Chapter 1

Introduction

Our thesis is comprised of five chapters including this introductory chapter. In this chapter, we have given the general introduction of the field of study, which is followed by the brief history of development of the subject. We have ended the chapter with a few important definitions and results of ring theory used in the thesis.

1.1 General introduction

Ring theory is a vast area in Algebra. Mathematicians have always tried to understand rings by defining a subclass, for example commutative rings and non commutative rings. Exchange rings are rings which satisfy exchange property. Clean ring was introduced by W. K. Nicholson in 1977, as a subclass of exchange(suitable) rings [36]. Since then the study on class of clean rings, its subclass and its generalization have become an area of interest for many mathematicians. Our work is mainly on a subclass of the class of clean rings, called nil clean rings, introduced by A. J. Diesl [23] and its generalizations.

1.2 A brief historical overview

The class of clean rings was introduced by W. K. Nicholson [36] in 1977 as a subclass of suitable rings or exchange rings which opened a new area of study for rings. By Nicholson [39], the endomorphism ring of a module is an exchange ring if and only if the module satisfies the finite exchange property. A module is defined to be clean if its endomorphism ring is clean. Han and Nicholson [27] in 2001 proved that direct sum of two clean modules is clean. Further, the authors in [15] proved that continuous modules are clean. Lam proved that all semiperfect rings are clean and Camillo and Khurana proved that the unit regular rings are clean (see [13, 29]). The notion of uniquely clean rings was defined by Anderson and Camillo [3] in the commutative case as those rings in which every element is uniquely the sum of a unit and an idempotent. Later on, Nicholson and Zhou [37] studied arbitrary uniquely clean rings. Ahn and Anderson [2] introduced the class of weakly clean rings, which includes clean rings as a subclass. Diesl [23] introduced nil clean rings and strongly nil clean rings as a subclass of clean rings and also developed a general theory based on idempotents and direct sum decompositions of modules, that unifies several of these existing concepts such as clean rings, $\pi-$regular rings and regular rings. Further, Danchev and McGovern [21] studied commutative weakly nil clean rings in the line of what Ahn and Anderson [2] did for clean rings.

In ring theory, defining a graph for a ring is not a new thing, in the past many mathematicians have studied graph theoretic properties of graphs associated with rings. In 1988, Istavan Beck studied colouring of graph of a finite commutative ring R, where vertices are elements of R and xy is an edge if and only if $xy = 0$ (see [12]). For a positive integer n, let $\mathbb{Z}_n$ be the ring of integers modulo n. P. Grimaldi [26] defined and studied various properties of the unit graph $G(\mathbb{Z}_n)$, with vertex set $\mathbb{Z}_n$ and two distinct vertices x and y are adjacent if $x + y$ is a unit. Further, Ashrafi, Maimani, Pournaki and Yassemi generalized $G(\mathbb{Z}_n)$ to unit graph $G(R)$, where R is an arbitrary associative ring with non zero identity (see [5]). They have studied properties like diameter, girth, chromatic index, etc. For more work on graphs associated with rings, we refer to [1, 5, 12, 38]. We have introduced the nil clean graph $G_N(R)$ associated with a finite commutative ring R.

1.3 Preliminaries

Throughout this thesis, rings R are associative rings with unity unless otherwise indicated and modules are unitary. The Jacobson radical, group of units, set of idempotents and set of nilpotent elements of a ring R are denoted by $\mathrm{J}(R), \mathrm{U}(R), \mathrm{Idem}(R)$ and $\mathrm{Nil}(R)$ respectively. Cyclic group of order m is denoted by C_m.

In the following subsections, we have stated definitions and a few basic results needed in the thesis. We begin with the definition of a formal matrix ring.

1.3.1 Formal matrix rings

Let there be given two rings A and B, an $A-B-$bimodule M, a $B-A-$bimodule N, bimodule homomorphisms

$$\phi : M \otimes_B N \longrightarrow A,$$

$$\psi : N \otimes_A M \longrightarrow B$$

for which the following associativity equalities hold:

$$\phi(m,n)m' = m\psi(n,m')$$

and

$$\psi(n,m)n' = n\phi(m,n')$$

for all $m, m' \in M$ and $n, n' \in N$. In what follows we will use the following notations: $\phi(m,n) = mn$ and $\psi(n,m) = nm$ for all $m \in M, n \in N$. A direct verification shows that the set of matrices of the form

$$R = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} \mid a \in A, b \in B, m \in M, n \in N \right\}$$

with matrix operations of addition and multiplication is a ring. The ring R is called a *ring of formal matrices* and is denoted by $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$. The ordered collection (A, B, M, N, ϕ, ψ) is called a Morita context or a situation of pre-equivalence. If $N = 0$, we have $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, and R is called *a formal triangular matrix ring.*

1.3.2 Some properties of idempotents

Peirce decomposition theorem

American algebraist Benjamin O. Peirce observed following important facts about idempotents of an arbitrary ring. These results are known as Peirce decomposition of a ring.

Proposition 1.3.1. *let R be a ring and $e = e^2 \in R$. If $f = 1 - e$, then the following are true*

(i) $R = Re \oplus Rf$.

(ii) $R = eR \oplus fR$.

(iii) $R = eRe \oplus eRf \oplus fRe \oplus fRf$.

The formal matrix version of Peirce decomposition (iii) is stated in the following theorem. This theorem will be used in the succeeding chapters.

Theorem 1.3.2. *Let R be a ring and $e \in \mathrm{Idem}(R)$. Then $R = \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix}$, where $f = 1 - e$.*

Lifting of idempotents

Let I be an ideal of a ring R. We say idempotent $\bar{a} \in R/I$ can be lifted to R if there exists an idempotent $e \in R$ whose image under natural map $R \longrightarrow R/I$ is $\bar{a}$.

1.3.3 Clean ring

Definition 1.3.3. *Let r be an element of a ring R. Then r is said to be a clean element of R, if r can be expressed as $r = e + u$, where $e \in \mathrm{Idem}(R)$ and $u \in \mathrm{U}(R)$. Further if e and u commute we say r is strongly clean. A ring R is called a clean ring (respectively strongly clean ring), if each $r \in R$ is clean (respectively strongly clean).*

It is well known that every ring can be represented as an endomorphism ring of an appropriate module. Here we present a characterization of a strongly clean endomorphism as documented and proved in [34], followed by a characterization of a clean endomorphism from [15].

Lemma 1.3.4. *For a ring R, let $End(M_R)$ be the ring of endomorphisms of a right R-module M_R. An element $f \in End(M_R)$ is strongly clean if and only if there exists a direct sum decomposition $M = A \oplus B$, where A and B are right R-submodules of M_R such that A and B are f-invariant and both $f : A \to A$ and $1 - f : B \to B$ are isomorphisms.*

Above decomposition of module M is known as the $ABAB$ decomposition for M. Above Lemma can be represented by the following diagram.

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & f \downarrow \cong & & (1-f) \downarrow \cong \\ M & = & A & \oplus & B \end{array}$$

Lemma 1.3.5. *For a ring R, let $End(M_R)$ be the ring of endomorphisms of a right R-module M_R. An element $f \in End(M_R)$ is clean if and only if there exist direct sum decompositions $M = A \oplus B$ and $M = C \oplus D$, where A, B, C and D are right R-submodules of M_R such that $f(A) \subseteq C$ and $(1 - f)(B) \subseteq D$ and both $f : A \to C$ and $1 - f : B \to D$ are isomorphisms.*

The $ABAB$ decomposition for M in the form of diagram is

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & f \downarrow \cong & & (1-f) \downarrow \cong \\ M & = & C & \oplus & D \end{array}$$

1.3.4 Clean index of a ring

Lee and Zhou [30, 31], defined clean index of a ring and characterized rings of clean index 1, 2, 3, 4. They also characterized Abelian rings with finite index.

Definition 1.3.6. *For $a \in R$, let $\xi(a) = \{e \in R : e^2 = e, a - e \in U(R)\}$. The clean index of R, denoted by $\mathrm{In}(R)$, is defined as*

$$\mathrm{In}(R) = \sup\{|\xi(a)| : a \in R\}.$$

Motivated by this concept, we have introduced nil clean index of rings in Chapter 2 of our thesis.

1.3.5 Nil clean rings

Diesl[23], introduced nil clean rings and strongly nil clean rings as a subclass of clean rings.

Definition 1.3.7. *Let r be an element of a ring R. Then r is said to be a nil clean element of R, if r can be expressed as $r = e + n$, where $e \in \mathrm{Idem}(R)$ and $n \in \mathrm{Nil}(R)$. Further if e and n commute, we say r is a strongly nil clean. The ring R is said to be a nil clean ring, (respectively a strongly nil clean ring) if each $r \in R$ is nil clean (respectively strongly nil clean).*

Here is a characterization of a nil clean endomorphism of a module from [23].

Lemma 1.3.8. *For a ring R, let $End(M_R)$ be the ring of endomorphisms of a right R-module M_R. An element $f \in End(M_R)$ is strongly nil clean if and only if there exists a direct sum decomposition $M = A \oplus B$, where A and B are right R-submodules of M_R such that A and B are f-invariant and both $f : A \to A$ and $1 - f : B \to B$ are nilpotents.*

The Lemma can be represented by the following diagram.

$$\begin{array}{ccccc} M = & A & \oplus & B \\ & f \downarrow nilpotent & & (1-f) \downarrow nilpotent \\ M = & A & \oplus & B \end{array}$$

For further development in the study of nil clean rings, we refer to [19, 20, 22, 23, 27].

1.3.6 Weakly clean ring

Ahn and Anderson[2] introduced the class of weakly clean rings, in such a way that the class of clean rings is a subclass of it.

Definition 1.3.9. *A ring R is said to be a weakly clean ring, if each element $r \in R$ can be written as $r = u + e$ or $r = u - e$, for some $u \in \mathrm{U}(R)$ and $e \in \mathrm{Idem}(R)$. For a nonempty subset $S \subset \mathrm{Idem}(R)$, R is said to be $S-$weakly clean if each $x \in R$ can be written in the form $x = u + e$ or $x = u - e$ where $u \in U(R)$ and $e \in S$.*

On the notion of weakly clean rings we have introduced the concept of weak clean index of a ring.

Other general ring theory terms can be found in [29].