Chapter 1

Introduction

Our thesis comprised of six chapters including this introductory chapter. In this chapter, we define various partition functions and a brief account of our work. In the remaining five chapters, we find several new congruences for some partition functions.

1.1 Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a non-negative integer n is a finite sequence of non-increasing positive integer *parts* λ_i such that $n = \sum_{i=1}^{k} \lambda_i$. The partition function $p(n)$ is the number of partitions of a non-negative integer n, with the convention that $p(0) = 1$. For example, we have $p(4) = 5$, as there are five partitions of 4, namely, (4), $(3,1)$, $(2,2)$, $(2,1,1)$ and $(1,1,1,1)$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, here and throughout the thesis, for $|q| < 1$, $(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n)$.

1.2 ℓ-regular partition function

A partition of a positive integer n is said to be an ℓ-regular partition, $\ell > 1$, if none of its parts is divisible by ℓ. For example, $(8,6,5,1)$ is a 7-regular partition of 20 as none of its parts is divisible by 7. If $b_\ell(n)$ denotes the number of ℓ-regular partitions of n then, with the convention that $b_\ell(0) = 1$, the generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}. \tag{1.2.1}$$

Recently, several mathematicians studied the congruence properties for ℓ-regular partitions for certain values of ℓ. We refer to Hou, Sun, Zhang [42] and the references listed there for details. Further work can be found in [33, 56, 23, 13, 46, 47]. In particular, by using the theory of modular forms, Carlson and Webb [23] obtained the following congruences.

Theorem 1.2.1. *If p is a prime such that $p \equiv 5$ or $7 \pmod 8$ and $1 \le j \le p-1$, then for all $\alpha, n \ge 0$,*

$$b_{10}\left(p^{2\alpha+1}(pn+j) + 3\cdot\frac{p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod 5. \tag{1.2.2}$$

Theorem 1.2.2. *If p is a prime such that $p \equiv 5 \pmod 6$ and $1 \le j \le p-1$, then for all $\alpha, n \ge 0$,*

$$b_{15}\left(p^{2\alpha+1}(pn+j) + 7\cdot\frac{p^{2\alpha+2}-1}{24}\right) \equiv 0 \pmod 5. \tag{1.2.3}$$

Theorem 1.2.3. *If p is a prime such that $p \equiv 5 \pmod 6$ and $1 \le j \le p-1$, then for all $\alpha, n \ge 0$,*

$$b_{20}\left(p^{2\alpha+1}(pn+j) + 19\cdot\frac{p^{2\alpha+2}-1}{24}\right) \equiv 0 \pmod 5. \tag{1.2.4}$$

In Chapter 2 of this thesis, we find alternative proofs of (1.2.2) and (1.2.4) by employing p-dissections of some q-products. Note that we could not apply these p-dissections to effect a proof of (1.2.3). We also find new infinite families of congruences for ℓ-regular partitions for $\ell \in \{5, 6, 7, 49\}$. For example, we have the following results.

Theorem 1.2.4. *If p is a prime such that $p \equiv -1 \pmod 6$ and $1 \le j \le p-1$, then for all $\alpha, n \ge 0$,*

$$b_5\left(25p^{2\alpha+1}(pn+j) + \frac{25p^{2\alpha+2}-1}{6}\right) \equiv 0 \pmod{25}.$$

Theorem 1.2.5. *If $p \ge 11$ is a prime such that -7 is a quadratic nonresidue modulo p, i.e., in Legendre symbol $\left(\frac{-7}{p}\right) = -1$, and $1 \le j \le p-1$, then for all $\alpha, n \ge 0$,*

$$b_{49}\left(7p^{2\alpha+1}(pn+j) + 7(p^{2\alpha+2}-1) + 5\right) \equiv 0 \pmod{49}.$$

1.3 2-color partition function

Ramanujan's so-called "most beautiful identity" for the partition function $p(n)$ is

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.3.1}$$

which readily implies one of his three famous partition congruences modulo 5, 7 and 11, namely,

$$p(5n+4) \equiv 0 \pmod 5. \tag{1.3.2}$$

The other two famous partition congruences found by Ramanujan are

$$p(7n+5) \equiv 0 \pmod 7 \tag{1.3.3}$$

and

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3.4}$$

We refer to a recent paper by Bruinier, Folsom, Kent and Ono [22] for further references on the partition function.

Now, let $p_0(n) := p(n)$ and for a positive integer k, let $p_k(n)$ denote the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of k. Then the generating function for $p_k(n)$ is given by

$$\sum_{n=0}^{\infty} p_k(n)q^n = \frac{1}{(q;q)_\infty(q^k;q^k)_\infty}.$$

It is clear from (1.3.2), with n replaced by $5n+4$ that

$$p_0(25n+24) \equiv 0 \pmod 5. \tag{1.3.5}$$

A stronger version

$$p_0(25n+24) \equiv 0 \pmod{25},$$

can also be deduced easily from (1.3.1) (see [20, p. 38]).

For $k = 1$, it is known that [18, Eq. (5.4)]

$$p_1(25n + 23) \equiv 0 \pmod{5}, \tag{1.3.6}$$

which can also be shown to be true for modulo 25.

For $k = 2$, Chan [25] found an analog of (1.3.1), namely,

$$\sum_{n=0}^{\infty} p_2(3n + 2)q^n = 3\frac{(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q; q)_\infty^4 (q^2; q^2)_\infty^4},$$

which immediately implies $p_2(3n+2) \equiv 0 \pmod{3}$. By using the theory of modular forms, Chen and Lin [27] found four new congruences for $p_2(n)$ modulo 7 as well as the congruence

$$p_2(25n + 22) \equiv 0 \pmod{5}. \tag{1.3.7}$$

Furthermore, when $k = 5, 10, 15, 20$, it follows from (1.3.1) and the generating function for $p_k(n)$ that

$$p_5(25n + 19) \equiv 0 \pmod{5}, \tag{1.3.8}$$

$$p_{10}(25n + 14) \equiv 0 \pmod{5}, \tag{1.3.9}$$

$$p_{15}(25n + 9) \equiv 0 \pmod{5}, \tag{1.3.10}$$

and

$$p_{20}(25n + 4) \equiv 0 \pmod{5}. \tag{1.3.11}$$

It can be easily seen that (1.3.8) also holds for modulo 25.

In Chapter 3, we give an alternative proof of (1.3.7) and also find two new congruences for $p_k(n)$ modulo 5 for $k = 3$ and 4:

$$p_3(25n + 21) \equiv 0 \pmod{5} \tag{1.3.12}$$

and

$$p_4(25n + 20) \equiv 0 \pmod{5}. \tag{1.3.13}$$

We employ Ramanujan's simple theta function identities and some other known identities for the Rogers-Ramanujan continued fraction $R(q)$, defined by

$$R(q) := \frac{q^{1/5}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \cdots = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}, \ |q|<1.$$

Interestingly, all nine congruences (1.3.5)–(1.3.13) can be written as a combined result.

Theorem 1.3.1. *If $k \in \{0,1,2,3,4,5,10,15,20\}$, then for any non-negative integer n,*

$$p_k(25n+\ell) \equiv 0 \pmod 5, \tag{1.3.14}$$

where $k+\ell = 24$.

Note that congruences (1.3.7), (1.3.12) and (1.3.13) do not hold for higher powers of the modulus 5 as $p_2(22) = 5630$, $p_3(21) = 2035$ and $p_4(20) = 1110$. It seems that (1.3.14) may hold for some more values of $k > 5$. We make the following conjecture.

Conjecture 1.3.1. *If $k \in \{7,8,17\}$, then for any non-negative integer n,*

$$p_k(25n+\ell) \equiv 0 \pmod 5,$$

where $k+\ell = 24$.

We acknowledge that work on Theorem 1.3.1 for $k \in \{3,4\}$ and Conjecture 1.3.1 was first initiated when Mr. Manosij Ghosh Dastidar, Department of Mathematical Sciences, Pondicherry University, India, visited my supervisor Prof. Nayandeep Deka Baruah as a Winter Intern in December, 2014.

1.4 t-cores

The Ferrers-Young diagram of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a left-aligned array of nodes with λ_i nodes in the ith row. The conjugate of a partition λ, denoted by λ', is the partition whose Ferrers-Young diagram is the reflection along the main diagonal of the diagram of λ. A partition λ is self-conjugate if $\lambda = \lambda'$. Let λ'_j denote the number of nodes in column j in the Ferrers-Young diagram of λ. The hook number of the (i,j)-node in the Ferrers-Young diagram of λ is given by $\lambda_i + \lambda'_j - i - j + 1$. A partition of n is called a t-core of n if none of the hook numbers is a multiple of t. For example, the Ferrers-Young diagram of the partition $\lambda = (4, 3, 1)$ of 8 is

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array}$$

The nodes $(1, 1)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, $(2, 3)$ and $(3, 1)$ have hook numbers $6,\ 4,\ 3,\ 1,\ 4,\ 2,\ 1$ and 1, respectively. Therefore, λ is a 5-core and a 7-core but not a 3-core. It is obvious that it is a t-core for $t \geq 8$.

If $a_t(n)$ denotes the number of t-cores of n, then the generating function for $a_t(n)$ is [36]

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty}. \tag{1.4.1}$$

Garvan, Kim and Stanton [36] gave some arithmetic properties for 5-cores and 7-cores by using combinatorial method. There are several arithmetic properties for 3- and 5-cores given by Baruah and Berndt [12], Hirschhon and Sellers [40] and Baruah and Nath [14, 15].

In the last section of Chapter 3, we find that

$$a_5(n-1) \equiv \tau(n) \pmod{25},$$

where $\tau(n)$ is Ramanujan's famous tau function defined by

$$q(q; q)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n, \tag{1.4.2}$$

and find several congruences for 3- and 5-cores. Some of them are given below.

Theorem 1.4.1. *For any integer $k \geq 2$ and any non-negative integer n, we have*

$$a_3\left(2^{2k} \cdot n + \frac{3 \cdot 2^{2k-1} + 4^k - 1}{3}\right) \equiv 0 \pmod{9}.$$

Theorem 1.4.2. *For any positive integer k and any non-negative integer n , we have*

$$a_3\left(5^{2k} \cdot n + \frac{5^{2k} - 1}{3}\right) \equiv (-5)^{3k} a_3(n) \pmod{9},$$

$$a_3\left(2^{2k} \cdot n + \frac{2^{2k} - 1}{3}\right) \equiv (-8)^k a_3(n) \pmod{9}.$$

Theorem 1.4.3. *For any non-negative integers k and n, we have*

$$a_5\left(2^{k+2}\cdot n + 2^{k+2} - 1\right) \equiv r_k a_5(2n+1) + s_k a_5(n) \pmod{25},$$

where $r_k = -24r_{k-1} + s_{k-1},\ s_k = -2048r_{k-1}$ *with* $r_0 = -24$ *and* $b_0 = -2048$.

Recently, some more partition functions with some restrictions on the parts are studied by various mathematicians. We introduce some of those functions in the next three sections and present a brief description of our work.

1.5 Broken k-diamond partitions

MacMahon in his renowned book "Combinatory Analysis" [49] introduced the partition analysis as the most important tool for solving combinatorial problems which are related with the system of linear diophantine inequalities and equations. MacMahon commenced with the most simplest case of plane partitions where the non-negative integers a_i of the partitions placed at the corners of a square such that the following order relations are satisfied:

$$a_1 \geq a_2,\ a_1 \geq a_3,\ a_2 \geq a_4 \text{ and } a_3 \geq a_4. \tag{1.5.1}$$

To represent $\geq$ relation, an arrow can be used as an alternative, for instance Fig.1 represents (1.5.1). Here and throughout the thesis, an arrow pointing from a_i to a_j is interpreted as $a_i \geq a_j$.

By using partition analysis, MacMahon derived the generating function

$$\begin{aligned}\varphi &:= \sum x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4},\\ &= \frac{1-x_1^2x_2x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)},\end{aligned}$$

where the sum is taken over all non-negative integers a_i satisfying (1.5.1). MacMahon also observed that, by putting $x_1 = x_2 = x_3 = x_4 = q$, the generating function becomes

$$\frac{1}{(1-q)(1-q^2)^2(1-q^3)}.$$

By using MacMahon's partition analysis, Andrews, Paule and Riese [7] introduced partition diamonds as new variations of plane partitions as shown in Fig.2.

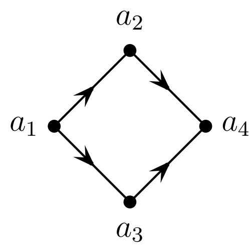


Fig.1. The inequality (1.5.1).

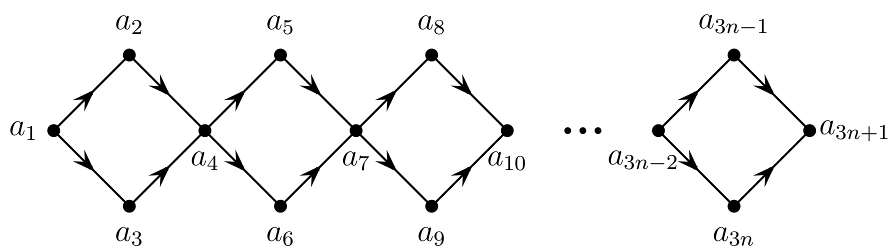


Fig.2. A plane partition diamond of length n.

In 2007, Andrews and Paule [8] studied the generalization of this partition diamonds by introducing k-elongated partition diamonds as shown in Fig.3, as the building blocks of the chain.

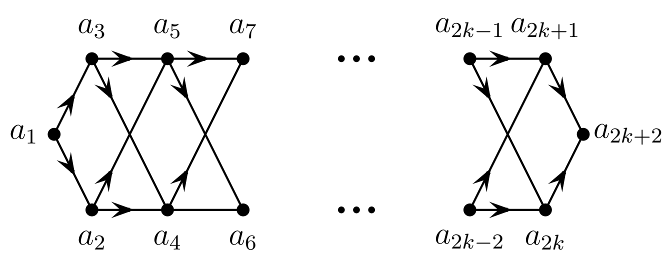


Fig.3. A k-elongated partition diamond of length 1.

Andrews and Paule [8] also introduced Broken k-diamonds. Broken k-diamonds consist of two separated k-elongated partition diamonds of length n where in one of them, the source is deleted, as shown in Fig.4.

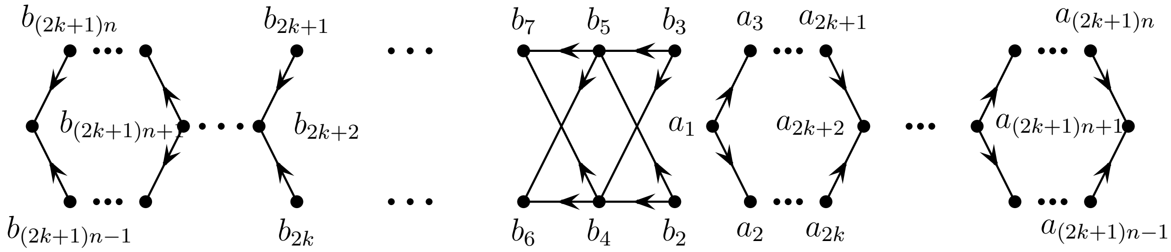


Fig.4. A broken k-diamond of length $2n$.

Definition 1.5.1. *For $n,\ k \geq 1$, define*

$$\begin{aligned} H^{\diamondsuit}_{n,k} &:= \{(b_2, \ldots, b_{(2k+1)n+1}, a_1, a_2, \ldots, a_{(2k+1)n}) \in \mathbb{N}^{(4k+1)n}, \\ &\qquad \text{the } a_i \text{ and } b_i \text{ satisfy all order relations in Fig.4}\} \\ h^{\diamondsuit}_{n,k} &:= h^{\diamondsuit}_{n,k}(x_2, \ldots, x_{(2k+1)n+1}, y_1, y_2, \ldots, y_{(2k+1)n+1}) \\ &:= \sum_{(b_2,\ldots,b_{(2k+1)n+1},a_1,a_2,\ldots,a_{(2k+1)n}) \in H^{\diamondsuit}_{n,k}} x_2^{b_2} \cdots x_{(2k+1)n+1}^{b_{(2k+1)n+1}} y_1^{a_1} y_2^{a_2} \cdots y_{(2k+1)n+1}^{a_{(2k+1)n+1}} \end{aligned}$$

and

$$h^{\diamondsuit}_{n,k}(q) := h^{\diamondsuit}_{n,k}(q, q, \ldots, q).$$

Andrews and Paule [8] also found the generating function for the number of broken k-diamond partitions of n as given in the next theorem.

Theorem 1.5.2. *Let for $n \geq 0$ and $k \geq 1$, $\Delta_k(n)$ denote the total number of broken k-diamond partitions of n. Then*

$$h^{\diamondsuit}_{\infty,k}(q) = \sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(q^2;q^2)_\infty}{(q;q)^3_\infty(-q^{2k+1};q^{2k+1})_\infty}. \tag{1.5.2}$$

For $k = 1$, they also proved the congruence

$$\Delta_1(2n+1) \equiv 0 \pmod 3 \tag{1.5.3}$$

and stated three more conjectures. Hirschhorn and Sellers [39] provided a new proof of (1.5.3) as well as elementary proofs of congruences modulo 2 for $k = 1$ and 2. Combinatorial proofs of (1.5.3) were given by Mortenson [50] and Fu [34]. There are a number of other congruences for $\Delta_2(n)$ in [51, 25, 28, 53]. Radu and Sellers [52] found parity results for broken k-diamond partitions for some values of k. Paule and Radu [51] conjectured four congruences for broken 3- and 5-diamond partitions. Two of those congruences were proved by Xiong [60] and the remaining two were proved by Jameson [43]. Radu and Sellers [53] found some parity results for broken 3-diamond partitions by using the theory of modular forms and subsequently, Lin [45] found the elementary proofs of those parity results. Cui and Gu [30] and Wang [55] also found more parity results for broken 3- and 8-diamond partitions respectively. Recently, Xia [58] found infinite families of congruences modulo 7 for broken 3-diamond partitions.

In Chapter 4, we find parity results for broken 5-, 7- and 11-diamond partitions by employing p-dissection of Ramanujan's theta functions. Some of the results are given in the following theorems.

Theorem 1.5.3. *For any odd prime p, $\alpha \geq 0$ and if n is not a triangular number, then*

$$\Delta_5\left(396 \cdot p^{2\alpha} \cdot n + \frac{99 \cdot p^{2\alpha}+1}{2}\right) \equiv 0 \pmod 2.$$

Theorem 1.5.4. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\Delta_7\left(8 \cdot 5^{2\alpha+1} \cdot n + 8 \cdot r \cdot 5^{2\alpha} + \frac{16 \cdot 5^{2\alpha}+2}{3}\right) \equiv 0 \pmod 2, \tag{1.5.4}$$

for $r = 3,\ 4,\ 8,\ 9,\ 13,$ *and* 14.

Theorem 1.5.5. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\Delta_{11}(2 \cdot 23^{\alpha+1} \cdot n + 2 \cdot r \cdot 23^{\alpha} + 1) \equiv 0 \pmod 2, \tag{1.5.5}$$

for $r = 5,\ 7,\ 10,\ 11,\ 14,\ 15,\ 17,\ 19,\ 20,\ 21,\ 22.$

1.6 k dots bracelet partitions

In 2011, Fu [34] gave a combinatorial proof of (1.5.3) and also applied the combinatorial approach to generalise the broken k-diamond partitions which he called k dots bracelet partitions. Before defining k dots bracelet partitions, Fu defined infinite bracelet partitions which consist of repeating diamonds and dots with $k-2$ dots between two consecutive diamonds as shown in Fig.5 and we see that an infinite bracelet partitions can be cut into $k-1$ different ways with k dots in half. For any

$k \geq 3$, a k dots bracelet partitions consist of $k-1$ different half bracelet as shown in Fig.6.

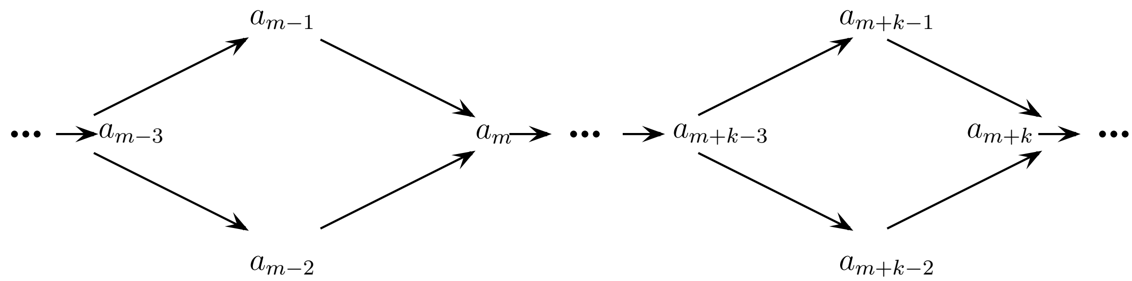
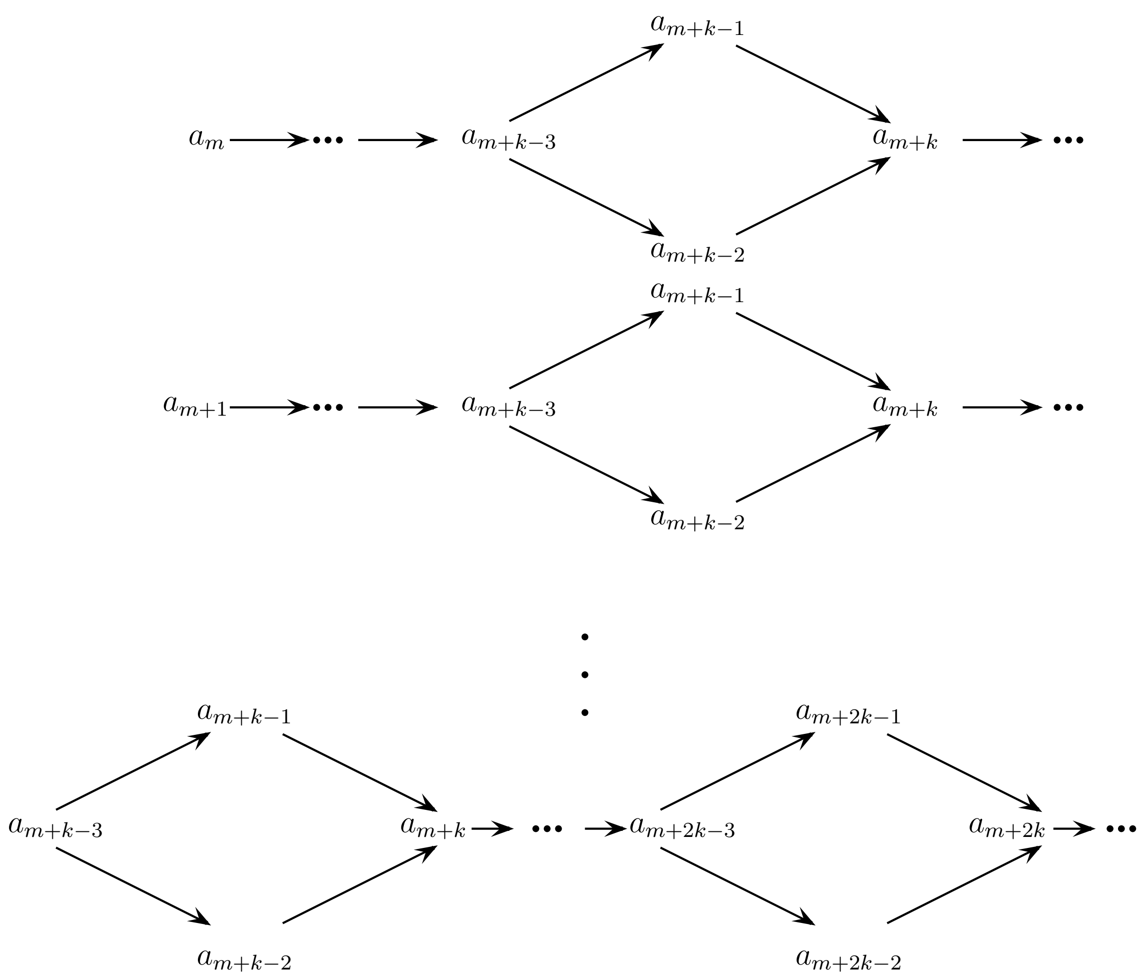


Fig.5. Infinite bracelet with k dots.

Fu [34] denoted the number of k dots bracelet partitions for a positive integer n by $\mathfrak{B}_k(n)$. The generating function for $\mathfrak{B}_k(n)$ is given by

$$\sum_{0}^{\infty} \mathfrak{B}_k(n) q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^k(-q^k;q^k)_\infty}. \tag{1.6.1}$$



$$\begin{tikzcd}
a_{m+k-1} \arrow[dr] & & & & a_{m+2k-1} \arrow[dr] & & \\
& a_{m+k} \arrow[r] & \cdots \arrow[r] & a_{m+2k-3} \arrow[ur] \arrow[dr] & & a_{m+2k} \arrow[r] & \cdots \\
a_{m+k-2} \arrow[ur] & & & & a_{m+2k-2} \arrow[ur] & &
\end{tikzcd}$$

Fig.6. $k-1$ different half bracelet.

He also proved the following congruences for k dots bracelet partitions:

(i) for $n \geq 0, k \geq 3$ if $k = p^r$ is a prime power,

$$\mathfrak{B}_k(2n+1) \equiv 0 \pmod{p},$$

(ii) for any $k \geq 3$, s an integer between 1 and $p-1$ such that $12s+1$ is a quadratic nonresidue modulo p and any $n \geq 0$, if $p\big|k$ for some prime $p \geq 5$, say $k = pm$, then

$$\mathfrak{B}_k(pn+s) \equiv 0 \pmod{p},$$

(iii) for any $n \geq 0, k \geq 3$ even, say $k = 2^m l$, where l is odd,

$$\mathfrak{B}_k(2n+1) \equiv 0 \pmod{2^m}.$$

Radu and Sellers [54] extended the set of congruences given by Fu. They proved that for all $n \geq 0$

$$\mathfrak{B}_5(10n+7) \equiv 0 \pmod{5^2},$$
$$\mathfrak{B}_7(14n+11) \equiv 0 \pmod{7^2},$$

and

$$\mathfrak{B}_{11}(22n+21) \equiv 0 \pmod{11^2}.$$

More recently, Cui and Gu [29] found several congruences modulo 2 for 5 dots bracelet partitions and congruences modulo p for any prime $p \geq 5$ for k dots bracelet partitions. Xia and Yao [59] also found several congruences modulo powers of 2 for 5 dots bracelet partitions. Recently, Yao [62] established the generating functions of $\mathfrak{B}_9(An+B)$ modulo 4 for some values of A and B and hence obtained congruences for modulo 2 and 4.

In Chapter 5 of this thesis, we find several new congruences modulo 2 for 7 and 11 dots bracelet partitions and also find congruences modulo p^2 and p^3 for k dots bracelet partitions for any prime $p>3$ by employing Ramanujan's theta functions and by finding the binomial expansion of $(q;q)_\infty^{p^n}$ congruent modulo p^n for $n=2$ and $n=3$ respectively. A few examples of our results are given below.

Theorem 1.6.1. *For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, where $n \neq \dfrac{k(3k-1)}{2}$, we have*

$$\mathfrak{B}_{11}\left(4 \cdot p^{2\alpha} \cdot n + \frac{p^{2\alpha}+5}{6}\right) \equiv 0 \pmod{2}.$$

Theorem 1.6.2. *Let $k = mp^r$, where $m \in \mathbb{N}$, $p \geq 5$ and $r \geq 2$. Then for any positive integer n, we have*

$$\mathfrak{B}_k(pn+\ell) \equiv 0 \pmod{p^2},$$

where $1 \leq \ell \leq p-1$ and $12\ell+1$ is quadratic nonresidue modulo p, i.e., in Legendre symbol $\left(\dfrac{12\ell+1}{p}\right) = -1$.

Theorem 1.6.3. *Let $k = mp^s$, where $m \in \mathbb{N}$, $p \geq 5$ and $s \geq 3$. Then for any positive integer n, we have*

$$\mathfrak{B}_k(pn+j) \equiv 0 \pmod{p^3},$$

where $1 \leq j \leq p-1$ and $12j+1$ is quadratic nonresidue modulo p, i.e., in Legendre symbol $\left(\dfrac{12j+1}{p}\right) = -1$.

Theorem 1.6.4. *Let $k = mp^s$, where $m \in \mathbb{N}$, $p \geq 5$ and $s \geq 3$. Then for any positive integer n, we have*

$$\mathfrak{B}_k\left(p(pn+j) + \frac{p^2-1}{12}\right) \equiv 0 \pmod{p^2},$$

for $j = 1,\ 2,\ \ldots,\ p-1$.

1.7 Singular overpartitions

Recently, Andrews [5] introduced singular overpartitions. To introduce singular overpartitions, first he defined some properties of the entries in a Frobenius symbol for n, which is of the form

$$\begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_r \\ b_1 & b_2 & \cdot & \cdot & \cdot & b_r \end{pmatrix}$$

where the rows are strictly decreasing sequences of non-negative integers and $\sum_{i=1}^{r}(a_i + b_i + 1) = n$. Andrews defined a column $\begin{matrix} a_j \\ b_j \end{matrix}$ in a Frobenius symbol as (k,i)-positive if $a_j - b_j \geq k - i - 1$, (k,i)-negative if $a_j - b_j \leq -i + 1$, and (k,i)-neutral if $-i + 1 < a_j - b_j < k - i - 1$, where we have corrected the misprint in Andrews [5] on the last expression by replacing $k - i + 1$ with $k - i - 1$. He then divided the Frobenius symbol into (k,i)-parity blocks, where if two coloumns $\begin{matrix} a_n \\ b_n \end{matrix}$ and $\begin{matrix} a_j \\ b_j \end{matrix}$ are both (k,i)-positive or both (k,i)-negative, then they have the same (k,i)-parity. These blocks are the sets of contiguous columns maximally extended to the right:

$$\begin{matrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_j \\ b_n & b_{n+1} & \cdot & \cdot & \cdot & b_j \end{matrix}$$

where all the entries have either the same (k,i)-parity or are (k,i)-neutral. The first non-neutral column in each parity block is called the anchor of the block.

A Frobenius symbol is said to be (k,i)-singular, if the following properties hold

(i) there are no overlined entries, or

(ii) the one overlined entry on the top row occurs in the anchor of a (k,i)-positive block, or

(iii) the one overlined entry on the bottom row occurs in an anchor of a (k,i)-negative block, and

(iv) if there is one overlined entry in each row, then they occur in adjacent (k,i)-parity blocks.

Andrews denoted the number of such singular overpartitions of n as $\overline{Q}_{k,i}(n)$. He found that $\overline{Q}_{k,i}(n)$ is equal to $\overline{C}_{k,i}(n)$, the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined, i.e.,

$$\sum_{n=0}^{\infty} \overline{Q}_{k,i}(n)q^n = \sum_{n=0}^{\infty} \overline{C}_{k,i}(n)q^n = \frac{(q^k, -q^i, -q^{k-i}; q^k)_\infty}{(q;q)_\infty},$$

where $(a_1, a_2, \ldots, a_k; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty$. Andrews also found the following two congruences

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}. \tag{1.7.1}$$

Chen, Hirschhorn and Sellers [26] found some infinite families of congruences modulo 3 for $\overline{C}_{3,1}$, $\overline{C}_{6,1}$ and $\overline{C}_{6,2}$. For example, they found the following congruences for $\overline{C}_{3,1}$.

Theorem 1.7.1. *For all k, $m \geq 0$,*

$$\overline{C}_{3,1}(2^k(4m+3)) \equiv 0 \pmod{3}.$$

Theorem 1.7.2. *Let $p \equiv 1 \pmod{4}$ be prime. Then for all k, $m \geq 0$ with $p \nmid m$,*

$$\overline{C}_{3,1}(p^{3k+2}m) \equiv 0 \pmod{3}.$$

Theorem 1.7.3. *Let $p \equiv 3 \pmod{4}$ be prime. Then for all k, $m \geq 0$ with $p \nmid m$,*

$$\overline{C}_{3,1}(p^{2k+1}m) \equiv 0 \pmod{3}.$$

From the above theorem for $p = 3$, $k = 0$ and $m \equiv 1, 2 \pmod{3}$, one can easily arrive at (1.7.1). In the same paper, they also found parity results for $\overline{C}_{3,1}$, $\overline{C}_{4,1}$ and $\overline{C}_{6,1}$.

In the last chapter of this thesis, we obtain several new congruences for $\overline{C}_{k,i}(n)$ for certain values of k and i by employing simple p-dissections of Ramanujan's theta functions. For example, we have the following results.

Theorem 1.7.4. *If $p \geq 5$ is a prime and $1 \leq j \leq p-1$, then for any non-negative integers α and n, we have*

$$\overline{C}_{3,1}\left(24p^{(2\alpha+1)}(pn+j) + p^{2(\alpha+1)}\right) \equiv 0 \pmod{4}.$$

Theorem 1.7.5. *For any non-negative integer n, we have*

$$\begin{aligned}\overline{C}_{3,1}(48n+12) &\equiv 0 \pmod{18},\\ \overline{C}_{3,1}(12n+7) &\equiv 0 \pmod{36},\\ \overline{C}_{3,1}(12n+11) &\equiv 0 \pmod{36},\\ \overline{C}_{3,1}(24n+14) &\equiv 0 \pmod{36}\end{aligned}$$

and

$$\overline{C}_{3,1}(24n+22) \equiv 0 \pmod{36}.$$

Theorem 1.7.6. *If p is a prime such that $p \equiv 3 \pmod 4$ and $1 \leq j \leq p-1$, then for all non-negative integers α and n, we have*

$$\overline{C}_{8,2}\left(p^{2\alpha+1}(pn+j)+\frac{5(p^{2(\alpha+1)}-1)}{24}\right) \equiv 0 \pmod{2}.$$

Theorem 1.7.7. *If p is a prime such that $p \equiv 13,\ 17,\ 19,$ or $23 \pmod{24}$ and $1 \leq j \leq p-1$, then for any non-negative integers α and n, we have*

$$\overline{C}_{8,2}\left(p^{2\alpha+1}(pn+j)+\frac{5(p^{2(\alpha+1)}-1)}{24}\right) \equiv 0 \pmod{4}.$$

Since our proofs mainly rely on various properties of Ramanujan's theta functions and dissections of certain q-products, in the last section of this chapter, we define a t-dissection and Ramanujan's general theta function and some of its special cases.

1.8 t-dissection and Ramanujan's theta functions

If $P(q)$ denotes a power series in q, then a t-dissection of $P(q)$ is given by

$$[P(q)]_{t-\text{dissection}} = \sum_{k=0}^{t-1} q^k P_k(q^t),$$

where P_k are power series in q^t.

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1.$$

Three special cases of $f(a,b)$ are [19, p. 36, Entry 22]

$$\begin{aligned}\varphi(q) &:= f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,\\ \psi(q) &:= f(q,q^3) = \frac{1}{2}f(1,q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},\end{aligned} \tag{1.8.1}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2} = (q;q)_\infty,$$

where the product representations in the above arise from Jacobi's famous triple product identity [19, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.8.2}$$

Also define

$$\chi(q) := (-q;q^2)_\infty,$$

where $(-q;q^2)_\infty$ generates partitions into distinct odd parts.

By manipulating the q-products, one can easily arrive at the following representations:

$$\begin{aligned}&\varphi(q) = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2},\ \varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty},\ \psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty},\\ &\psi(-q) = \frac{(q;q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty}, \chi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty},\ \chi(-q) = \frac{(q;q)_\infty}{(q^2;q^2)_\infty}.\end{aligned} \tag{1.8.3}$$