$

Proof. Setting $k = 24$ and $i = 8$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{24,8}(n)q^n = \frac{(q^{24}, -q^8, -q^{16}; q^{24})_\infty}{(q;q)_\infty}.$$

Thus

$$\sum_{n=0}^{\infty} \overline{C}_{24,8}(n)q^n \equiv \psi(q)(q^4;q^4)_\infty \pmod 2,$$

which is the $\alpha = 0$ case of (6.6.1). Now suppose that (6.6.1) holds for some $\alpha \geq 0$. With the aid of (2.2.2) and (2.2.1), we rewrite (6.6.1) as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{C}_{24,8}\left(p^{2\alpha}n + 7\cdot\frac{p^{2\alpha}-1}{24}\right)q^n \\
&\equiv \Big[\sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\Big] \\
&\quad\times\Big[\sum_{\substack{m=-\frac{p-1}{2}\\ m\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{4\cdot\frac{3m^2+m}{2}} f(-q^{4\cdot\frac{3p^2+(6m+1)p}{2}}, -q^{4\cdot\frac{3p^2-(6m+1)p}{2}}) \\
&\quad + (-1)^{\frac{\pm p-1}{6}} q^{4\cdot\frac{p^2-1}{24}} f(-q^{4p^2})\Big] \pmod 2.
\end{aligned} \tag{6.6.2}$$

Now consider the congruence

$$\frac{k^2+k}{2} + 4\cdot\frac{3m^2+m}{2} \equiv \frac{7(p^2-1)}{24}, \tag{6.6.3}$$

where $0 \leq k \leq p-1$ and $-(p-1)/2 \leq m \leq (p-1)/2$. Since the above congruence is equivalent to

$$3(2k+1)^2 + (12m+2)^2 \equiv 0 \pmod p,$$

and $\left(\frac{-3}{p}\right) = -1$ as $p \equiv -1 \pmod 6$, the only one solution of (6.6.3) is $k = \frac{p-1}{2}$ and $m = \frac{p-1}{6}$. Therefore, extracting the terms involving $q^{pn+7\frac{p^2-1}{24}}$ from both sides of (6.6.2) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{24,8}\left(p^{2\alpha}\left(pn + 7\cdot\frac{p^2-1}{24}\right) + 7\cdot\frac{p^{2\alpha}-1}{24}\right) q^n \equiv \psi(q^p)(q^{4p};q^{4p})_\infty \pmod 2. \tag{6.6.4}$$

Extracting the terms containing q^{pn} from both sides of the above and then replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} \overline{C}_{24,8}\left(p^{2(\alpha+1)}n + 7\cdot\frac{p^{2(\alpha+1)}-1}{24}\right) q^n \equiv \psi(q)(q^4;q^4)_\infty \pmod 2,$$

which is the $\alpha+1$ case of (6.6.1). □

We now prove Theorem 6.1.9.

Proof of Theorem 6.1.9. Comparing the coefficients of q^{pn+j}, $1 \le j \le p-1$, from both sides of(6.6.4), we immediately arrive at (6.1.14). □

Theorem 6.6.2. *If p is a prime such that $p \equiv 3 \pmod 4$, then for any non-negative integer α, we have*

$$\sum_{n=0}^{\infty} \overline{C}_{48,16}\left(p^{2\alpha}n + 5\cdot\frac{p^{2\alpha}-1}{8}\right) q^n \equiv \psi(q)\psi(q^4) \pmod 2.$$

Proof. Setting $k = 48$ and $i = 16$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{48,16}(n)q^n = \frac{(q^{48}, -q^{16}, -q^{32}; q^{24})_\infty}{(q;q)_\infty}.$$

Therefore,

$$\sum_{n=0}^{\infty} \overline{C}_{48,16}(n)q^n \equiv \psi(q)\psi(q^4) \pmod 2.$$

The rest of the proof is similar to that of Theorem 6.4.1, so we omit the proof. □

Theorem 6.1.10 is an easy consequence of an intermediate step of the proof of the above theorem. So we omit the proof.