Chapter 2

New congruences for ℓ-regular partitions for $\ell \in \{5, 6, 7, 49\}$

2.1 Introduction

As mentioned in the introductory chapter, this chapter includes several congruences for ℓ-regular partitions, for certain ℓ. In the next section, we state the p-dissections of $\psi(q)$, $f(-q)$, $f^3(-q)$ and $\psi(q^2)f^2(-q)$, where the p-dissections of $\psi(q)$ and $f(-q)$ are due to Cui and Gu [32] and the remaining two are new, which will be used in our subsequent sections. In Sections 2.3–2.5, we prove some theorems from which the following results are easily followed.

Theorem 2.1.1. *If $j \in \{0, 2, 3, 4, 5, 6\}$, then for any non-negative integers α and n,*

$$b_5\left(25 \cdot 7^{6\alpha+5}(7n+j) + \frac{25 \cdot 7^{6\alpha+5} - 1}{6}\right) \equiv 0 \pmod{25}. \tag{2.1.1}$$

Theorem 2.1.2. *If p is a prime such that $p \equiv -1 \pmod 6$ and $1 \le j \le p-1$, then for any non-negative integers α and n,*

$$b_5\left(25p^{2\alpha+1}(pn+j) + \frac{25p^{2\alpha+2} - 1}{6}\right) \equiv 0 \pmod{25}. \tag{2.1.2}$$

Theorem 2.1.3. *If p is a prime such that $\left(\frac{-6}{p}\right) = -1$ and $1 \le j \le p-1$, then for any non-negative integers α and n,*

$$b_6\left(p^{2\alpha+1}(pn+j) + 5 \cdot \frac{p^{2\alpha} - 1}{24}\right) \equiv 0 \pmod{3}. \tag{2.1.3}$$

Theorem 2.1.4. *For any non-negative integers α and n,*

$$b_7\left(7^3\cdot 3^{2\alpha+2}\cdot n+\frac{7^3\cdot 5\cdot 3^{2\alpha+1}-1}{4}\right)\equiv 0 \pmod{147} \tag{2.1.4}$$

and

$$b_7\left(7^3\cdot 3^{2\alpha+3}\cdot n+\frac{7^3\cdot 11\cdot 3^{2\alpha+2}-1}{4}\right)\equiv 0 \pmod{147}. \tag{2.1.5}$$

Theorem 2.1.5. *If $p\geq 11$ is a prime such that $\left(\frac{-7}{p}\right)=-1$ and $1\leq j\leq p-1$, then for any non-negative integers α and n,*

$$b_{49}\left(7p^{2\alpha+1}(pn+j)+7(p^{2\alpha+2}-1)+5\right)\equiv 0 \pmod{49}. \tag{2.1.6}$$

In the last section of this chapter, we find two theorems from which (1.2.2) and (1.2.4) follow immediately.

The contents of this chapter have been submitted [1].

2.2 Preliminary lemmas

Cui and Gu [32] found the following p-dissections of $\psi(q)$ and $f(-q)$.

Lemma 2.2.1. (Cui and Gu [32, Theorem 2.1]) *If p is an odd prime, then*

$$\psi(q)=\sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{2.2.1}$$

Furthermore, for $0\leq k\leq \frac{p-3}{2}$,

$$\frac{k^2+k}{2}\not\equiv\frac{p^2-1}{8} \pmod{p}.$$

Lemma 2.2.2. (Cui and Gu [32, Theorem 2.2]) *If $p\geq 5$ is a prime and*

$$\frac{\pm p-1}{6}:=\begin{cases}\dfrac{p-1}{6}, & \text{if } p\equiv 1 \pmod{6};\\[2mm] \dfrac{-p-1}{6}, & \text{if } p\equiv -1 \pmod{6},\end{cases}$$

then

$$(q;q)_\infty = \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2};q^{p^2})_\infty. \quad (2.2.2)$$

Furthermore, if $\frac{-(p-1)}{2} \leq k \leq \frac{(p-1)}{2}$, $k \neq \frac{(\pm p-1)}{6}$, *then*

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

In the following two lemmas, we present new p-dissections of $(q;q)^3_\infty$ and $\psi(q^2)(q;q)^2_\infty$.

Lemma 2.2.3. *If* $p \geq 3$ *is a prime, then*

$$(q;q)^3_\infty = \sum_{\substack{k=0 \\ k\neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn\cdot\frac{pn+2k+1}{2}} + p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} (q^{p^2};q^{p^2})^3_\infty. \quad (2.2.3)$$

Furthermore, if $k \neq \frac{p-1}{2}$, $0 \leq k \leq p-1$, *then*

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}.$$

Proof. From [20, p. 14], we recall Jacobi's identity

$$(q;q)^3_\infty = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{n(n+1)}{2}}.$$

Dissecting the above sum into p terms, we obtain

$$\begin{aligned}
(q;q)_\infty^3 &= \sum_{k=0}^{p-1}\sum_{n=0}^{\infty}(-1)^{pn+k}(2(pn+k)+1)q^{\frac{(pn+k)((pn+k)+1)}{2}} \\
&= \sum_{k=0}^{p-1}(-1)^k q^{\frac{k(k+1)}{2}}\sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{pn\cdot\frac{pn+2k+1}{2}} \\
&= \sum_{\substack{k=0 \\ k\neq\frac{p-1}{2}}}^{p-1}(-1)^k q^{\frac{k(k+1)}{2}}\sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{pn\cdot\frac{pn+2k+1}{2}} \\
&\quad + (-1)^{\frac{p-1}{2}}q^{\frac{p^2-1}{8}}\sum_{n=0}^{\infty}(-1)^n p(2n+1)q^{p^2\cdot\frac{n(n+1)}{2}} \\
&= \sum_{\substack{k=0 \\ k\neq\frac{p-1}{2}}}^{p-1}(-1)^k q^{\frac{k(k+1)}{2}}\sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{pn\cdot\frac{pn+2k+1}{2}} \\
&\quad + p(-1)^{\frac{p-1}{2}}q^{\frac{p^2-1}{8}}(q^{p^2};q^{p^2})_\infty^3.
\end{aligned}$$

If $\dfrac{k^2+k}{2}\equiv\dfrac{p^2-1}{8} \pmod p$, then we find that $k=\dfrac{p-1}{2}$, which completes the proof of (2.2.3). □

Lemma 2.2.4. *If $p\geq 5$ is a prime and*

$$\frac{\pm p-1}{3} := \begin{cases} \dfrac{p-1}{3}, & \text{if } p\equiv 1 \pmod 3; \\ \dfrac{-p-1}{3}, & \text{if } p\equiv -1 \pmod 3, \end{cases}$$

then

$$\begin{aligned}
\psi(q^2)(q;q)_\infty^2 = &\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq\frac{\pm p-1}{3}}}^{\frac{p-1}{2}} q^{3k^2+2k}\sum_{n=-\infty}^{\infty}(3pn+3k+1)q^{pn(3pn+6k+2)} \\
&\pm pq^{\frac{p^2-1}{3}}\psi(q^{2p^2})(q^{p^2};q^{p^2})_\infty^2,
\end{aligned} \tag{2.2.4}$$

Furthermore, if $k \neq \frac{\pm p-1}{3}$, $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}$, *then*

$$3k^2 + 2k \not\equiv \frac{p^2-1}{3} \pmod{p}.$$

Proof. From [19, p. 21], we recall that

$$\psi(q^2)(q;q)_\infty^2 = \sum_{n=-\infty}^{\infty} (3n+1)q^{3n^2+2n}.$$

Dissecting the right side into p terms, we find that

$$\begin{aligned}
\psi(q^2)(q;q)_\infty^2 &= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} \sum_{n=-\infty}^{\infty} (3(pn+k)+1)q^{3(pn+k)^2+2(pn+k)} \\
&= \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} q^{3k^2+2k} \sum_{n=-\infty}^{\infty} (3pn+3k+1)q^{pn(3pn+6k+2)}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{3}}}^{\frac{p-1}{2}} q^{3k^2+2k} \sum_{n=-\infty}^{\infty} (3pn+3k+1)q^{pn(3pn+6k+2)} \\
&\quad \pm q^{\frac{p^2-1}{3}} \sum_{n=-\infty}^{\infty} p(3n+1)q^{p^2(3n^2+2n)} \\
&= \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{3}}}^{\frac{p-1}{2}} q^{3k^2+2k} \sum_{n=-\infty}^{\infty} (3pn+3k+1)q^{pn(3pn+6k+2)} \\
&\quad \pm pq^{\frac{p^2-1}{3}} \psi(q^{2p^2})(q^{p^2};q^{p^2})_\infty^2.
\end{aligned}$$

Now, if $3k^2 + 2k \equiv \frac{p^2-1}{3} \pmod{p}$, then $k = \frac{\pm p-1}{3}$, which completes the proof of (2.2.4). □

We end this section by defining an operator H which acts on a Laurent series in one variable by picking out those terms in which the power is congruent to 0 modulo

7. If

$$\xi := \frac{(q;q)_\infty}{q^2(q^{49};q^{49})_\infty} \text{ and } T := \frac{(q^7;q^7)_\infty^4}{q^7(q^{49};q^{49})_\infty^4}, \tag{2.2.5}$$

then Garvan [36] proved that

$$H(\xi) = -1,\ H(\xi^2) = 1,\ H(\xi^3) = -7,\ H(\xi^4) = -4T - 7, H(\xi^5) = 10T + 49,$$
$$\text{and } H(\xi^6) = 49. \tag{2.2.6}$$

2.3 New congruences for 5-regular partitions

Theorem 2.3.1. *If p is a prime such that $p \equiv -1 \pmod 6$, then for all $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} b_5\left(25p^{2\alpha}n + \frac{25p^{2\alpha}-1}{6}\right) q^n \equiv (-1)^{\alpha\cdot\frac{p-2}{3}} 5p^\alpha (q;q)_\infty^4 \pmod{25}. \tag{2.3.1}$$

Proof. It is clear from the generating function (1.2.1) that

$$\sum_{n=0}^{\infty} b_5(n)q^n = \frac{(q^5;q^5)_\infty}{(q;q)_\infty} = (q^5;q^5)_\infty \sum_{n=0}^{\infty} P(n)q^n, \tag{2.3.2}$$

where $P(n)$ is the ordinary partition function, that is, the number of unrestricted partitions of the non-negative integer n.

It is well-known (for example, see [20]) that

$$\sum_{n=0}^{\infty} P(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}.$$

Therefore, from (2.3.2), we have

$$\sum_{n=0}^{\infty} b_5(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^5}.$$

Since $(q^5;q^5)_\infty \equiv (q;q)_\infty^5 \pmod 5$, we find that

$$\sum_{n=0}^{\infty} b_5(5n+4)q^n \equiv 5(q^5;q^5)_\infty^4 \pmod{25}.$$

Extracting the terms involving q^{5n} from both sides of the above and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} b_5(25n+4)q^n \equiv 5(q;q)_\infty^4 \pmod{25}, \tag{2.3.3}$$

which is the $\alpha = 0$ case of (2.3.1). Now suppose that (2.3.1) holds for some $\alpha \geq 0$. With the help of (2.2.2) and (2.2.3), we can rewrite (2.3.1) as

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_5\left(25p^{2\alpha}n + \frac{25p^{2\alpha}-1}{6}\right)q^n \\
&\equiv (-1)^{\alpha\cdot\frac{p-2}{3}} \cdot 5p^{\alpha}\Bigg[\sum_{\substack{m=-\frac{p-1}{2} \\ m\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{\frac{3m^2+m}{2}} f\left(-q^{\frac{3p^2+(6m+1)p}{2}}, -q^{\frac{3p^2-(6m+1)p}{2}}\right) \\
&\quad + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2})\Bigg] \\
&\quad \times \Bigg[\sum_{\substack{k=0 \\ k\neq\frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{pn\cdot\frac{pn+2k+1}{2}} \\
&\quad + p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} (q^{p^2};q^{p^2})_\infty^3\Bigg] \pmod{25}.
\end{aligned} \tag{2.3.4}$$

Now our objective is to find those terms above for which the powers of q satisfy the congruence

$$\frac{k^2+k}{2} + \frac{3m^2+m}{2} \equiv \frac{p^2-1}{6} \pmod{p}, \tag{2.3.5}$$

where $0 \leq k \leq p-1$ and $-(p-1)/2 \leq m \leq (p-1)/2$. Since the above is equivalent to

$$3(2k+1)^2 + (6m+1)^2 \equiv 0 \pmod{p}$$

and $\left(\frac{-3}{p}\right) = -1$ as $p \equiv -1 \pmod{6}$, it follows that the only solution of (2.3.5) is $k = \dfrac{p-1}{2}$ and $m = \dfrac{\pm p-1}{6}$. Therefore, extracting the terms containing $q^{pn+\frac{p^2-1}{6}}$ from both sides of (2.3.4) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} b_5\left(25p^{2\alpha+1}n + \frac{25p^{2\alpha+2}-1}{6}\right)q^n \equiv (-1)^{(\alpha+1)\frac{p-2}{3}} 5p^{\alpha+1}(q^p;q^p)_\infty^4 \pmod{25}. \tag{2.3.6}$$

Again extracting the terms containing q^{pn} from both sides of the above and replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} b_5\left(25p^{2\alpha+2}n+\frac{25p^{2\alpha+2}-1}{6}\right)q^n \equiv (-1)^{(\alpha+1)\frac{p-2}{3}}5p^{\alpha+1}(q;q)_\infty^4 \pmod{25},$$

which is clearly the $\alpha+1$ case of (2.3.1). This completes the proof. □

Theorem 2.3.2. *For $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} b_5\left(25\cdot 7^{6\alpha}n+\frac{25\cdot 7^{6\alpha}-1}{6}\right)q^n \equiv 2^\alpha\cdot 5(q;q)_\infty^4 \pmod{25}. \tag{2.3.7}$$

Proof. We again use induction on α. From (2.3.3), we have

$$\sum_{n=0}^{\infty} b_5(25n+4)q^n \equiv 5(q;q)_\infty^4 \pmod{25},$$

which is the $\alpha = 0$ case of (2.3.7). Now suppose that (2.3.7) holds for some $\alpha \geq 0$. With the aid of (2.2.5), we rewrite (2.3.7) as

$$\sum_{n=0}^{\infty} b_5\left(25\cdot 7^{6\alpha}n+\frac{25\cdot 7^{6\alpha}-1}{6}\right)q^n \equiv 2^\alpha\cdot 5q^8(q^{49};q^{49})_\infty^4\xi^4 \pmod{25}.$$

Extracting the terms containing q^{7n+1} from both sides of the above congruence, and then using (2.2.6), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_5\left(25\cdot 7^{6\alpha}(7n+1)+\frac{25\cdot 7^{6\alpha}-1}{6}\right)q^{7n+1}\\
&\equiv 2^\alpha\cdot 5q^8(q^{49};q^{49})_\infty^4 H(\xi^4)\\
&\equiv 2^\alpha\cdot 5q^8(q^{49};q^{49})_\infty^4\left(-4\frac{(q^7;q^7)_\infty^4}{q^7(q^{49};q^{49})_\infty^4}-7\right) \pmod{25}.
\end{aligned}$$

Dividing both sides by q and replacing q^7 by q, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_5\left(25\cdot 7^{6\alpha+1}\cdot n+25\cdot 7^{6\alpha}+\frac{25\cdot 7^{6\alpha}-1}{6}\right)q^n\\
&\equiv 2^\alpha\left(5(q;q)_\infty^4+15q(q^7;q^7)_\infty^4\right) \pmod{25}.
\end{aligned}$$

which can be rewritten with the help of (2.2.5) as

$$\sum_{n=0}^{\infty} b_5\left(25\cdot 7^{6\alpha+1}\cdot n+25\cdot 7^{6\alpha}+\frac{25\cdot 7^{6\alpha}-1}{6}\right)q^n$$
$$\equiv 2^{\alpha}\left(5q^8(q^{49};q^{49})_\infty^4\xi^4+15q(q^7;q^7)_\infty^4\right) \pmod{25}.$$

Extracting the terms containing q^{7n+1} from both sides of the above and then using (2.2.6), we deduce that

$$\sum_{n=0}^{\infty} b_5\left(25\cdot 7^{6\alpha+1}\cdot (7n+1)+25\cdot 7^{6\alpha}+\frac{25\cdot 7^{6\alpha}-1}{6}\right)q^n$$
$$\equiv 2^{\alpha}\left(20(q;q)_\infty^4+15q(q^7;q^7)_\infty^4\right) \pmod{25}.$$

Proceeding further in a similar way, we find that

$$\sum_{n=0}^{\infty} b_5\left(25\cdot 7^{6\alpha+4}\cdot (7n+1)+25\cdot 7^{6\alpha}(1+7+7^2+7^3)+\frac{25\cdot 7^{6\alpha}-1}{6}\right)q^{7n+1}$$
$$\equiv 2^{\alpha}\left(-75q(q^7;q^7)_\infty^4+10q^8(q^{49};q^{49})_\infty^4\right)$$
$$\equiv 2^{\alpha}\cdot 10q^8(q^{49};q^{49})_\infty^4 \pmod{25}.$$

Dividing both sides of the above by q and replacing q^7 by q, we obtain

$$\sum_{n=0}^{\infty} b_5\left(25\cdot 7^{6\alpha+5}\cdot n+25\cdot 7^{6\alpha}(1+7+7^2+7^3+7^4)+\frac{25\cdot 7^{6\alpha}-1}{6}\right)q^n$$
$$\equiv 2^{\alpha}\cdot 10q(q^7;q^7)_\infty^4 \pmod{25}. \tag{2.3.8}$$

Extracting the terms containing q^{7n+1} from both sides of the above and then simplifying, we arrive at

$$\sum_{n=0}^{\infty} b_5\left(25\cdot 7^{6\alpha+6}\cdot n+\frac{25\cdot 7^{6\alpha+6}-1}{6}\right)q^n\equiv 2^{\alpha+1}\cdot 5(q;q)_\infty^4 \pmod{25},$$

which is the $\alpha+1$ case of (2.3.7). □

Now we prove Theorem 2.1.1 and Theorem 2.1.2.

Proofs of Theorem 2.1.1 and Theorem 2.1.2. Comparing the coefficients of q^j, $j\in\{0,2,3,4,5,6\}$ on both sides of (2.3.8), we easily arrive at (2.1.1). Again, comparing the coefficients of q^{pn+j}, $1\le j\le p-1$, on both sides of (2.3.6), we readily deduce (2.1.2). □

2.4 New congruences for 6-regular partitions

Theorem 2.4.1. *If p is a prime such that $\left(\frac{-6}{p}\right) = -1$, then for all $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} b_6\left(p^{2\alpha}n + 5\cdot\frac{p^{2\alpha}-1}{24}\right)q^n \equiv (-1)^{\alpha\frac{\pm p-1}{6}}\psi(q)(q^2;q^2)_\infty \pmod 3. \qquad (2.4.1)$$

Proof. Once again we use induction on α. Since $(q;q)_\infty^3 \equiv (q^3;q^3)_\infty \pmod 3$, we have

$$\sum_{n=0}^{\infty} b_6(n)q^n = \frac{(q^6;q^6)_\infty}{(q;q)_\infty} \equiv \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty} \equiv \psi(q)(q^2;q^2)_\infty \pmod 3,$$

which is the $\alpha = 0$ case of (2.4.1). Now suppose that (2.4.1) holds for some $\alpha \geq 0$. Using (2.2.1) and (2.2.2), we rewrite (2.4.1) as

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_6\left(p^{2\alpha}n + 5\cdot\frac{p^{2\alpha}-1}{24}\right)q^n \\
&\equiv (-1)^{\alpha\frac{\pm p-1}{6}}\left[\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right] \\
&\quad\times\left[\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{3k^2+k} f\left(-q^{3p^2+(6k+1)p}, -q^{3p^2-(6k+1)p}\right)\right. \\
&\quad\left. + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{12}}(q^{2p^2};q^{2p^2})_\infty\right] \pmod 3. \qquad (2.4.2)
\end{aligned}$$

We now consider the congruence

$$3k^2 + k + \frac{m^2+m}{2} \equiv \frac{5(p^2-1)}{24} \pmod p, \qquad (2.4.3)$$

where $0 \leq m \leq (p-1)/2$ and $-(p-1)/2 \leq k \leq (p-1)/2$. Since the above is equivalent to

$$(12k+2)^2 + 6(2m+1)^2 \equiv 0 \pmod p$$

and $\left(\frac{-6}{p}\right) = -1$, it follows that the only solution of (2.4.3) is $k = \frac{\pm p-1}{6}$ and $m = \frac{p-1}{2}$. Therefore, extracting the terms containing $q^{pn+\frac{5p^2-5}{24}}$ from both sides of

(2.4.2) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} b_6\left(p^{2\alpha}\left(pn+\frac{5p^2-5}{24}\right)+5\cdot\frac{p^{2\alpha}-1}{24}\right)q^n \equiv (-1)^{(\alpha+1)\frac{\pm p-1}{6}}\psi(q^p)(q^{2p};q^{2p})_\infty \pmod 3. \tag{2.4.4}$$

Again extracting the terms containing q^{pn} from both sides of the above congruence and replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} b_6\left(p^{2\alpha+2}n+5\cdot\frac{p^{2\alpha+2}-1}{24}\right)q^n \equiv (-1)^{(\alpha+1)\frac{\pm p-1}{6}}\psi(q)(q^2;q^2)_\infty \pmod 3,$$

which is obviously the $\alpha+1$ case of (2.4.1). □

We now prove Theorem 2.1.3.

Proof of Theorem 2.1.3. Comparing the coefficients of q^{pn+j}, $1\le j\le p-1$, on both sides of (2.4.4), we easily arrive at (2.1.3). □

2.5 New congruences for 7- and 49-regular partitions

We first prove Theorem 2.1.4.

Proof of Theorem 2.1.4. We note that

$$\sum_{n=0}^{\infty} b_7(n)q^n = \frac{(q^7;q^7)_\infty}{(q;q)_\infty} = (q^7;q^7)_\infty \sum_{n=0}^{\infty} P(n)q^n, \tag{2.5.1}$$

where $P(n)$ is the ordinary partition function.

From [20, Equation 2.4.5, p. 40], we recall the well-known identity

$$\sum_{n=0}^{\infty} P(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}. \tag{2.5.2}$$

Employing the above in (2.5.1), we find that

$$\sum_{n=0}^{\infty} b_7(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^3} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^7}.$$

Therefore,

$$\sum_{n=0}^{\infty} b_7(7n+5)q^n \equiv 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^3} \equiv 7(q^7;q^7)_\infty^2(q;q)_\infty^4 \pmod{49},$$

which, by (2.2.5), is equivalent to

$$\sum_{n=0}^{\infty} b_7(7n+5)q^n \equiv 7q^8(q^7;q^7)_\infty^2(q^{49};q^{49})_\infty^4\xi^4 \pmod{49}.$$

Extracting the terms containing q^{7n+1} from the above, we have

$$\sum_{n=0}^{\infty} b_7(49n+12)q^{7n+1} \equiv 7q^8(q^7;q^7)_\infty^2(q^{49};q^{49})_\infty^4 H(\xi^4) \pmod{49},$$

which, by (2.2.6), reduces to

$$\begin{aligned}\sum_{n=0}^{\infty} b_7(49n+12)q^{7n+1} &\equiv 7q^8(q^7;q^7)_\infty^2(q^{49};q^{49})_\infty^4\left(-\frac{4(q^7;q^7)_\infty^4}{q^7(q^{49};q^{49})_\infty^4}-7\right)\\ &\equiv 21q(q^7;q^7)_\infty^6 \pmod{49}.\end{aligned}$$

Dividing both sides of the above by q and replacing q^7 by q and then again using (2.2.5), we find that

$$\sum_{n=0}^{\infty} b_7(49n+12)q^n \equiv 21q^{12}(q^{49};q^{49})_\infty^6\xi^6 \pmod{49}.$$

Extracting the terms containing q^{7n+5} from both sides of the above, we have

$$\sum_{n=0}^{\infty} b_7(343n+257)q^{7n+5} \equiv 21q^{12}(q^{49};q^{49})_\infty^6 H(\xi^6) \pmod{49}.$$

Employing once again (2.2.6), we arrive at

$$b_7(343n+257) \equiv 0 \pmod{49}. \tag{2.5.3}$$

Now, from Furcy and Penniston's paper [35], for all $\alpha, n \geq 0$, we note that

$$b_7\left(3^{2\alpha+2}n+\frac{11\cdot 3^{2\alpha+1}-1}{4}\right) \equiv 0 \pmod{3} \tag{2.5.4}$$

and

$$b_7\left(3^{2\alpha+3}n+\frac{11\cdot 3^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod{3}. \tag{2.5.5}$$

Again, replacing n by $3^{2\alpha+2}n+\dfrac{5\cdot 3^{2\alpha+1}-3}{4}$ in (2.5.3), we have

$$b_7\left(7^3\cdot\left(3^{2\alpha+2}n+\frac{5\cdot 3^{2\alpha+1}-3}{4}\right)+257\right)\equiv 0 \pmod{49},$$

that is,

$$b_7\left(3^{2\alpha+2}(7^3\cdot n+142)+\frac{11\cdot 3^{2\alpha+1}-1}{4}\right)\equiv 0 \pmod{49}.$$

Now from (2.5.4) and the above congruence, we easily deduce (2.1.4).

Similarly, replacing n by $3^{2\alpha+3}n+\dfrac{11\cdot 3^{2\alpha+2}-3}{4}$ in (2.5.3) and using (2.5.5) we easily arrive at (2.1.5) to finish the proof. □

Theorem 2.5.1. *If p is a prime such that $\left(\frac{-7}{p}\right)=-1$ and $p\geq 5$, then for all $\alpha\geq 0$,*

$$\sum_{n=0}^{\infty} b_{49}\left(7p^{2\alpha}n+7(p^{2\alpha}-1)+5\right)q^n\equiv 7p^{2\alpha}(q;q)_\infty^3(q^7;q^7)_\infty^3 \pmod{49}. \tag{2.5.6}$$

Proof. We note that

$$\sum_{n=0}^{\infty} b_{49}(n)q^n=\frac{(q^{49};q^{49})_\infty}{(q;q)_\infty}=(q^{49};q^{49})_\infty\sum_{n=0}^{\infty}P(n)q^n.$$

Employing (2.5.2), we have

$$\sum_{n=0}^{\infty} b_{49}(7n+5)q^n=(q^7;q^7)_\infty\left(7\,\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4}+49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}\right).$$

Therefore,

$$\sum_{n=0}^{\infty} b_{49}(7n+5)q^n\equiv 7(q;q)_\infty^{24}\equiv 7(q;q)_\infty^3(q^7;q^7)_\infty^3 \pmod{49}, \tag{2.5.7}$$

which is clearly the $\alpha=0$ case of (2.5.6). Now suppose that (2.5.6) holds for some $\alpha\geq 0$.

With the help of (2.2.3), we rewrite (2.5.6) as

$$\sum_{n=0}^{\infty} b_{49}\left(7p^{2\alpha}n + 14(p^{2\alpha}-1)+5\right)q^n$$
$$\equiv 7p^{2\alpha}\Bigg[\sum_{\substack{m=0\\ m\neq \frac{p-1}{2}}}^{p-1} (-1)^m q^{\frac{m(m+1)}{2}} \sum_{n=0}^{\infty}(-1)^n(2pn+2m+1)q^{pn\cdot\frac{pn+2m+1}{2}}$$
$$+ p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} (q^{p^2};q^{p^2})_\infty^3\Bigg]$$
$$\times\Bigg[\sum_{\substack{k=0\\ k\neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{7\frac{k(k+1)}{2}} \sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{7pn\cdot\frac{pn+2k+1}{2}}$$
$$+ p(-1)^{\frac{p-1}{2}} q^{7\cdot\frac{p^2-1}{8}} (q^{7p^2};q^{7p^2})_\infty^3\Bigg] \pmod{49}. \tag{2.5.8}$$

We now want to know when the exponents above satisfy the congruence

$$\frac{m^2+m}{2} + 7\,\frac{k^2+k}{2} \equiv p^2-1 \pmod{p}, \tag{2.5.9}$$

where $0 \leq k, m \leq p-1$. Since the above congruence is equivalent to

$$(2m+1)^2 + 7\,(2k+1)^2 \equiv 0 \pmod{p}$$

and since $\left(\frac{-7}{p}\right) = -1$, it follows that the only solution of (2.5.9) is $m = \dfrac{p-1}{2}$ and $k = \dfrac{p-1}{2}$. Therefore, extracting the terms containing q^{pn+p^2-1} from both sides of (2.5.8) and replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} b_{49}\left(7p^{2\alpha}(pn+p^2-1) + 7(p^{2\alpha}-1)+5\right)q^n$$
$$\equiv 7p^{2\alpha+2}(q^p;q^p)_\infty^3(q^{7p};q^{7p})_\infty^3 \pmod{49}. \tag{2.5.10}$$

Again extracting the terms containing q^{pn} from both sides of the above congruence and replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} b_{49}\left(7p^{2\alpha+2}n + 7(p^{2\alpha+2}-1)+5\right)q^n \equiv 7p^{2\alpha+2}(q;q)_\infty^3(q^7;q^7)_\infty^3 \pmod{49},$$

which is the case for $\alpha+1$ of (2.5.6). Thus we complete the proof. $\square$

We now prove Theorem 2.1.5.

Proof of Theorem 2.1.5. Comparing the coefficients of q^{pn+j}, $1 \le j \le p-1$, on both sides of (2.5.10), we can easily deduce (2.1.6). $\square$

Using (2.2.3) in (2.5.7) and then comparing the coefficients of q^{7n+j} for $j \in \{2,4,5\}$, we readily obtain the following result.

Corollary 2.5.2. *For any non-negative integer n and $j \in \{2,4,5\}$,*

$$b_{49}(7(7n+j)+5) \equiv 0 \pmod{49}.$$

2.6 Proofs of Theorem 1.2.1 and Theorem 1.2.3

Theorem 2.6.1. *If p is a prime such that $p \equiv 5$ or $7 \pmod 8$ and $\alpha \ge 0$, then*

$$\sum_{n=0}^{\infty} b_{10}\left(p^{2\alpha}n + 3 \cdot \frac{p^{2\alpha}-1}{8}\right) q^n \equiv p^{\alpha}(-1)^{\alpha\cdot\frac{p-1}{2}}\psi(q)(q^2;q^2)_\infty^3 \pmod 5. \tag{2.6.1}$$

Proof. We prove the theorem by induction on α. Note that

$$\sum_{n=0}^{\infty} b_{10}(n)q^n = \frac{(q^{10};q^{10})_\infty}{(q;q)_\infty}.$$

Since $(q;q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod 5$, we find that

$$\sum_{n=0}^{\infty} b_{10}(n)q^n \equiv \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty} \equiv \psi(q)(q^2;q^2)_\infty^3 \pmod 5,$$

which is the $\alpha = 0$ case of (2.6.1). Suppose (2.6.1) holds for some $\alpha \ge 0$. With the help of (2.2.1) and (2.2.3), we can rewrite (2.6.1) as

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_{10}\left(p^{2\alpha}n + 3\cdot\frac{p^{2\alpha}-1}{8}\right)q^n \\
&\equiv p^{\alpha}(-1)^{\alpha\cdot\frac{p-1}{2}}\Big[\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\Big] \\
&\quad\times\Big[\sum_{\substack{k=0\\ k\neq\frac{p-1}{2}}}^{p-1} (-1)^k q^{k(k+1)}\sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{pn(pn+2k+1)} \\
&\quad+ p(-1)^{\frac{p-1}{2}} q^{2\cdot\frac{p^2-1}{8}}(q^{2p^2};q^{2p^2})_\infty^3\Big] \pmod 5. \qquad (2.6.2)
\end{aligned}$$

We want those terms above for which the powers of q satisfy the congruence

$$k^2 + k + \frac{m^2+m}{2} \equiv 3\cdot\frac{p^2-1}{8} \pmod p,$$

where $0 \le k \le p-1$ and $0 \le m \le (p-1)/2$. The congruence is clearly equivalent to

$$2(2k+1)^2 + (2m+1)^2 \equiv 0 \pmod p. \qquad (2.6.3)$$

Since $\left(\frac{-2}{p}\right) = -1$ as $p \equiv 5$ or $7 \pmod 8$, the only solution of (2.6.3) is $k = \frac{p-1}{2}$ and $m = \frac{p-1}{2}$. Therefore, extracting the terms involving $q^{pn+3\cdot\frac{p^2-1}{8}}$ from both sides of (2.6.2) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} b_{10}\left(p^{2\alpha+1}n + 3\frac{p^{2\alpha+2}-1}{8}\right)q^n \equiv p^{\alpha+1}(-1)^{(\alpha+1)\frac{p-1}{2}}\psi(q^p)(q^{2p};q^{2p})_\infty^3 \pmod 5. \qquad (2.6.4)$$

Again extracting the terms containing q^{pn} from both sides of the above congruence and replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} b_{10}\left(p^{2\alpha+2}n + 3\frac{p^{2\alpha+2}-1}{8}\right)q^n \equiv p^{\alpha+1}(-1)^{(\alpha+1)\frac{p-1}{2}}\psi(q)(q^2;q^2)_\infty^3 \pmod 5,$$

which is the $\alpha+1$ case of (2.6.1). □

Theorem 2.6.2. *If p is a prime such that $p \equiv -1 \pmod 6$, then for all $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} b_{20}\left(p^{2\alpha}n + 19 \cdot \frac{p^{2\alpha}-1}{24}\right) q^n \equiv (-p)^{\alpha}\psi(q)\psi(q^4)(-q^2;-q^2)_\infty^2 \pmod 5. \quad (2.6.5)$$

Proof. We use induction on α. Clearly,

$$\sum_{n=0}^{\infty} b_{20}(n)q^n = \frac{(q^{20};q^{20})_\infty}{(q;q)_\infty} \equiv \frac{(q^4;q^4)_\infty^5}{(q;q)_\infty} \equiv \psi(q)\psi(q^4)(-q^2;-q^2)_\infty^2 \pmod 5,$$

which is $\alpha = 0$ case of (2.6.5). Now suppose (2.6.5) holds for some $\alpha \geq 0$. Using (2.2.1) and (2.2.4) with q replaced by $-q^2$ in (2.6.5), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_{20}\left(p^{2\alpha}n + 19 \cdot \frac{p^{2\alpha}-1}{24}\right) q^n \\
&\equiv (-p)^{\alpha}\Big[\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\Big] \\
&\quad \times \Big[\sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{3}}}^{\frac{p-1}{2}} (-1)^k q^{2(3k^2+2k)} \sum_{n=-\infty}^{\infty} (-1)^n (3pn+3k+1) q^{2pn(3pn+6k+2)} \\
&\quad - p(-1)^{\frac{p^2-1}{3}} q^{2\cdot\frac{p^2-1}{3}} \psi(q^{4p^2})(-q^{2p^2};-q^{2p^2})_\infty^2\Big] \pmod 5. \qquad (2.6.6)
\end{aligned}$$

Now consider the congruence

$$\frac{m^2+m}{2} + 6k^2 + 4k \equiv 19 \cdot \frac{p^2-1}{24} \pmod p, \qquad (2.6.7)$$

where $0 \leq m \leq (p-1)/2$ and $-(p-1)/2 \leq k \leq (p-1)/2$. Since the above congruence is equivalent to

$$(12k+4)^2 + 3(2m+1)^2 \equiv 0 \pmod p$$

and $\left(\frac{-3}{p}\right) = -1$ as $p \equiv -1 \pmod 6$, it follows that the only solution of (2.6.7) is $k = \dfrac{\pm p - 1}{3}$ and $m = \dfrac{p-1}{2}$. So, extracting the terms containing $q^{pn+19\cdot\frac{p^2-1}{24}}$ from both sides of (2.6.6) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} b_{20}\left(p^{2\alpha+1}n + 19 \cdot \frac{p^{2\alpha+2}-1}{24}\right) q^n \equiv (-p)^{\alpha+1}\psi(q^p)\psi(q^{4p})(-q^{2p};-q^{2p})_\infty^2 \pmod 5. \qquad (2.6.8)$$

Again extracting the terms containing q^{pn} from both sides of the above congruence and replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} b_{20}\left(p^{2\alpha+2}n + 19 \cdot \frac{p^{2\alpha+2}-1}{24}\right) q^n \equiv (-p)^{\alpha+1}\psi(q)\psi(q^4)(-q^2;-q^2)_\infty^2 \pmod 5,$$

which is the $\alpha + 1$ case of (2.6.5). □

Now we are in a position to prove Theorem 1.2.1 and Theorem 1.2.3.

Proofs of Theorem 1.2.1 and Theorem 1.2.3. Comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$, from both sides of (2.6.4), we immediately obtain (1.2.2). On the other hand, comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$ from both sides of (2.6.8), we readily arrive at (1.2.4). □

2.7 Table of congruences for ℓ-regular partitions found in the literature

Carlson and Webb [23] have found congruences for ℓ-regular partitions when ℓ =10, 15 and 20 modulo 5. In this Chapter, we found congruences for ℓ-regular partitions when ℓ =5, 6, 7 and 49 modulo 25, 3, 147 and 49, respectively. In the following table, we list other values of ℓ for which congruences for ℓ-regular partitions have been found in the literature.

Authors' names and Source	Year	Values of ℓ	Modulo
Hirschhorn and Sellers [41]	2010	5	2
Furcy and Penniston [35]	2012	$\ell \equiv 1 \pmod 3$, $\ell \leq 49$	3
Cui and Gu [32]	2013	2, 4, 5, 8, 13, 16	2
Xia and Yao [57]	2013	9	2
Carlson and Webb [23]	2014	10, 15, 20	5
Cui and Gu [31]	2014	9	3

Authors' names and Source	Year	Values of ℓ	Modulo
Lin and Wang [48]	2014	9	3
Yao [63]	2014	9	4, 8, 9
Ahmed and Baruah [1]	2015	5, 6, 7, 49	25, 3, 147, 49
Lin [47]	2015	13	3
Baruah and Das [13]	2015	7, 23	2
Lin [46]	2015	7	3
Hou, Sun and Zhang [42]	2015	3, 5, 6, 7, 10	3, 5, 7
Webb [56]	2015	13	3