Chapter 5

Nil clean graph of rings

5.1 Introduction

In this chapter we have introduced the nil clean graph $G_N(R)$ associated with a finite commutative ring R. The properties on girth, diameter, dominating sets etc. of $G_N(R)$ have been studied. The set of nil clean elements of a ring R is denoted by $NC(R)$.

5.2 Basic properties

In this section we defined the nil clean graph of a finite commutative ring and discuss its basic properties.

Definition 5.2.1. *The nil clean graph of a ring R, denoted by $G_N(R)$, is defined by setting R as vertex set and defining two distinct verities x and y to be adjacent if and only if $x + y$ is a nil clean element in R. Here we are not considering loops at a point (vertex) in the graph.*

[2]The contents of this chapter have been accepted for publication in *Algebra Colloquium* (2017)

For illustration below is the nil clean graph of $GF(25)$, where $GF(25)$ is the finite field with 25 elements.

$$\begin{aligned} GF(25) &\cong \mathbb{Z}_5[x]/\langle x^2+x+1\rangle \\ &= \{ax+b+\langle x^2+x+1\rangle : a,b\in\mathbb{Z}_5\}. \end{aligned}$$

Let us define $\alpha := x+\langle x^2+x+1\rangle$. Then we have

$$\begin{aligned} GF(25) = \{\ & \overline{0},\overline{1},\overline{2},\overline{3},\overline{4}, \\ & \alpha, 1+\alpha, 2+\alpha, 3+\alpha, 4+\alpha, \\ & 2\alpha, 1+2\alpha, 2+2\alpha, 3+2\alpha, 4+2\alpha, \\ & 3\alpha, 1+3\alpha, 2+3\alpha, 3+3\alpha, 4+3\alpha, \\ & 4\alpha, 1+4\alpha, 2+4\alpha, 3+4\alpha, 4+4\alpha\ \}. \end{aligned}$$

Observe that $NC(GF(25)) = \{\overline{0},\overline{1}\}$.

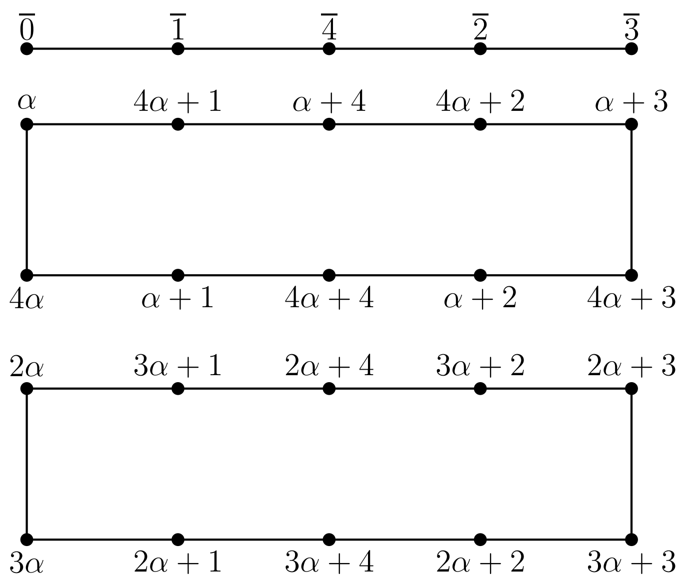


Figure 5.1: Nil clean graph of $GF(25)$

In graph theory, a *complete graph* is a simple undirected graph (with no loops and no multiple edges between two given vertices) in which every pair of distinct vertices is connected by a unique edge.

So by this definition the following theorem follows.

Theorem 5.2.2. *The nil clean graph $G_N(R)$ is a complete graph if and only if R is a nil clean ring.*

Proof. Let $G_N(R)$ be a complete nil clean graph of a ring R. For $r \in R$, r is adjacent to 0, so $r = r + 0$ is nil clean. Hence R is nil clean. The converse is clear from the definition of the nil clean graph. □

Two graphs G_1 and G_2 are said to be isomorphic if there exists an isomorphism from G_1 to G_2, i.e., a bijective mapping $f : V(G_1) \to V(G_2)$, such that two vertices u_1 and v_1 are adjacent in G_1 if and only if the vertices $f(u_1)$ and $f(v_1)$ are adjacent in G_2 [16]. For rings R and S if $R \cong S$ it is easy to see that $G_N(R) \cong G_N(S)$.

Lemma 5.2.3. *Let R be a ring and idempotents lift modulo $\mathrm{Nil}(R)$. If $x + \mathrm{Nil}(R)$ and $y + \mathrm{Nil}(R)$ are adjacent in $G_N(R/\mathrm{Nil}(R))$ then every element of $x + \mathrm{Nil}(R)$ is adjacent to every element of $y + \mathrm{Nil}(R)$ in the nil clean graph $G_N(R)$.*

Proof. Let $x + \mathrm{Nil}(R)$ and $y + \mathrm{Nil}(R)$ be adjacent in $G_N(R/\mathrm{Nil}(R))$. Then

$$(x + \mathrm{Nil}(R)) + (y + \mathrm{Nil}(R)) = e + \mathrm{Nil}(R),$$

where e is an idempotent in R, as idempotents lift modulo $\mathrm{Nil}(R)$. Thus we have $x + y = e + n$, for some $n \in \mathrm{Nil}(R)$ and hence x and y are adjacent in $G_N(R)$. Now for $a \in x + \mathrm{Nil}(R)$ and $b \in y + \mathrm{Nil}(R)$, we have $a = x + n_1$ and $b = y + n_2$, for some n_1, $n_2 \in \mathrm{Nil}(R)$. Therefore $a + b = e + (n - n_1 - n_2)$. Hence, a and b are adjacent in $G_N(R)$. □

Let G be a graph. For $x \in V(G)$, the degree of x, denoted by $deg(x)$, is defined to be the number of edges of G for which x is an end point. The neighbor set of $x \in V(G)$, is defined to be $N_G(x) := \{y \in V(G) | y \text{ is adjacent to } x\}$. Let $N_G[x] = N_G(x) \cup \{x\}$.

Lemma 5.2.4. *Let $G_N(R)$ be the nil clean graph of a ring R and let $x \in R$.*

(i) If $2x$ is nil clean, then $deg(x) = |NC(R)|-1$.

(ii) If $2x$ is not nil clean, then $deg(x) = |NC(R)|$.

Proof. Let $x \in R$. Observe that $x + R = R$. So for every $y \in NC(R)$, there exists a unique element $x_y \in R$, such that $x + x_y = y$. Thus we have

$$deg(x) \leq |NC(R)|.$$

Now if $2x \in NC(R)$, define

$$f : NC(R) \to N_{G_N(R)}[x]$$

by

$$f(y) = x_y.$$

It is easy to see that f is a bijection and therefore

$$deg(x) = |N_{G_N(R)}(x)| = |N_{G_N(R)}[x]|-1 = |NC(R)|-1.$$

If $2x \notin NC(R)$, define

$$f : NC(R) \to N_{G_N(R)}(x)$$

by

$$f(y) = x_y.$$

Then f is a bijection and therefore

$$deg(x) = |N_{G_N(R)}(x)| = |NC(R)|.$$

□

A graph G is said to be *connected* if for any two distinct vertices of G, there is a path in G connecting them.

Theorem 5.2.5. *For a ring R, the following hold:*

(i) $G_N(R)$ *need not be connected.*

(ii) *Let* $R = \mathbb{Z}_n$*. For* $\overline{a} \in \mathbb{Z}_n$ *there is a path from* $\overline{a}$ *to* $\overline{0}$*.*

(iii) $G_N(\mathbb{Z}_n)$ *is connected.*

(iv) *Let* $R = \mathbb{Z}_n$*. For* $A \in M_n(\mathbb{Z}_n)$ *there is a path from* A *to* 0*, where* 0 *is the zero matrix of* $M_n(\mathbb{Z}_n)$*.*

(v) $G_N(M_n(\mathbb{Z}_n))$ *is connected.*

Proof. (i) is clear by the graph $G_N(GF(25))$, figure 5.1. For (ii) and (iii) if n is odd replacing p by n in the figure 5.5, we get a Hamiltonian path in $G_N(\mathbb{Z}_n)$; if n is even the following is a Hamiltonian path in $G_N(\mathbb{Z}_n)$.

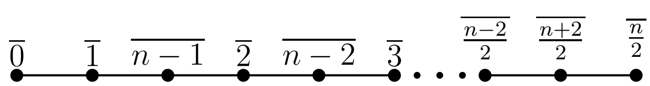


Figure 5.2: Hamiltonian path in $G_N(\mathbb{Z}_n)$, when n is an even natural number.

Now for the proof of (iv), let $A = [a_{ij}] \in M_n(\mathbb{Z}_n)$, now we define

$$A_1 = [a1_{ij}] = \begin{cases} -a_{ij}, & i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $A_1 + A$ is nilpotent, hence nil clean. Thus there exists an edge between A and A_1. Again define

$$A_2 = [a2_{ij}] = \begin{cases} a_{ij}, & i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have an edge between A_1 and A_2 in $G_N(M_n(\mathbb{Z}_n))$. For each element a_{ii} of A_2, by (ii) we have a path

$$\{a_{ii}, b_{i1}, b_{i2}, b_{i3}, \ldots, b_{ik_i} = \overline{0}\}$$

of length $k_i \in \mathbb{N}$ to $\overline{0}$. Now let

$$K = \max\{k_i : 1 \leq i\},$$

and we can construct a path of length K from A_2 to 0, as follows. For $1 \leq i \leq k$, define

$$B_i = [b1_{jl}] = \begin{cases} b_{ji}, & \text{if } b1_{ij} \text{ appears in some above paths;} \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\{A, A_1, A_2, B_1, B_2, \ldots, B_K = 0\}$$

is a path from A to 0 in $G_N(M_n(\mathbb{Z}_n))$. Lastly (v) follows from (iv). □

The following result is a corollary of the Wedderburn's Theorem [32].

Lemma 5.2.6. *A ring R is a finite commutative reduced ring with no non trivial idempotents if and only if R is a finite field.*

Proof.(⇒) Let $0 \neq x \in R$. Observe the set $A = \{x^k : k \in \mathbb{N}\}$ is a finite set. Therefore there exist $m > l$ such that $x^l = x^m$. Note that

$$\begin{aligned} x^l &= x^m \\ &= x^{m-l+l} \\ &= x^{m-l}.x^l \\ &= x^{m-l}.x^m \\ &= x^{2m-l+l-l} \\ &= x^{2(m-l)+l} \\ &= \vdots \\ &= x^{k(m-l)+l}. \end{aligned}$$

Where k is a natural number. Now we have

$$\begin{aligned}[x^{l(m-l)}]^2 &= x^{l(m-l)}.x^{l(m-l)} \\ &= x^{l(m-l)+l(m-l)+l-l} \\ &= x^{l(m-l)+l}.x^{l(m-l)-l} \\ &= x^{l}.x^{l(m-l)-l} \\ &= x^{l(m-l)},\end{aligned}$$

that is $x^{l(m-l)}$ is an idempotent. Thus $x^{l(m-l)} = 1$, which gives that x is a unit, therefore R is a finite field. ($\Leftarrow$) Obvious. $\square$

5.3 Invariants of a nil clean graph

In this section, we prove some results related to several invariants of a nil clean graph. Following subsection is for girth of $G_N(R)$.

5.3.1 Girth of $G_N(R)$

For a graph G, the *girth* of G is the length of the shortest cycle in G.

Theorem 5.3.1. *The following hold for the nil clean graph $G_N(R)$ of R:*

(i) If R is not a field, then the girth of $G_N(R)$ is equal to 3.

(ii) Suppose that R is a field.

(a) The girth of $G_N(R)$ is $2p$ if $R \cong GF(p^k)$ (field of order p^k), where p is a odd prime and $k > 1$.

(b) The girth of $G_N(R)$ is infinite, in fact $G_N(R)$ is a path, otherwise.

Proof. (i) Let R have at least one non-trivial idempotent or non trivial nilpotent. If $e \in R$ is a nontrivial idempotent, then we have

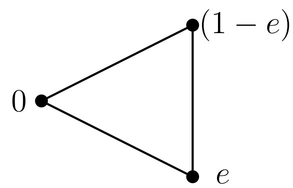


Figure 5.3: A cycle of length 3 in $G_N(R)$ for an idempotent e

so the girth of $G_N(R)$ is 3. Again if R contains a nontrivial nilpotent $n \in R$, then we have the cycle

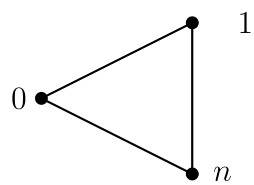


Figure 5.4: A cycle of length 3 in $G_N(R)$ for a nilpotent e

so the girth is 3. By **Lemma 5.2.6** rings without non trivial idempotents and nilpotents are field. This proves (i).

(ii) The set of nil clean elements of a finite field is $\{0, 1\}$, so the nil clean graph of $\mathbb{F}_p$, where p is a prime, is

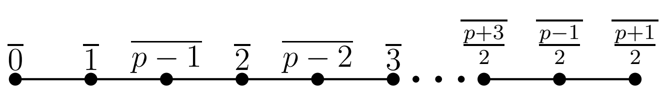


Figure 5.5: Nil clean graph of $\mathbb{Z}_p$

From the graph, it is clear that the grith of $G_N(\mathbb{F}_p)$ is infinite, which proves (b). It is easy to observe that the nil clean graph of $GF(p^k)$ for $p > 2$, is a disconnected graph consisting of a path of length p and $(\frac{p^{k-1}-1}{2})$ $2p-$cycles. For the proof, let

$$GF(p^k) = Z_p[X]/\langle f(x)\rangle,$$

where $f(x)$ is an irreducible polynomial of degree k over $\mathbb{Z}_p$. Let $A \subseteq GF(p^k)$, such that A consists of all linear combinations of $x, x^2, \ldots, x^{k-1}$ with coefficients from $\mathbb{Z}_p$ such that if $g(x) + \langle f(x)\rangle \in A$ then $-g(x) + \langle f(x)\rangle \notin A$. Clearly, A can be written as

$$A = \{g_i(x) + \langle f(x)\rangle \mid 1 \leq i \leq (\frac{p^{k-1}-1}{2})\}.$$

Let $\overline{g_i(x)} = g_i(x) + \langle f(x)\rangle$ for $1 \leq i \leq (\frac{p^{k-1}-1}{2})$. So (a) follows from
Figure 5.6.

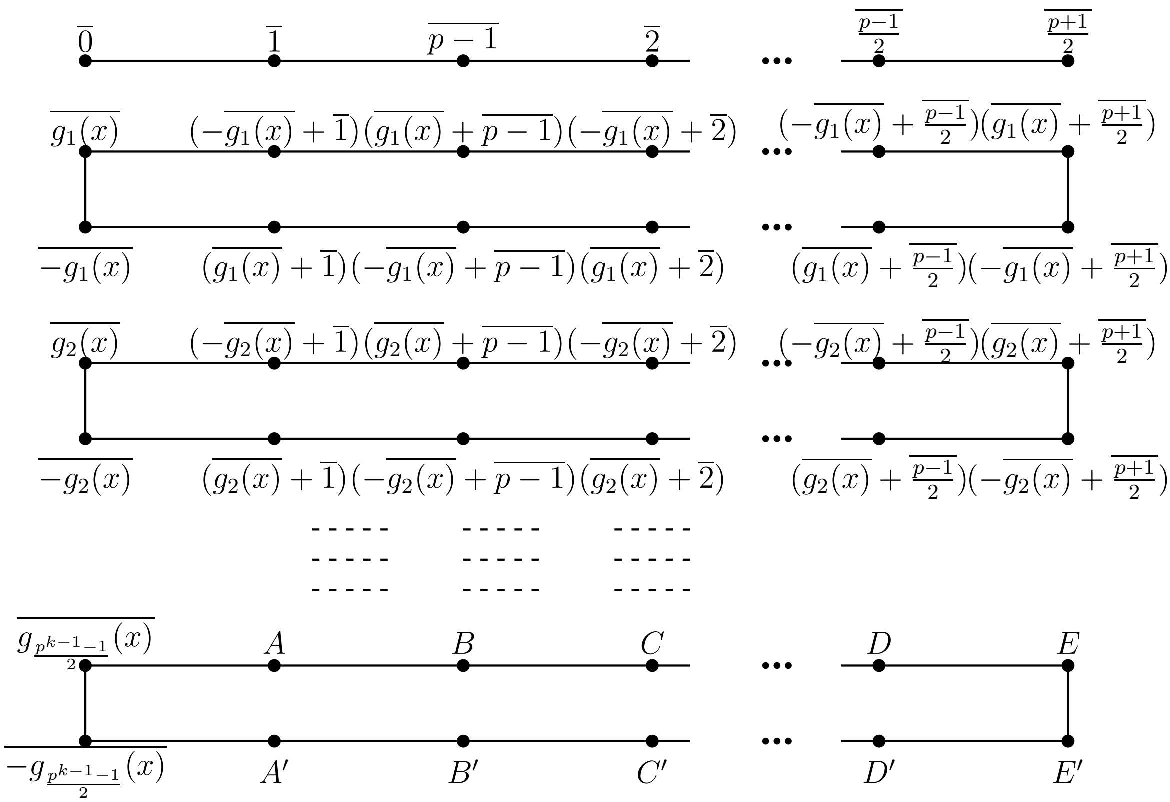


Figure 5.6: Nil clean graph of $GF(p^k)$

Here, $A = -\overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{1}$, $B = -\overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{p-1}$, $C = \overline{-g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{2}$,

$D = \overline{-g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p-1}{2}}$, $E = -\overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p+1}{2}}$,

$A' = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{1}$, $B' = -\overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{p-1}$, $C' = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{2}$,

$D' = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p-1}{2}}$, $E' = -\overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p+1}{2}}$. □

Corollary 5.3.2. *$G_N(R)$ is not cyclic.*

A graph G is said to be bipartite if its vertex set can be partitioned into two disjoint subsets V_1 and V_2, such that $V(G) = V_1 \cup V_2$ and every edge in G has the form $e = (x, y) \in E(G)$, where $x \in V_1$ and $y \in V_2$. Note that no two vertices both in V_1 or both in V_2 are adjacent.

Theorem 5.3.3. *$G_N(R)$ is bipartite if and only if R is a field.*

Proof. Let $G_N(R)$ be bipartite. Therefore the girth is of $G_N(R)$ is not an odd number. Hence by **Theorem 5.3.1**. R must be a field. Now if R is a field, it is clear from the nil clean graph of R that $G_N(R)$ is bipartite. □

5.3.2 Dominating set

Let G be a graph. A subset $S \subseteq V(G)$ is said to be a dominating set for G if for each $x \in V(G)$, $x \in S$ or there exists $y \in S$ such that x is adjacent to y. We show that for a finite commutative weak nil clean ring, the dominating number is 2, where the dominating number is the carnality of the smallest dominating set.

Theorem 5.3.4. *Let R be a weak nil clean ring such that R has no non trivial idempotents. Then $\{1, 2\}$ is a dominating set for $G_N(R)$.*

Proof. Let $a \in R$. For some $n \in \mathrm{Nil}(R)$, a is in one of following forms

$$a = \begin{cases} n + 0 & ; \\ n + 1 & ; \\ n - 1 & . \end{cases}$$

Now if $a = n$, then $n + 1 \in \mathrm{NC}(R)$ implies a is adjacent to 1.
If $a = n - 1$, then $n - 1 + 1 = n \in \mathrm{NC}(R)$ implies a is adjacent to 1.
If $a = n + 1$ and $2 = n_1$ for some nilpotent $n_1 \in R$, then a is adjacent to 2.
If $a = n + 1$ and $2 = n_1 - 1$ for some nilpotent $n_1 \in R$, then a is adjacent to 2.
If $a = n+1$ and $2 = n_1+1$ for some nilpotent $n_1 \in R$, then $a+2 = (n+1)+(n_1+1) = (n + n_1) + 2 = n + n_1 + (n_1 + 1) = (n + 2n_1) + 1$ is nil clean. Hence a is adjacent to 2. Thus $\{1, 2\}$ is a dominating set for $G_N(R)$. □

Theorem 5.3.5. *Let $R = A \times B$, such that A is nil clean and B is weak nil clean with no non trivial idempotents. Then $\{(1_A, 1_B), (2_A, 2_B)\}$ is a dominating set for $G_N(R)$.*

Proof. Let $(a, b) \in R$, where $a \in A$ and $b \in B$. For $n_1 \in \text{Nil}(A)$, $n_2 \in \text{Nil}(B)$ and $0 \neq e \in \text{Idem}(A)$, (a, b) has one of the following forms

$$(a, b) = \begin{cases} (n_1, n_2) + (e, 1_B) & ; \\ (n_1, n_2) + (e, 0) & ; \\ (n_1, n_2) - (e, 1_B) & ; \\ (n_1, n_2) + (0, 0) & . \end{cases}$$

If $(a, b) = (n_1, n_2) + (e, 1_B)$, we have

$$(a, b) + (2_A, 2_B) = (n_1 + e + 2_A, n_2 + 1_B + 2_B).$$

Since A is nil clean,

$$n_1 + e + 2_A = n_1' + f \text{ for some } n_1' \in \text{Nil}(A) \text{ and } f \in \text{Idem}(A).$$

As B is weak nil clean, we have

$$2_B = n_2', \text{ or } n_2' - 1_B, \text{ for some } n_2' \in \text{Nil}(B).$$

If $2_B = n_2'$, we have

$$(a, b) + (2_A, 2_B) = (n_1', n_2 + n_2') + (f, 1_B),$$

which is a nil clean expression; hence (a, b) is adjacent to $(2_A, 2_B)$.
If $2_B = n_2' - 1_B$, we have

$$(a, b) + (2_A, 2_B) = (n_1', n_2 + n_2') + (f, 0),$$

so (a, b) is adjacent to $(2_A, 2_B)$. In other three cases it is easy to see that $(a, b) + (1_A, 1_B)$ is nil clean hence (a, b) is adjacent to $(1_A, 1_B)$. Therefore $\{(1_A, 1_B), (2_A, 2_B)\}$ is a dominating set for $G_N(R)$. ∎

Theorem 5.3.6. *Let R be a weak nil clean ring. The $\{1, 2\}$ is a dominating set for $G_N(R)$.*

Proof. If R has no non trivial idempotents, then by **Theorem 5.3.4** we are done. If R has a non trivial idempotent, say e, then $R = eR \oplus (1-e)R$. By **Theorem 2.3** of [9], either eR or $(1-e)R$ is a nil clean ring. Without lost of generality suppose that eR is a nil clean ring and $(1-e)R$ is a weak nil clean ring. Now if $(1-e)R$ has no non trivial idempotents, then we have the result by **Theorem 5.3.5**. Otherwise, repeating as above we get a direct sum decomposition of R where only one summand is weak nil clean. As R is a finite ring, after repeating above finite number of times we will have a direct sum decomposition of R, where idempotents of weak nil clean summand of R is trivial. Then again by **Theorem 5.3.5** we have the result. □

5.3.3 Chromatic index

An edge colouring of a graph G is a map $C : E(G) \to S$, where S is a set of colours such that for all $e, f \in E(G)$, if e and f are adjacent, then $C(e) \neq C(f)$. The *chromatic index* of a graph, denoted by $\chi'(G)$; is defined as the minimum number of colours needed for a proper colouring of G. Let $\triangle$ be the maximum vertex degree of G. Vizings theorem [24] gives $\triangle \leq \chi'(G) \leq \triangle + 1$. Vizings theorem divides the graphs into two classes according to their chromatic index; graphs satisfying $\chi'(G) = \triangle$ are called graphs of class 1, and those with $\chi'(G) = \triangle + 1$ are graphs of class 2.

We show that $G_N(R)$ is of class 1.

Theorem 5.3.7. *Let R be a finite commutative ring. Then the nil clean graph of R is of class* 1*.*

Proof. We colour the edge ab by the colour $a+b$. By this colouring, every two distinct edges ab and ac have different colours and

$$C = \{a+b | ab \text{ is an edge in } G_N(R)\}$$

is the set of colours. Therefore the nil clean graph has a $|C|$-edge colouring and so

$$\chi'(G_N(R)) \leq |C|.$$

But $C \subset NC(R)$ and

$$\chi'(G_N(R)) \leq |C| \leq |NC(R)|.$$

By **Lemma 5.2.4**, $\triangle \leq |NC(R)|$, and so by Vizing's theorem, we have

$$\chi'(G_N(R)) \geq \triangle = |NC(R|.$$

Therefore we have

$$\chi'(G_N(R)) = |NC(R| = \triangle,$$

i.e., $G_N(R)$ is of class 1. □

5.3.4 Diameter

For a graph G, the number of edges on the shortest path between vertices x and y is called the *distance* between x and y and is denoted by $d(x,y)$. If there is no path between x and y then we say $d(x,y)=\infty$. The diameter of a graph $diam(G)$ is the maximum of distances of each pair of distinct vertices in G.

The following are some results related to diameter of the nil clean graph of a ring.

Lemma 5.3.8. *R is a nil clean ring if and only if $diam(G_N(R)) = 1$.*

Theorem 5.3.9. *Let R be a non nil clean, weak nil clean ring with no non trivial idempotents. Then $diam(G_N(R)) = 2$.*

Proof. Let $a, b \in R$ and let $n_1, n_2 \in \text{Nil}(R)$. Since R is a weak nil clean ring with no non trivial idempotents, so for any $r \in R$ there exists $n \in \text{Nil}(R)$ such that r is in one of following forms

$$r = \begin{cases} n + 0 & , \\ n + 1 & , \\ n - 1 & . \end{cases}$$

If $a = n_1 + 1$ and $b = n_2 - 1$ or n_2, then $a + b$ is nil clean. Thus ab is an edge in $G_N(R)$, so $d(a, b) = 1$.
If $a = n_1 + 1$ and $b = n_2 + 1$, we have the path

$$a \;—\!—\; (-1) \;—\!—\; b$$

in $G_N(R)$, so $d(a, b) \leq 2$.
If $a = n_1 - 1$ and $b = n_2 - 1$ or n_2. Then, as above, there is a path of length 2 from a to b through 1. So in this case also $d(a, b) \leq 2$. Finally if $a = n_1$ and $b = n_2$, then $d(a, b) = 2$. Thus from above we conclude that

$$diam(G_N(R)) \leq 2.$$

Now as R is a non nil clean, weak nil clean ring, we have at least one $x \in R$, such that $x = n - 1$ but $x \neq n + 1$, i.e., x is not nil clean. Thus we have $d(0, x) \geq 2$, so

$$diam(G_N(R)) \geq 2.$$

Hence the result follows. □

Theorem 5.3.10. *Let $R = A \times B$, such that A is nil clean and B weak nil clean with no non trivial idempotents. Then $diam(G_N(R)) = 2$.*

Proof. We have $\text{Idem}(R) = \{(e, 0_B), (e, 1_B) | e \in \text{Idem}(A)\}$. Now let $(a_1, b_1), (a_2, b_2) \in R$. In case $(a_1, b_1) + (a_2, b_2)$ is nil clean then $d((a_1, b_1), (a_2, b_2)) = 1$ in $G_N(R)$. If

$(a_1, b_1) + (a_2, b_2)$ is not nil clean, then $b_1 + b_2$ is not nil clean. So we have following cases:

Case I:

If $b_1 = n_1 + 1$ and $b_2 = n_2 + 1$, we have the path

$$(a_1, b_1) \text{ —— } (0, -1) \text{ —— } (a_2, b_2)$$

in $G_N(R)$, thus $d((a_1, b_1), (a_2, b_2)) \leq 2$.

Case II:

If $b_1 = n_1 - 1$ and $b_2 = n_2 - 1$, we have the path

$$(a_1, b_1) \text{ —— } (0, 1) \text{ —— } (a_2, b_2)$$

in $G_N(R)$, thus $d((a_1, b_1), (a_2, b_2)) \leq 2$.

Case III:

If $b_1 = n_1 - 1$ and $b_2 = n_2$, we have the path

$$(a_1, b_1) \text{ — } (0, 1) \text{ — } (a_2, b_2)$$

in $G_N(R)$, thus $d((a_1, b_1), (a_2, b_2)) \leq 2$.

Case IV:

If $b_1 = n_1$ and $b_2 = n_2 - 1$, by **Case III** and symmetry we have $d((a_1, b_1), (a_2, b_2)) \leq 2$.

Therefore combining above Cases, we have

$$diam(R) \leq 2.$$

But, R not nil clean implies

$$diam(R) \geq 2.$$

Thus $diam(R) = 2$. □

Theorem 5.3.11. *If R is weak nil clean but not nil clean then diam$(G_N(R)) = 2$.*

Proof. If R has no non trivial idempotents, then by **Theorem 5.3.10** we are done. If R has a non trivial idempotent say e, then $R = eR \oplus (1-e)R$. So by **Theorem 2.3** of [9] we have one of eR or $(1-e)R$ must be a nil clean ring. Without lost of generality suppose that eR is a nil clean ring and $(1-e)R$ is a weak nil clean ring. Now if $(1-e)R$ has no non trivial idempotents, then we have the result by **Theorem 5.3.10**. Otherwise repeating as above we get a direct sum decomposition of R where only one summand is weak nil clean. As R is a finite ring, so after repeating above to the weak nil clean summand of R for finite number of times, we will have a direct sum decomposition of R, where idempotents of weak nil clean summand of R are trivial. Thus again by **Theorem 5.3.10**, we have the result. □

Theorem 5.3.12. *Let n be a positive integer. The following hold for $\mathbb{Z}_n$:*

(i) If $n = 2^k$, for some integer $k \geq 1$, then diam$(G_N(\mathbb{Z}_n)) = 1$.

(ii) If $n = 2^k 3^l$, for some integer $k \geq 0$ and $l \geq 1$, then diam$(G_N(\mathbb{Z}_n)) = 2$.

(iii) For a prime p, diam$(G_N(\mathbb{Z}_p)) = p - 1$.

(iv) If $n = 2p$, where p is an odd prime, then diam$(G_N(\mathbb{Z}_{2p})) = p - 1$.

(v) If $n = 3p$, where p is an odd prime, then diam$(G_N(\mathbb{Z}_{3p})) = p - 1$.

Proof. (i) and (ii) follow from **Lemma 5.3.8** and **Theorem 5.3.11** respectively. (iii) follows from $G_N(\mathbb{Z}_p)$ in Figure 5.5, (iv) follows from graph $G_N(\mathbb{Z}_{2p})$ in Figure 5.7 and (v) follows from graphs in Figure 5.9 and Figure 5.8.

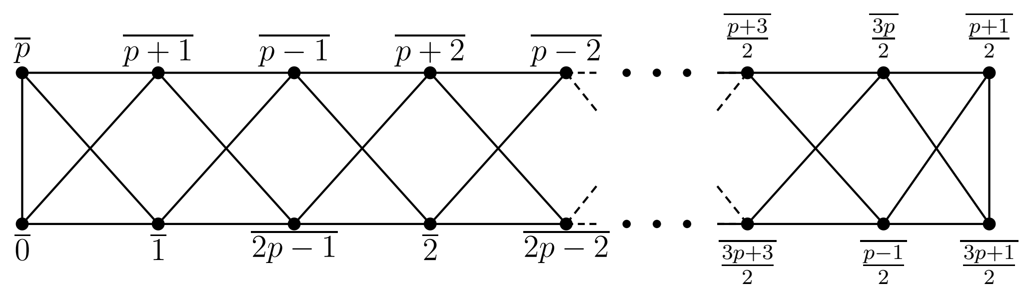
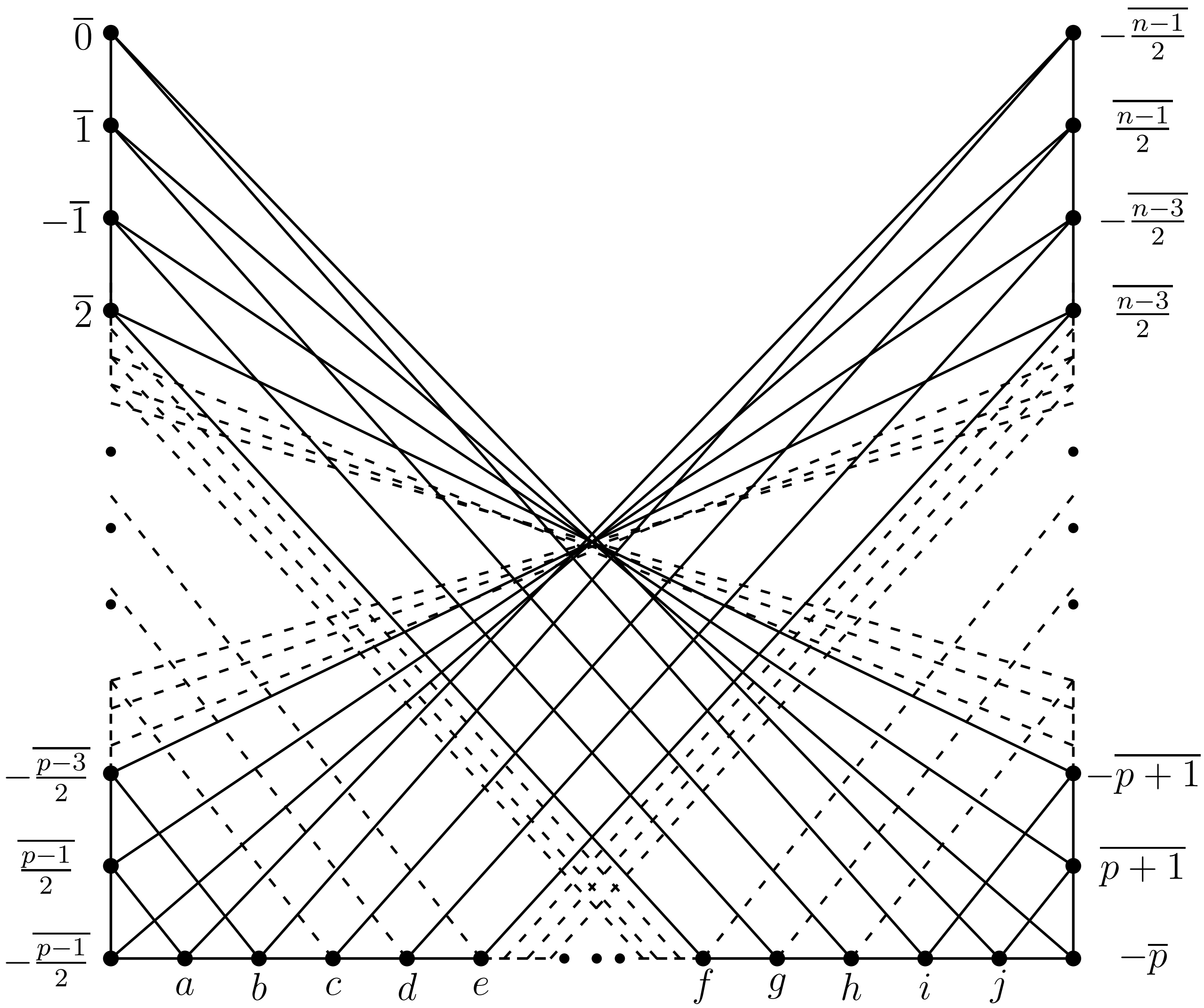
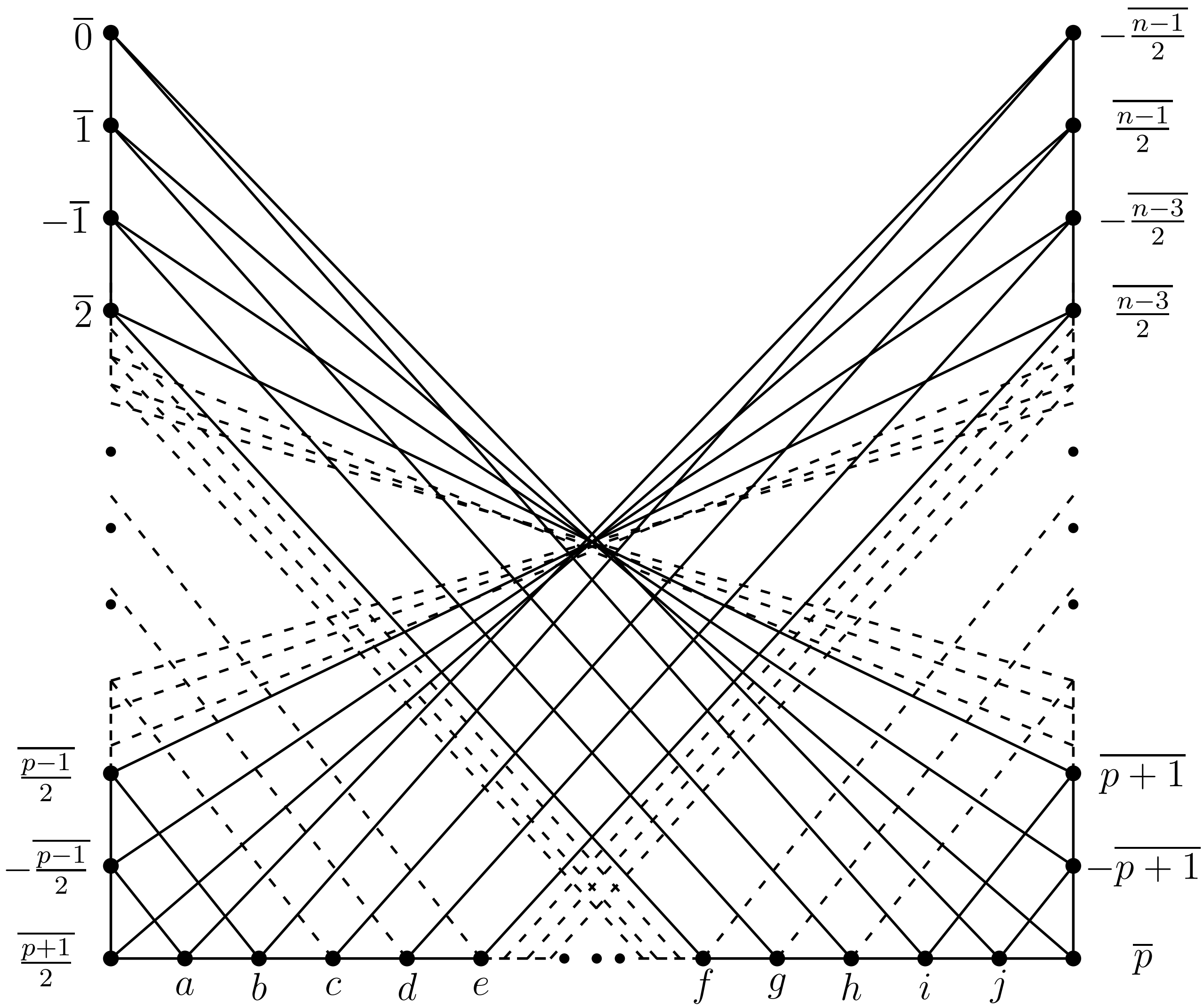


Figure 5.7: Nil clean graph of $\mathbb{Z}_{2p}$



$$a = \overline{\frac{p+1}{2}}, b = -\overline{\frac{p+1}{2}}, c = \overline{\frac{p+3}{2}}, d = -\overline{\frac{p+3}{2}}, e = \overline{\frac{p+5}{2}},$$
$$j = \overline{p}, i = -\overline{p-1}, h = \overline{p-1}, g = -\overline{p+2}, f = \overline{p-2}$$

Figure 5.8: Nil clean graph of $\mathbb{Z}_{3p}$ where $p \equiv 1(mod 3)$



$$a = -\overline{\tfrac{p+1}{2}}, b = \overline{\tfrac{p+3}{2}}, c = -\overline{\tfrac{p+3}{2}}, d = \overline{\tfrac{p+5}{2}}, e = -\overline{\tfrac{p+5}{2}},$$
$$j = -\overline{p}, i = \overline{p-1}, h = -\overline{p-1}, g = \overline{p+2}, f = -\overline{p-2}$$

Figure 5.9: Nil clean graph of $\mathbb{Z}_{3p}$ where $p \equiv 2(mod 3)$