Chapter 4

Weak nil clean rings

4.1 Introduction

In the year 2006, Ahn and Anderson defined a ring R to be weakly clean if each element $r \in R$ can be written as $r = u + e$ or $r = u - e$ for some $u \in \mathrm{U}(R)$ and $e \in \mathrm{Idem}(R)$ [2]. Motivated by this concept, we observe the example $\mathbb{Z}_6 = \{\overline{0},\ \overline{1},\ \overline{2},\ \overline{3},\ \overline{4},\ \overline{5}\}$, here $\mathrm{Idem}(\mathbb{Z}_6) = \{\overline{0},\ \overline{1},\ \overline{3},\ \overline{4}\}$ and $\mathrm{Nil}(\mathbb{Z}_6) = \{\overline{0}\}$. So clearly $\mathbb{Z}_6$ is not a nil clean ring as $\overline{2}$ and $\overline{5}$ can not be written as a sum of an idempotent and a nilpotent of $\mathbb{Z}_6$. But we see that every elements $r \in \mathbb{Z}_6$ can be written as $r = n - e$ or $r = n + e$ for $e \in \mathrm{Idem}(\mathbb{Z}_6)$ and $n \in \mathrm{Nil}(\mathbb{Z}_6)$, which led us to introduce weak nil clean ring. A weak nil clean ring is a ring with unity in which each element of the ring can be expressed as a sum or difference of a nilpotent and an idempotent. A study on commutative weak nil clean rings have been done by Peter V. Danchev and W. Wm. McGovern (see [21]). Here we have given a stronger version of a few of their results along with some new results. We have also determined all natural numbers n, for which $\mathbb{Z}_n$ is a weak nil clean ring but not nil clean ring. Further we have discussed an S-weak nil clean ring, a ring in which each element can be expressed as a sum or difference of a nilpotent and an element of S, where $S \subseteq \mathrm{Idem}(R)$, and have shown that if $S = \{0, 1\}$, then an S-weak nil clean ring contains a unique maximal ideal. Finally we have shown that weak* nil clean rings (Definition 4.2.1) are exchange rings and strongly nil clean rings provided $2 \in R$ is nilpotent in the later case. We have ended the chapter by introducing weak J-clean rings and obtain a few results on weak J-clean rings as an effort to answer **Problem** 5 of [21].

4.2 Weak nil clean rings

Definition 4.2.1. *An element $r \in R$ is said to be a weak nil clean element of the ring R, if $r = n + e$ or $r = n - e$, for some $n \in \mathrm{Nil}(R)$, $e \in \mathrm{Idem}(R)$, and a ring is said to be a weak nil clean ring if each of its elements is weak nil clean. Further if $r = n - e$ or $n + e$ with $ne = en$, then r is called weak* nil clean.*

Obviously every nil clean ring is weak nil clean, but the above example denies the converse. Also if R is a weak nil clean ring or a weak* nil clean ring then for $n \geq 2$,

$$S = \{A = (a_{ij}) \in T_n(R) : a_{11} = a_{22} = \cdots = a_{nn}\},$$

is a weak nil clean ring which is not weak* nil clean, where $T_n(R)$ is the ring of upper triangular matrices of order n over R. Analogous to the concept of clean and nil clean rings, it is easy to see that every weak nil clean ring is weakly clean and the converse is not true. The following theorem is easy to see.

Theorem 4.2.2. *Every homomorphic image of a weak nil clean ring is weak nil clean.*

However the converse is not true as $\mathbb{Z}_6 \cong \mathbb{Z}/\langle 6 \rangle$ is a weak nil clean ring, but $\mathbb{Z}$ is not a weak nil clean ring. A finite direct product $\prod R_\alpha$ of rings is nil clean if and only if each R_α is nil clean. The next result shows that this is not true for weak nil clean rings (the following result generates (ii) of **Proposition 1.9** of [21]).

Theorem 4.2.3. *Let $\{R_\alpha\}$ be a finite collection of rings. Then the direct product $R = \prod R_\alpha$ is weak nil clean if and only if each R_α is weak nil clean and at most one R_α is not nil clean.*

Proof. ($\Rightarrow$) Let R be weak nil clean. Then each R_α being a homomorphic image of R is weak nil clean. Suppose for some α_1 and α_2, $\alpha_1 \neq \alpha_2$, R_{α_1} and R_{α_2} are not nil clean. Since R_{α_1} is not nil clean, not all elements $x \in R_{\alpha_1}$ are of the form $n - e$, where $n \in \mathrm{Nil}(R_{\alpha_1})$ and $e \in \mathrm{Idem}(R_{\alpha_1})$. But R_{α_1} is weak nil clean, so there exists $x_{\alpha_1} \in R_{\alpha_1}$, with $x_{\alpha_1} = n_{\alpha_1} + e_{\alpha_1}$, where $e_{\alpha_1} \in \mathrm{Idem}(R_{\alpha_1})$ and $n_{\alpha_1} \in \mathrm{Nil}(R_{\alpha_1})$, but $x_{\alpha_1} \neq n - e$ for any $n \in \mathrm{Nil}(R_{\alpha_1})$ and $e \in \mathrm{Idem}(R_{\alpha_1})$. Likewise there exists $x_{\alpha_2} \in R_{\alpha_2}$, with $x_{\alpha_2} = n_{\alpha_2} - e_{\alpha_2}$, where $e_{\alpha_2} \in \mathrm{Idem}(R_{\alpha_2})$ and $n_{\alpha_2} \in \mathrm{Nil}(R_{\alpha_2})$, but $x_{\alpha_2} \neq n + e$ for any $n \in \mathrm{Nil}(R_{\alpha_2})$ and $e \in \mathrm{Idem}(R_{\alpha_2})$.

Define $x = (x_\alpha) \in R,$ such that
$$x_\alpha = x_\alpha \text{ if } \alpha \in \{\alpha_1, \alpha_2\}$$
$$= 0 \quad \text{if } \alpha \notin \{\alpha_1, \alpha_2\}.$$

Then clearly $x \neq n \pm e$ for any $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$. Hence at most one R_α is not nil clean.

($\Leftarrow$) If each R_α is nil clean, then $R = \prod R_\alpha$ is nil clean, so weak nil clean. So assume some R_{α_0} is weak nil clean but not nil clean and that all other R_α's are nil clean. Let $x = (x_\alpha) \in R$. In R_{α_0} we can write $x_{\alpha_0} = n_{\alpha_0} + e_{\alpha_0}$ or $x_{\alpha_0} = n_{\alpha_0} - e_{\alpha_0}$, where $n_{\alpha_0} \in \mathrm{Nil}(R_{\alpha_0})$, $e_{\alpha_0} \in \mathrm{Idem}(R_{\alpha_0})$. If $x_{\alpha_0} = n_{\alpha_0} + e_{\alpha_0}$, for $\alpha \neq \alpha_0$, let $x_\alpha = n_\alpha + e_\alpha$ and if $x_{\alpha_0} = n_{\alpha_0} - e_{\alpha_0}$, for $\alpha \neq \alpha_0$, let $x_\alpha = n_\alpha - e_\alpha$ then $n = (n_\alpha) \in \mathrm{Nil}(R)$ and $e = (e_\alpha) \in \mathrm{Idem}(R)$ and $x = n + e$ or $x = n - e$ respectively. Hence R is weak nil clean. □

Proposition 4.2.4. *If R be a weak nil clean ring, then $J(R) \subseteq \mathrm{Nil}(R)$.*

Proof. Let $x \in J(R)$, and $x = n - e$ or $x = n + e$, where $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$. If $x = n - e$ then there exists a $k \in \mathbb{N}$ such that $(x + e)^k = 0$, which gives $e \in J(R) \cap \mathrm{Idem}(R)$. Hence $e = 0$ i.e., $x = n \in \mathrm{Nil}(R)$. Similarly for $x = n + e$, we get $x = n \in \mathrm{Nil}(R)$. Thus $J(R) \subseteq \mathrm{Nil}(R)$. □

Proposition 4.2.5. *If a commutative ring R is weak nil clean then $R/\mathrm{Nil}(R)$ is weak nil clean. The converse holds if idempotents can be lifted modulo* $\mathrm{Nil}(R)$.

Proof. ($\Rightarrow$) Follows from Theorem (4.2.2).
($\Leftarrow$) Let $x \in R$. Since $R/\mathrm{Nil}(R)$ is weak nil clean, so $x+\mathrm{Nil}(R) = y+\mathrm{Nil}(R)$ or $(-y)+\mathrm{Nil}(R)$, where $y^2 - y \in \mathrm{Nil}(R)$ (as $R/\mathrm{Nil}(R)$ is a reduced ring). Since idempotents of R lift modulo $\mathrm{Nil}(R)$, there exists $e \in \mathrm{Idem}(R)$ such that $y - e \in \mathrm{Nil}(R)$, which implies $x - e \in \mathrm{Nil}(R)$ or $x + e \in \mathrm{Nil}(R)$ i.e., $x - e = n$ or $x + e = m$ for some $m, n \in \mathrm{Nil}(R)$, which proves the result. □

For more examples of weak nil clean rings, we consider the method of idealization. Let R be a commutative ring and M an $R-$module. The trivial extension of R and M is the ring

$$R(M) = R \oplus M$$

with product defined as

$$(r, m)(r', m') = (rr', rm' + r'm)$$

and sum as

$$(r, m) + (r', m') = (r + r', m + m'),$$

for $(r, m), (r', m') \in R(M)$.

Theorem 4.2.6. *Let R be a ring and M be an R-module. Then R is weak nil clean if and only if $R(M)$ is weak nil clean.*

Proof. ($\Leftarrow$) Note that $R \approx R(M)/(0 \oplus M)$ is a homomorphic image of $R(M)$. Hence by Theorem (4.2.2), R is a weak nil clean ring.
($\Rightarrow$) Let R be a weak nil clean ring and $(r, m) \in R \oplus M$, where $r \in R$ and $m \in M$. We have $r = n + e$ or $n - e$ for $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$. Then

$$\begin{aligned}(r, m) &= (n + e, m) \text{ or } (n - e, m)\\ &= (n, m) + (e, 0) \text{ or } (n, m) - (e, 0)\end{aligned}$$

is a weak nil clean expression of (r,m), where $(n,m) \in \text{Nil}(R)$ and $(e,0) \in \text{Idem}(R)$. Hence $R(M) = R \oplus M$ is weak nil clean. □

Now we try to determine all n for which $\mathbb{Z}_n$ is weak nil clean but not nil clean. We recall that, $\text{Idem}\,(\mathbb{Z}_{p^k}) = \{0,1\}$, for any prime $p \in \mathbb{N}$ and $k \in \mathbb{N}$.

Lemma 4.2.7. *$\mathbb{Z}_{3^k}$, is weak nil clean but not nil clean for every $k \in \mathbb{N}$.*

Proof. The proof follows from the fact that

$$\text{Idem}\,(\mathbb{Z}_{3^k}) = \{0,1\} \text{ and } \text{Nil}\,(\mathbb{Z}_{3^k}) = \{0,3,6,...,3(3^{k-1}-1)\}.$$

So if $a \in \mathbb{Z}_{3^k}$

$$a \equiv \begin{cases} 0(\text{mod } 3) & ; \\ 1(\text{mod } 3) & ; \\ 2(\text{mod } 3) & . \end{cases}$$

If $a \equiv 1(mod 3)$, $a = n+1$ where $n \in \text{Nil}\,(\mathbb{Z}_{3^k})$; If $a \equiv 2(mod 3)$, $a = n-1$ where $n \in \text{Nil}\,(\mathbb{Z}_{3^k})$; If $a \equiv 0(mod 3)$, $a = n$ where $n \in \text{Nil}\,(\mathbb{Z}_{3^k})$. □

Lemma 4.2.8. *Let p be a prime and $k \in \mathbb{N}$. Then $\mathbb{Z}_{p^k}$ is weak nil clean but not nil clean iff $p = 3$.*

Proof. ($\Leftarrow$) It follows from Lemma 4.2.7.
($\Rightarrow$) We know that $\mathbb{Z}_{2^k}$ is nil clean $\forall k \in \mathbb{N}$ and $\mathbb{Z}_{3^k}$ is weak nil clean $\forall k \in \mathbb{N}$ but not nil clean. Now consider $p > 3$. We have

$$\text{Idem}\,(\mathbb{Z}_{p^k}) = \{0,1\} \text{ and } \text{Nil}\,(\mathbb{Z}_{p^k}) = \{0, p, 2p, \ldots, (p^{k-1}-1)p\}.$$

So if we consider the sum or difference of nilpotents and idempotents of $\mathbb{Z}_{p^k}$ respectively, then at most $4p^{k-1}$ elements can be obtained, but $p > 4$, so $p^k > 4p^{k-1}$. Hence not all elements of $\mathbb{Z}_{p^k}$ can be written as a sum or difference of a nilpotent and an idempotent of $\mathbb{Z}_{p^k}$. So $p = 3$. □

Theorem 4.2.9. *The only n for which $\mathbb{Z}_n$ is weak nil clean but not nil clean is of the form 2^r3^t, where $t \in \mathbb{N}, r \in \mathbb{N} \cup \{0\}$.*

Proof. We have already seen that $\mathbb{Z}_{3^t}$ is weak nil clean but not nil clean. Next let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

with $\alpha_i \in \mathbb{N},\ 1 \leq i \leq k$ and p_i's are distinct primes such that

$$p_1 \leq p_2 \leq ... \leq p_n.$$

If $k > 2$, then there exists some i with $1 \leq i \leq k$ such that $p_i > 3$. Then $\mathbb{Z}_{p_i^{\alpha_i}}$ is not weak nil clean. Hence $\mathbb{Z}_n$ can not be weak nil clean as

$$\mathbb{Z}_n = \mathbb{Z}_{p_1^{\alpha_1}} \oplus \mathbb{Z}_{p_2^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{\alpha_k}}.$$

So $k \leq 2$ and $p_i \leq 3$ i.e., $n = p_1^{\alpha_1} p_2^{\alpha_2}$. If $k = 1$, then p_1 must be 3 as $\mathbb{Z}_{2^r}$ is nil clean. Again if $k = 2$, then since p_i's are distinct so $p_1 = 2$ and $p_2 = 3$. Also if $n = 2^{\alpha_1}3^{\alpha_2}$, then $\mathbb{Z}_n = \mathbb{Z}_{2^{\alpha_1}} \oplus \mathbb{Z}_{3^{\alpha_2}}$. Since $\mathbb{Z}_{2^{\alpha_1}}$ is nil clean and $\mathbb{Z}_{3^{\alpha_2}}$ is weak nil clean but not nil clean, so $\mathbb{Z}_n$ is weak nil clean but not nil clean. This completes the proof. □

If R is commutative then $R[x]$ is never weak nil clean. For if $x \in R[x]$ is of the form $\sum_i a_i x^i - e$ or $\sum_i a_i x^i + e$, where $a_i \in \mathrm{Nil}(R), e \in \mathrm{Idem}(R)$, then $a_0 - e = 0$ or $a_0 + e = 0$, which is absurd.

However if R is weak nil clean and $\sigma : R \to R$ is a ring endomorphism then for any $n \in \mathbb{N}$, the quotient

$$S = R[x;\sigma]/< x^n >,$$

where $R[x;\sigma]$ is the Hilbert twist, is a weak nil clean ring. Indeed if

$$f = a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1} \in S$$

and $a_0 = n + e$ or $a_0 = n - e$, where $n \in \mathrm{Nil}(R), e \in \mathrm{Idem}(R)$, then $f = (f - e) + e$ or $f = (f + e) - e$ is a weak nil clean decomposition of f in S.

In order to show that, weak* nil cleanness penetrates to corner, we need the following lemmas.

Lemma 4.2.10. *Let R be a ring and $x = n + e$ or $n - e$ be weak* nil clean decomposition of $x \in R$ with $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$. Then $\mathrm{ann}_l(x) \subseteq \mathrm{ann}_l(e)$ and $\mathrm{ann}_r(x) \subseteq \mathrm{ann}_r(e)$, where $\mathrm{ann}_l(a)$ and $\mathrm{ann}_r(a)$ denote the left and right annihilator of an element a in R respectively.*

Proof. Let $r \in \mathrm{ann}_l(x)$. Then $rx = 0$. Now if $x = e + n$, then $rn + re = 0$ and so $rne + re = 0$, i.e., $re(n + 1) = 0$, implying $re = 0$. Hence $r \in \mathrm{ann}_l(e)$.
Again if $x = n - e$, then $rn - re = 0$ and so $rne - re = 0$, i.e., $re(n - 1) = 0$, implying $re = 0$. So $r \in \mathrm{ann}_l(e)$. Hence $\mathrm{ann}_l(x) \subseteq \mathrm{ann}_l(e)$. Similarly we have $\mathrm{ann}_r(x) \subseteq \mathrm{ann}_r(e)$. □

Lemma 4.2.11. *Let R be a ring and $x = n + e$ or $n - e$ be weak* nil clean decomposition of $x \in R$ with $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$. Then $\mathrm{ann}_l(x) \subseteq R(1 - e)$ and $\mathrm{ann}_r(x) \subseteq (1 - e)R$.*

Proof. Straightforward.

Theorem 4.2.12. *Let R be a ring and $f \in Idem(R)$. Then $x \in fRf$ is weak* nil clean in R if and only if x is weak* nil clean in fRf.*

Proof.($\Leftarrow$) If $x \in fRf$ is weak* nil clean in fRf, then by the same weak* nil clean decomposition, x is weak* nil clean in R.
($\Rightarrow$) Let x be weak* nil clean in R. Then $x = n + e$ or $n - e$ for some $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$ with $ne = en$. First let $x = n + e$. Since $x \in fRf$, we have

$$\begin{aligned}(1 - f) &\in \mathrm{ann}_l(x) \cap \mathrm{ann}_r(x)\\ &\subseteq R(1 - e) \cap (1 - e)R\\ &= (1 - e)R(1 - e) \text{ [by Lemma 4.2.11].}\end{aligned}$$

Thus we have $(1-f)e = 0 = e(1-f)$, giving $fe = e = ef$, and consequently $fef \in \text{Idem}(fRf)$. Also $xf = fx$, therefore we have $nf = fn$, i.e., $fnf \in \text{Nil}(fRf)$. Hence $x = fnf + fef$. Similarly if $x = n - e$ then $x = fnf - fef$. Hence x is weak* nil clean in fRf. □

The following is an immediate consequence of **Theorem 4.2.12**.

Corollary 4.2.13. *Let R be a weak* nil clean ring and $e \in \text{Idem}(R)$. Then the corner ring eRe is also weak* nil clean.*

4.3 S-weak nil clean rings

An S-weak nil clean ring is a generalization of a weak nil clean ring, which is defined as follows:

Definition 4.3.1. *Let S be a non-empty set of idempotents of R. The ring R is called $S-weak\ \ nil\ \ clean$ if each $r \in R$ can be written as $r = n + e$ or $n - e$, where $n \in \text{Nil}(R)$ and $e \in S$. Further if $ne = en$, then R is called S-weak* nil clean.*

Proposition 4.3.2. *If R is a $\{0,1\}$-weak nil clean ring, then R has exactly one maximal ideal.*

Proof. Since R is a $\{0,1\}$-weak nil clean ring. We have

$$R = \text{U}(R) \cup \text{Nil}(R)$$

and

$$\text{U}(R) = (1 + \text{Nil}(R)) \cup (-1 + \text{Nil}(R)).$$

It follows that for any $x \in \text{Nil}(R)$ and any $r \in R$, we have $xr, rx \in \text{Nil}(R)$. Next if possible let $n_1 - n_2 = u$, where $n_1, n_2 \in \text{Nil}(R)$ and $u \in \text{U}(R)$. Then $u^{-1}n_1 - u^{-1}n_2 = 1$ i.e., $n_3 = 1 + n_4$, where $u^{-1}n_1 = n_3 \in \text{Nil}(R)$ and $u^{-1}n_2 = n_4 \in \text{Nil}\,(R)$, which is a contradiction. Thus $n_1 - n_2 \in \text{Nil}(R)$, for any $n_1,\ \ n_2 \in \text{Nil}(R)$, implying that

Nil(R) is an ideal. Hence, by proposition 4.2.4. $J(R) = \text{Nil}(R)$. This completes the proof. $\square$

From above theorem it is clear that $\{0,1\}-$ nil clean rings are local rings. The converse is not true.

Theorem 4.3.3. *If a ring R is S-weak* nil clean for $S \subseteq \text{Idem}(R)$, then $S = \text{Idem}(R)$.*

Proof. Let $e' \in \text{Idem}(R)$. Since R is S-weak* nil clean, $-e' = n + e$ or $-e' = n - e$ for some $n \in \text{Nil}(R)$, and $e \in S$, with $ne = en$. If $-e' = n + e$, then

$$
\begin{aligned}
& 1 - e' = 1 + n + e \\
\Rightarrow \quad & (1+n+e)^2 = 1 + n + e \; [1 - e' \in \text{Idem}(R)] \\
\Rightarrow \quad & 1 + n^2 + e + 2n + 2e + 2ne = 1 + n + e \\
\Rightarrow \quad & n^2 + n + 2e(1+n) = 0 \\
\Rightarrow \quad & (n+2e)(1+n) = 0.
\end{aligned}
$$

But $1 + n \in U(R)$, so $n = -2e$, giving

$$-e' = n + e = -2e + e = -e.$$

Thus $e' = e \in S$. Again if $-e' = n - e$, then

$$
\begin{aligned}
& (-e')^2 = e'^2 = e', \\
\Rightarrow \quad & (n-e)^2 = -n + e \\
\Rightarrow \quad & n^2 - 2ne + e = -n + e \\
\Rightarrow \quad & n^2 + n(1-2e) = 0 \\
\Rightarrow \quad & n\{n + (1-2e)\} = 0.
\end{aligned}
$$

But $n + (1 - 2e) \in U(R)$, so $n = 0$, i.e., $e' = e \in S$. Hence $\text{Idem}(R) = S$. $\square$

But in case of a weak clean ring, it is possible that R is $S-$weak clean and $S \subsetneq \text{Idem}(R)$ [2].

4.4 More results on weak nil clean rings

It is well known that $\mathbb{Z}_3$ is clean, so the upper triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_3)$ is clean and hence exchange, but $\mathbb{T}_2(\mathbb{Z}_3)$ is not weak nil clean. So in general, exchange rings are not weak nil clean rings. But one can see that weak* nil clean rings are exchange.

Theorem 4.4.1. *Let R be a weak* nil clean ring. Then R is an exchange ring.*

Proof. Let R be a weak* nil clean ring and $x \in R$. Then $x = n + e$ or $x = n - e$, where $n \in \mathrm{Nil}(R)$ and $e \in \mathrm{Idem}(R)$.
If $x = n - e$, then

$$\begin{aligned}(1-n)[x-(1-n)^{-1}e(1-n)] &= (1-n)[(n-e)-(1-n)^{-1}e(1-n)]\\ &= n-e-n^2+ne-e+en\\ &= x-(n-e)^2 = x-x^2\end{aligned}$$

implying $$[x-(1-n)^{-1}e(1-n)] = (1-n)^{-1}(x-x^2).$$

Similarly if $x = n + e$, we have $x - e = u^{-1}(x^2 - x)$ for $u = (2e - 1) + n \in \mathrm{U}(R)$. Then by condition (1) of Proposition 1.1 of [36], R is exchange. □

Finally we take the question " under what condition a weak* nil clean ring is strongly nil clean ring?" To answer this question we need the following Lemma.

Lemma 4.4.2. *Let R be a ring with $2 \in \mathrm{Nil}(R)$ and M_R a right R-module. If an endomorphism $\phi \in \mathrm{End}(M_R)$ is a sum or difference of a nilpotent n and an idempotent e that commute. Then there exists a direct sum decomposition $M = A \oplus B$ such that $\phi|_A$ is an element of $\mathrm{End}(A)$ which is nilpotent and $(1-\phi)|_B$ is an element of $\mathrm{End}(B)$ which is nilpotent.*

Proof. Suppose $\phi = a - e$, where $e \in \mathrm{Idem}(\mathrm{End}(M_R)), a \in \mathrm{Nil}(\mathrm{End}(M_R))$ and $ea = ae$. We have decomposition $M = A \oplus B$, where
$A = (1 - e)M$ and $B = eM$. Then A and B are $\phi-$invariant.

Now $\phi|_A = (a-e)|_A = a|_A - e|_A = a|_A$ and so $\phi|_A$ is nilpotent.

And $(1-\phi)|_B = (1-(a-e))|_B = (1-a+e)|_B = (2-a-(1-e))|_B = (2-a)|_B$ is nilpotent as 2 is nilpotent.

Again, if $\phi = a+e$, where $e \in \text{Idem}(\text{End}(M_R))$ and $a \in \text{Nil}(\text{End}(M_R))$, then by **Definition** 1.2.8 and **Lemma** 1.2.3 of [22] such a decomposition exists. □

Now we can state the following theorem.

Theorem 4.4.3. *A ring R is strongly nil clean if and only if R is weak* nil clean with $2 \in \text{Nil}(R)$.*

Proof.(⇒) It is by the definition of a weak* nil clean ring.

(⇐) The result follows from **Lemma** 1.2.6 of [22] and **Lemma** (4.4.2).

Corollary 4.4.4. *A weak* nil clean ring R with $2 \in \text{Nil}(R)$ is strongly $\pi-$regular.*

4.5 Weak J-clean rings

In this section we have defined a weak J-clean ring, as a generalization of J-clean rings introduced by Chen [19].

Definition 4.5.1. *An element a in a ring R is said to be weak J-clean if a can be written as $a = j + e$ or $a = j - e$ for some $j \in J(R)$ and $e \in \text{Idem}(R)$. Moreover if $ae = ea$ we say a is weak* J-clean.*

Below are some of the preliminary results related to weak J-clean rings.

Lemma 4.5.2. *Every weak* J-clean element in a ring is strongly clean.*

Proof. Let $a \in R$, $e \in \text{Idem}(R)$ and $w \in J(R)$. If $a = w + e$ we have $a = (1-e) + (2e - 1 + w)$; else if $a = w - e$ we have $a = (1-e) - (1-w)$. □

Lemma 4.5.3. *Let R be a ring and $a = w + e$ or $a = w - e$ be a weak* J-clean decomposition of a in R, where $e \in \text{Idem}(R)$ and $w \in J(R)$. Then $\text{ann}_l(a) \subseteq \text{ann}_l(e)$ and $\text{ann}_r(a) \subseteq \text{ann}_r(e)$.*

Proof. Let $r \in \operatorname{ann}_l(a)$. Then $ra = 0$. If $a = w + e$, then $re = -rw$, so $re = -rwe = -rew$. It follows that $re = 0$, i.e., $r \in \operatorname{ann}_l(e)$. Similarly $\operatorname{ann}_r(a) \subseteq \operatorname{ann}_r(e)$ holds. □

Theorem 4.5.4. *Let R be a ring and let $f \in R$ be an idempotent. Then $a \in fRf$ is weak* J-clean in R if and only if a is weak* J-clean in fRf.*

Proof. Let $a \in fRf$ and $a = w+e$ or $w-e$ for $w \in J(R)$ and $e \in \operatorname{Idem}(R)$. We show that $e \in \operatorname{Idem}(fRf)$, and it will follow that $w \in J(fRf)$. This will show that the above weak* J-clean expression of a is also a weak* J-clean expression of a in fRf. To show $e \in \operatorname{Idem}(fRf)$ observe that $1-f \in \operatorname{ann}_l(a) \cap \operatorname{ann}_r(a) \subseteq \operatorname{ann}_l(e) \cap \operatorname{ann}_r(e)$, implying $ef = e = fe$. So $e \in \operatorname{Idem}(fRf)$. The other implication is easy to see. □

Corollary 4.5.5. *Let R be a weak* J-clean ring and $e \in R$ be an idempotent. Then eRe is weak* J-clean.*

Before proceeding further we have generalized one popular concept of idempotent lifting modulo ideal I of a ring R.

Definition 4.5.6. *Let I be an ideal of R. We say idempotents lift weakly modulo I, if for each idempotent $\overline{e} \in R/I$, there exists an idempotent $e \in R$ such that $e - f \in I$ or $e + f \in I$.*

Theorem 4.5.7. *If R is a ring such that $R/J(R)$ is boolean and each idempotent lifts weakly modulo $J(R)$, then R is weak J-clean.*

Proof. For $a \in R$, $\overline{a} \in R/J(R)$ is an idempotent. By assumption we can find an idempotent $e \in R$, such that $a - e \in R/J(R)$ or $a + e \in J(R)$. In both cases we get a weak J-clean expression for a in R, so R is weak J-clean. □