Chapter 2

Nil clean index of rings

2.1 Introduction

As mentioned in the introductory chapter, in this chapter we have introduced the notion of nil clean index of a ring and characterized arbitrary rings with nil clean index 1 and 2. Also a few results for rings with indices 3 and 4 are listed.

For an element $a \in R$, if $a - e \in \text{Nil}(R)$ for some $e^2 = e \in R$, then $a = e + (a - e)$ is called a nil clean expression of a in R and a is called a nil clean element. The ring R is called nil clean if each of its elements is nil clean. A ring R is uniquely nil clean if every element of R has a unique nil clean expression in R. For an element a of R, we denote

$$\eta(a) = \{e \in R \mid e^2 = e \text{ and } a - e \in \text{Nil}(R)\}$$

and the nil clean index of R, denoted by $\text{Nin}(R)$ is defined as

$$\text{Nin}(R) = \sup\{\mid \eta(a) \mid : a \in R\}$$

where $\mid \eta(a) \mid$ denotes the cardinality of the set $\eta(a)$. Thus R is uniquely nil clean if and only if R is a nil clean ring of nil clean index 1.

[1]The contents of this chapter have been published in *International Electronic Journal of Algebra* (2014).

2.2 Elementary properties

Some basic properties related to nil clean index are presented here as a preparation for the study on nil clean index of a ring.

Lemma 2.2.1. *Let R be a ring and let $e, a, b \in R$. Then the following hold:*

(i) If $e \in R$ is a central idempotent or a central nilpotent element, then $\mid \eta(e) \mid = 1$, so $Nin(R) \geq 1$.

(ii) $e \in \eta(a)$ iff $1 - e \in \eta(1 - a)$ and so $\mid \eta(a) \mid = \mid \eta(1 - a) \mid$.

(iii) If $f : R \to R$ is a homomorphism, then $e \in \eta(a)$ implies $f(e) \in \eta(f(a))$ and for the converse f must be a monomorphism.

(iv) If R has at most n idempotents or at most n nilpotent elements, then $Nin(R) \leq n$.

Proof. (i) Let e be a central idempotent. So we have $e = e+0$, a nil clean expression of e. If possible let $e = a + n$ be another nil clean expression of e in R, where $a \in \text{idem}(R)$, $n \in \text{nil}(R)$ and $n^k = 0$ for some positive integer k. Then $(e - a)^{2k-1} = 0$, which gives

$$e^{2k-1} - \binom{2k-1}{1} e^{2k-2} a + \cdots + \binom{2k-1}{2k-2} (-1)^{2k-2} e a^{2k-2} + (-1)^{2k-1} a^{2k-1} = 0,$$

so $(e + (-1)^{2k-1} a) - \left\{ \binom{2k-1}{1} - \binom{2k-1}{2} + \cdots + (-1)^{(2k-3)} \binom{2k-1}{2k-2} \right\} ea = 0.$

Using elementary result of binomial coefficients, we get

$$(e - a) - (1 + (-1)^{2k-3}) ea = 0.$$

Hence $e = a$, i.e, $\mid \eta(e) \mid = 1$.

(ii)

$$
\begin{aligned}
& e \in \eta(a), \\
\Leftrightarrow \quad & a - e \text{ is nilpotent}, \\
\Leftrightarrow \quad & e - a \text{ is nilpotent}, \\
\Leftrightarrow \quad & (1-a) - (1-e) \text{ is nilpotent}, \\
\Leftrightarrow \quad & 1 - e \in \eta(1-a),
\end{aligned}
$$

so we get $\mid \eta(a) \mid = \mid \eta(1-a) \mid$.

(iii) is straightforward and *(iv)* is clear from the definition of nil clean index. □

Lemma 2.2.2. *If S is a subring of a ring R, where S and R may or may not share the same identity, then $Nin(S) \leq Nin(R)$.*

Proof. Since S is a subring of R, all the idempotents and nilpotent elements of S are also idempotents and nilpotent elements of R. Let $a \in S$ and $e \in \eta_S(a)$ i.e., $e^2 = e \in S$ and $a - e \in \text{nil}(S)$. Obviously this implies e is an idempotent in R and $a - e \in \text{nil}(R),$ i.e., $e \in \eta_R(a)$. Therefore we have

$$
\begin{aligned}
& \eta_S(a) \subseteq \eta_R(a) \text{ for all } a \in S, \\
\Rightarrow \quad & \mid \eta_S(a) \mid \leq \mid \eta_R(a) \mid \text{ for all } a \in S, \\
\Rightarrow \quad & \sup_{a \in S} \mid \eta_S(a) \mid \leq \sup_{a \in S} \mid \eta_R(a) \mid \leq \sup_{a \in R} \mid \eta_R(a) \mid .
\end{aligned}
$$

So we get $\text{Nin}(S) \leq \text{Nin}(R)$. □

Lemma 2.2.3. *Let $R = S \times T$ be the direct product of two rings S and T. Then $Nin(R) = Nin(S)Nin(T)$.*

Proof. Since S and T are subrings of R, so

$$\mathrm{Nin}(S) \leq \mathrm{Nin}(R) \text{ and } \mathrm{Nin}(T) \leq \mathrm{Nin}(R).$$

If $\mathrm{Nin}(S) = \infty$ or $\mathrm{Nin}(T) = \infty$, then $\mathrm{Nin}(R) = \infty$ and hence, $\mathrm{Nin}(R) = \mathrm{Nin}(S)\mathrm{Nin}(T)$ holds. Now let

$$\mathrm{Nin}(S) = n < \infty \text{ and } \mathrm{Nin}(T) = m < \infty.$$

As $n,\ m \geq 1$ and there exist elements $s \in S$ and $t \in T$, such that

$$\mid \eta_S(s) \mid = n \text{ and } \mid \eta_T(t) \mid = m.$$

Thus $s = e_i + n_i$, for $i = 1, 2, \ldots, n$ and $t = f_j + m_j$ for $j = 1, 2, ..., m$, where e_i's, f_j's are distinct idempotents and n_i's, m_j's are distinct nilpotent elements of S and T respectively. Therefore $(s, t) \in R$, can be expressed as

$$(s, t) = (e_i, f_j) + (n_i, m_j),$$

which are mn distinct nil clean expressions of $(s, t) \in R$. Hence

$$\mathrm{Nin}(R) \geq mn.$$

If possible let $\mathrm{Nin}(R) > nm$, say $nm + 1$, then there exists an element $(a, b) \in R$, such that it has at least $nm + 1$ nil clean expressions in R. That is

$$(a, b) = (g_i, h_i) + (c_i, d_i),$$

where $i = 1, 2, \ \ldots, mn + 1,\ \ (g_i, h_i)^2 = (g_i, h_i)$ and $(c_i, d_i) \in \mathrm{nil}(R)$. So $a = g_i + c_i$ and $b = h_i + d_i$ are nil clean expressions for a and b respectively. Let

$$K = \{(g_i, h_i) \mid i = 1,\ \ 2,\ \ 3,\ \ldots, mn,\ \ mn + 1\}.$$

Now we have

$$\begin{aligned} & \mid K \mid = nm + 1 \\ \Rightarrow \quad & \mid \{g_i\} \mid . \mid \{h_i\} \mid = nm + 1 \\ \Rightarrow \quad & \mid \{g_i\} \mid > n \text{ or } \mid \{h_i\} \mid > m, \end{aligned}$$

which gives $\text{Nin}(S) > n$ or $\text{Nin}(T) > m$, which is absurd. □

Lemma 2.2.4. *Let I be an ideal of R with $I \subseteq nil(R)$ and let $n \geq 1$ be an integer. Then the following hold:*

(i) *$Nin(R/I) = Nin(R)$.*

(ii) *If $Nin(R) \leq n$, then every idempotent of R/I can be lifted to at most n idempotents of R.*

Proof. (i) Let $a \in R$. Then any idempotent $x+I \in \eta(a+I)$ is lifted to an idempotent e_x of R. Now from

$$(a + I) - (x + I) \in \text{nil}(R/I),$$

we get

$$(a + I) - (e_x + I) \in \text{nil}(R/I),$$

which means there exists some positive integer k, such that

$$\begin{aligned} & (a - e_x)^k + I = I \\ \Rightarrow \quad & a - e_x \in \text{nil}(R) \\ \text{i.e.,} \quad & e_x \in \eta(a). \end{aligned}$$

So the mapping $\eta(a) \to \eta(a + I)$ is onto, i.e.,

$$\mid \eta(a) \mid \geq \mid \eta(a + I) \mid \quad \text{for all } a \in R.$$

Conversely, if $e \in \eta(a)$ then $a - e \in \text{nil}(R)$; so there exists some positive integer k, such that

$$(a-e)^k = 0 \in I.$$

This implies

$$(a-e)^k + I = I$$

and so

$$(a - I) - (e + I) \in \text{nil}(R/I)$$

which gives

$$e + I \in \eta(a + I).$$

Therefore the mapping

$$\eta(a+I) \to \eta(a) \text{ is onto.}$$

i.e., $\mid \eta(a+I) \mid \geq \mid \eta(a) \mid, \text{ for all } a \in R$. Hence

$$\mid \eta(a) \mid = \mid \eta(a+I) \mid, \text{ for all } a \in R,$$

which implies

$$\sup_{a \in R} \mid \eta(a) \mid = \sup_{(a+I) \in R/I} \mid \eta(a+I) \mid,$$

consequently

$$\text{Nin}(R) = \text{Nin}(R/I).$$

(ii) Let $a \in R$ such that $a^2 - a \in I$. If $a - e \in I \subseteq \text{nil}(R)$, for some $e^2 = e \in R$, then $e \in \eta(a)$. But $\mid \eta(a) \mid \leq \text{Nin}(R) \leq n$. So there are at most n such elements. $\square$

Lemma 2.2.5. *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* A *and* B *are rings,* ${}_AM_B$ *is a bimodule. Let* $Nin(A) = n$ *and* $Nin(B) = m$. *Then*

(i) $Nin(R) \geq |M|$.

(ii) If $(M,+) \cong C_{p^k}$, *where* p *is a prime and* $k \geq 1$, *then* $Nin(R) \geq n + [\frac{n}{2})(|M|-1)$, *where* $[\frac{n}{2})$ *denotes the least integer greater than or equal to* $\frac{n}{2}$.

(iii) Either $Nin(R) \geq nm + |M|-1$ *or* $Nin(R) \geq 2nm$.

Proof. (i) Let $\alpha = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$. Then we have

$$\left\{ \begin{pmatrix} 1_A & w \\ 0 & 0 \end{pmatrix} \mid w \in M \right\} \subseteq \eta(\alpha)$$

as

$$\begin{pmatrix} 1_A & w \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \text{ is nilpotent.}$$

So we have

$$\text{Nin}(R) \geq |\eta(\alpha)| \geq |M|.$$

(ii) Let $q = p^k$ and $a = e_i + n_i, \quad i = 1, 2, \dots n$ be n distinct nil clean expressions of a in A. For any $e = e^2 \in A$

$$(M,+) = eM \oplus (1-e)M.$$

Since $(M,+) \cong C_{p^k}$, so $(M,+)$ is indecomposable and hence

$$M = eM \text{ or } = (1-e)M.$$

Assume that

$$(1-e_1)M = \cdots = (1-e_s)M = M \text{ and } e_{s+1}M = \cdots = e_nM = M.$$

If $s \geq (n-s)$ (i.e., $s \geq [\frac{n}{2}]$), then for $\alpha = \begin{pmatrix} 1_A - a & 0 \\ 0 & 0 \end{pmatrix}$, we have

$$\eta(\alpha) \supseteq \left\{ \begin{pmatrix} 1_A - e_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1_A - e_j & w \\ 0 & 0 \end{pmatrix} : 1 \leq i \leq n, 1 \leq j \leq s, 0 \neq w \in M \right\},$$

so

$$|\eta(\alpha)| \geq n + s(q-1).$$

If $s < (n-s)$ (i.e., $n - s \geq [\frac{n}{2}]$), for $\beta = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$

$$\eta(\beta) \supseteq \left\{ \begin{pmatrix} e_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e_j & w \\ 0 & 0 \end{pmatrix} : 1 \leq i \leq n, s+1 \leq j \leq n, 0 \neq w \in M \right\},$$

therefore

$$|\eta(\beta)| \geq n + (n-s)(q-1).$$

Hence

$$\text{Nin}(R) \geq n + \left[\frac{n}{2}\right)(q-1).$$

(iii) Let $a = e_i + n_i, \quad i = 1, 2, \ldots n$ and $b = f_j + m_j, \quad j = 1, 2, \ldots m$ be distinct nil clean expressions of a and b in A and B respectively.

Case I:

If $e_{i_0} M(1 - f_{j_0}) + (1 - e_{i_0}) M f_{j_0} = 0$ for some i_0 and j_0. Then $e_{i_0} w = w f_{i_0}$ for all $w \in M$. Thus for $\alpha = \begin{pmatrix} 1_A - a & 0 \\ 0 & b \end{pmatrix}$

$$\eta(\alpha) \supseteq \left\{ \begin{pmatrix} 1_A - e_i & 0 \\ 0 & f_j \end{pmatrix}, \begin{pmatrix} 1_A - e_{i_0} & w \\ 0 & f_{j_0} \end{pmatrix} ; 1 \leq i \leq n, 1 \leq j \leq m; 0 \neq w \in M \right\},$$

so we have $|\eta(\alpha)| \geq mn + |M| - 1$.

Case II:

If $e_iM(1-f_j)+(1-e_i)Mf_j \neq 0$ for all i and j. Take

$$0 \neq w_{ij} \in e_iM(1-f_j)+(1-e_i)Mf_j \text{ for each pair } (i,j).$$

For $\alpha = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, we have

$$\eta(\alpha) \supseteq \left\{ \begin{pmatrix} e_i & 0 \\ 0 & f_j \end{pmatrix}, \begin{pmatrix} e_i & w_{ij} \\ 0 & f_j \end{pmatrix}; 1 \leq i \leq n, 1 \leq j \leq m; 0 \neq w_{ij} \in M \right\},$$

thus $|\eta(\alpha)| \geq 2mn$.

Cases I and **II** imply, either

$$\text{Nin}(R) \geq nm + |M| - 1 \text{ or } \text{Nin}(R) \geq 2nm.$$

□

Lemma 2.2.6. *Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where A and B are rings, ${}_AM_B$ is a bimodule with $(M,+) \cong C_{2^r}$. Then $Nin(R) = 2^r Nin(A) Nin(B)$.*

Proof. Let $k = \text{Nin}(A)$ and $l = \text{Nin}(B)$. For $e_i \in \text{Idem}(A), f_j \in \text{Idem}(B), n_i \in \text{Nil}(A)$ and $m_j \in \text{Nil}(B)$, let $a = e_i + n_i,\ \ i = 1, 2, \ldots k$ and $b = f_j + m_j,\ \ j = 1, 2, \ldots l$ be distinct nil clean expressions of $a \in A$ and $b \in B$ respectively. Write $M = \{0,\ x,\ 2x, \ldots, (2^r-1)x\}$, for any $e = e^2 \in A$, either $M = eM$ or $M = (1_A - e)M$; so $ex \in \{0, x\}$. Suppose $e_1x \neq e_2x$, say $e_1x = 0$ and $e_2x = x$. Then

$$ax = n_1x = x + n_2x = (1+n_2)x$$

Because $ax \in M$, $\ \ ax = ix$ for some $2 \leq i \leq 2^k$. So

$$\begin{aligned} & n_1x = ix, \\ \Rightarrow \quad & 0 = i^px \text{ [Since } n^p = 0 \text{ for some } p \in \mathbb{N}], \end{aligned}$$

which gives i is even, so let $i = 2j$. Now

$$\begin{aligned} & (1+n_2)x = (2j)x \\ \Rightarrow \quad & (1+n_2)^r x = (2j)^r x = j^r(2^r)x = 0, \\ \Rightarrow \quad & x = 0 \text{ [as } n+1 \in \mathrm{U}(A)], \end{aligned}$$

a contradiction, as $x \neq 0$. So $e_1 x = e_2 x = \cdots = e_k x$. Similarly $xf_1 = xf_2 = \cdots = xf_l$.

Case I:

If $e_i x = 0$ and $xf_j = 0$, for $\alpha = \begin{pmatrix} 1_A - a & 0 \\ 0 & b \end{pmatrix}$ we have

$$\alpha = \begin{pmatrix} 1_A - e_i & w \\ 0 & f_j \end{pmatrix} + \begin{pmatrix} -n_i & -w \\ 0 & m_j \end{pmatrix},$$

$i = 1, 2, \ldots, k, j = 1, 2, \ldots, l$ and $\forall w \in M$, therefore in this case

$$\mathrm{Nin}(R) \geq |\eta(\alpha)| \geq 2^r kl.$$

Case II:

If $e_i x = x, \quad xf_j = x$, for $\beta = \begin{pmatrix} 1_A - a & 0 \\ 0 & b \end{pmatrix}$, we have

$$\beta = \begin{pmatrix} 1_A - e_i & w \\ 0 & f_j \end{pmatrix} + \begin{pmatrix} -n_i & -w \\ 0 & m_j \end{pmatrix},$$

$i = 1, 2, \ldots, k, j = 1, 2, \ldots, l,$ and $\forall w \in M$, therefore in this case also

$$\mathrm{Nin}(R) \geq |\eta(\alpha)| \geq 2^r kl.$$

Case III:

If $e_i x = x, \quad xf_j = 0$, for $\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ we have

$$\gamma = \begin{pmatrix} e_i & w \\ 0 & f_j \end{pmatrix} + \begin{pmatrix} n_i & -w \\ 0 & m_j \end{pmatrix},$$

$i = 1, 2, \ldots, k, j = 1, 2, \ldots, l$ and $\forall w \in M$, therefore in this case $\mathrm{Nin}(R) \geq |\eta(\alpha)| \geq 2^r kl$.

Case IV:

If $e_i x = 0, \quad x f_j = x$, for $\delta = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, we have

$$\delta = \begin{pmatrix} e_i & w \\ 0 & f_j \end{pmatrix} + \begin{pmatrix} n_i & -w \\ 0 & m_j \end{pmatrix}$$

$i = 1, 2, \ldots, k, j = 1, 2, \ldots, l$ and $\forall w \in M$, therefore in this case also

$$\mathrm{Nin}(R) \geq |\eta(\alpha)| \geq 2^r kl.$$

On the other hand for $\alpha = \begin{pmatrix} c & z \\ 0 & d \end{pmatrix} \in R$, we have

$$\eta(\alpha) = \left\{ \begin{pmatrix} e & w \\ 0 & f \end{pmatrix} \in R, \quad e \in \eta(c), \quad f \in \eta(d), \quad w = ew + we \right\}.$$

So, $|\eta(\alpha)| \leq |M||\eta(c)||\eta(d)| \leq 2^r kl$ and therefore

$$\mathrm{Nin}(R) \leq 2^r kl.$$

Hence $\mathrm{Nin}(R) = 2^r kl = 2^r \mathrm{Nin}(A)\mathrm{Nin}(B)$. $\square$

Lemma 2.2.7. *Let A and B be rings and ${}_AM_B$ a nontrivial bimodule. If $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is a formal triangular matrix ring, then $Nin(A) < Nin(R)$ and $Nin(B) < Nin(R)$.*

Proof. Let $k = \text{Nin}(A)$, for $n_i \in \text{Nil}(R)$, $e_i \in \text{Idem}(R)$; let $a = e_i + n_i \quad (i = 1,\ 2, \ldots, k)$ be k distinct nil clean expressions of a in A. If $e_1 M = 0$, then

$$\begin{aligned} \begin{pmatrix} 1_A - a & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 1_A - e_i & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -n_i & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1_A - e_1 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -n_1 & -x \\ 0 & 0 \end{pmatrix} \qquad \forall\ 0 \neq x \in M \end{aligned}$$

are at least $k+1$ distinct nil clean expressions of $\begin{pmatrix} 1_A - a & 0 \\ 0 & 0 \end{pmatrix}$ in R.

If $e_1 M \neq 0$, then $e_1 x \neq 0$ for some $x \in M$. So we have

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} e_i & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_i & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & e_1 x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_1 & -e_1 x \\ 0 & 0 \end{pmatrix} \qquad \forall\ 0 \neq x \in M \end{aligned}$$

are at least $k+1$ distinct nil clean expressions of $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ in R.

So in any case

$$\text{Nin}(R) \geq k + 1 > k = \text{Nin}(A).$$

Similarly

$$\text{Nin}(R) > \text{Nin}(B).$$

□

Lemma 2.2.8. *Let R be a ring with unity. Then $In(R) \geq Nin(R)$, where $In(R)$ is the clean index of R.*

Proof. Let $\text{Nin}(R) = k$. Then there is an element $a \in R$, such that it has k nil clean expressions in R, i.e.,

$$a = e_i + n_i, \text{ for } i = 1, \ 2, \cdots, k,$$

where $e_i \in \text{idem}(R)$ and $n_i \in \text{nil}(R)$. From this we get,

$$a - 1 = e_i + (n_i - 1)$$

are k clean expression for $(a-1) \in R$, and therefore $\text{In}(R) \geq k$. □

2.3 Rings of nil clean index 1

Lemma 2.3.1. *$Nin(R) = 1$, if and only if R is abelian and for any $0 \neq e^2 = e \in R, e \neq n + m$ for $n, m \in nil(R)$.*

Proof. Let $e^2 = e \in R$. Then for any $r \in R$, we have

$$e + 0 = [e + er(1-e)] + [-er(1-e)],$$

where

$$(e + er(1-e))^2 = e + er(1-e) \in \text{Idem}(R)$$

$$(-er(1-e))^2 = er(1-e)er(1-e) = 0 \text{ i.e., } -er(1-e) \in \text{nil}(R).$$

Since $\text{Nin}(R) = 1$, so

$$e = e + er(1-e) \text{ gives } er = ere.$$

Similarly $re = ere$, hence $er = re$, thus R is abelian. For last part, if $e = n + m$ for some $n, m \in \text{nil}(R)$ then

$$e + (-m) = 0 + n,$$

since $\mathrm{Nin}(R) = 1$, this is not possible.

Conversely, suppose R is abelian and no nonzero idempotent of R can be written as a sum of two nilpotent elements. We know that $\mathrm{Nin}(S) \geq 1$ for any ring S. Assume that $a \in R$ has two nil clean expressions

$$a = e_1 + n_1 = e_2 + n_2, \tag{2.3.1}$$

where $e_1, e_2 \in \mathrm{idem}(R)$ and $n_1, n_2 \in \mathrm{nil}(R)$. If $e_1 = e_2$ we have nothing to prove. So let $e_1 \neq e_2$. Now multiplying equation (2.3.1) by $(1 - e_1)$ we get,

$$e_1(1 - e_1) + n_1(1 - e_1) = e_2(1 - e_1) + n_2(1 - e_2)$$

$$e_2(1 - e_1) = n_1(1 - e_1) - n_2(1 - e_2). \tag{2.3.2}$$

Since R is abelian, therefore

$$e_2(1 - e_1) \in \mathrm{Idem}(R) \text{ and } n_1(1 - e_1),\ n_2(1 - e_2)\,\mathrm{Nil}(R).$$

So (2.3.2) gives a contradiction if $e_2(1 - e_1) \neq 0$. $e_2(1 - e_1) = 0$, i.e., $e_2 - e_1e_2$. Similarly, $e_1 - e_1e_2$. Hence $e_1 = e_2$. This shows $|\eta(a)| \leq 1$ for all $a \in R$, hence $\mathrm{Nin}(R) = 1$. □

Theorem 2.3.2. *$Nin(R) = 1$ if and only if R is an abelian ring.*

Proof. (⇒) This is by **Lemma 2.3.1**.

(⇐) Let R be an abelian ring and e a non zero idempotent of R. We claim that e can not be written as sum of two nilpotent elements. Suppose $e = a + b$ where $a, b \in \mathrm{Nil}(R)$ and for positive integers $n < m$, $a^n = 0 = b^m$. Then $(e - a)^m = 0$, using binomial theorem we get

$$e^m - \binom{m}{1} a e^{(m-1)} + \binom{m}{2} a^2 e^{(m-2)} - \cdots + (-1)^{(n-1)} \binom{m}{n-1} a^{(n-1)} e^{(m-n+1)} = 0$$

which gives

$$e\left[1-\binom{m}{1}a+\binom{m}{2}a^2-\cdots+(-1)^{(n-1)}\binom{m}{n-1}a^{(n-1)}\right.$$
$$\left.+(-1)^n\binom{m}{n}a^n+(-1)^{(n+1)}\binom{m}{n+1}a^{(n+1)}+\cdots+(-1)^m a^m\right]=0.$$

This implies

$$e(1-a)^m=0,$$

therefore we get, $e=0$ [since $1-a\in \mathrm{U}(R)$].

Similarly, if $n>m$, then $(e-b)^n=0$ and so $e=0$, a contradiction. Hence, no nonzero idempotent can be written as sum of two nilpotent elements and therefore $\mathrm{Nin}(R)=1$. □

Above theorem gives the following observations:

(i) A ring R with $\mathrm{Nin}(R)=1$ is always Dedekind finite, but the converse is not true by **Example 2.5.2**.

(ii) Rings with trivial idempotents have nil clean index one and consequently the local rings are of nil clean index one. If $\mathrm{Nin}(R)=1$, then it is easy to see that idempotents of $R[[x]]$ are idempotents of R, and for any

$$\alpha=\alpha_0+\alpha_1x+\cdots\in R[[x]],$$

we have

$$\eta_{R[[x]]}(\alpha)\subseteq\eta_R(\alpha_0).$$

This gives

$$\mathrm{Nin}(R[x])=\mathrm{Nin}(R[[x]])=1.$$

But if $\mathrm{Nin}(R)>1$ then, there is some noncentral idempotent $e\in R$, such that $er\neq re$ for some $r\in R$. So either

$$er(1-e)\neq 0 \text{ or } (1-e)re\neq 0.$$

Let $er(1-e) \neq 0$. Then we have

$$\begin{aligned} a &:= e + er(1-e) \\ &= [e + er(1-e)x^i] + [er(1-e)(1-x^i)], \end{aligned}$$

where i is a positive integer, are infinitely many nil clean expressions of a in $R[x]$, which implies

$$\mathrm{Nin}(R[x]) = \infty.$$

Now we have the following theorem.

Theorem 2.3.3. *$Nin(R[[x]])$ is finite iff $Nin(R) = 1$.*

Proof. If $\mathrm{Nin}(R) = 1$, then it is easy to see that idempotents of $R[[x]]$ are idempotents of R. For any $\alpha = \alpha_0 + \alpha_1 x + \cdots \in R[[x]]$, it is easy to see that $\eta_{R[[x]]}(\alpha) \subseteq \eta_R(\alpha_0)$. This gives $\mathrm{Nin}(R[x]) = \mathrm{Nin}(R[[x]]) = 1$. But if $\mathrm{Nin}(R) > 1$ then, there is some noncentral idempotent $e \in R$, such that $er \neq re$ for some $r \in R$. So either $er(1-e) \neq 0$ or $(1-e)re \neq 0$. Let $er(1-e) \neq 0$. Then we have $a := e + er(1-e) = [e + er(1-e)x^i] + [er(1-e)(1-x^i)]$ where i is a positive integer, are infinitely many nil clean expressions of a in $R[x]$ which implies $\mathrm{Nin}(R[x]) = \infty$. □

Corollary 2.3.4. *$Nin(R[[x]])$ is 1 or infinite.*

2.4 Rings of nil clean index 2

In this section, we characterize rings of nil clean index 2. From the discussion above we see that such rings must be non abelian.

Theorem 2.4.1. *Nin*$(R) = 2$ *if and only if* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where Nin*$(A) =$ *Nin*$(B) = 1$ *and* ${}_AM_B$ *is a bimodule with* $|M| = 2$.

Proof. ($\Leftarrow$) For $\alpha_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} \in R$, we have

$$\left\{ \begin{pmatrix} 0 & \omega \\ 0 & 1_B \end{pmatrix}; \quad \omega \in M \right\} \subseteq \eta(\alpha_0).$$

Therefore

$$\mathrm{Nin}(R) \geq \mid \eta(\alpha_0) \mid \geq \mid M \mid = 2.$$

For any $\alpha = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in R$

$$\eta(\alpha) = \left\{ \begin{pmatrix} e & w \\ 0 & f \end{pmatrix}; \ e \in \eta(a), \ f \in \eta(b), \ w = ew + wf \right\}.$$

Because $\mid M \mid = 2, \ \mid \eta(a) \mid \leq 1, \ \mid \eta(b) \mid \leq 1$, it follows that $\mid \eta(\alpha) \mid \leq 2$. Hence $\mathrm{Nin}(R) = 2$.

($\Rightarrow$) Suppose R is non abelian and let $e^2 = e \in R$ be a non central idempotent. If neither $eR(1-e)$ nor $(1-e)Re$ is zero, then take $0 \neq x \in eR(1-e)$ and $0 \neq y \in (1-e)Re$. Then

$$\begin{aligned} e &= e + 0 \\ &= (e + x) - x \\ &= (e + y) - y \end{aligned}$$

are three distinct nil clean expressions of e in R. So without loss of generality, we can assume that

$$eR(1-e) \neq 0 \text{ but } (1-e)Re = 0.$$

The Peirce decomposition of R gives

$$R = \begin{pmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{pmatrix}.$$

As above $2 = \text{Nin}(R) \geq \mid eR(1-e) \mid$; so $\mid eR(1-e) \mid = 2$. Write

$$eR(1-e) = \{0, x\}.$$

If possible let $a = e_1 + n_1 = e_2 + n_2$ be two distinct nil clean expressions of a in eRe. If $e_1 x = x$, then

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_1 & x \\ 0 & 0 \end{pmatrix} \end{aligned}$$

are three distinct nil clean expressions of $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R$. If $e_1 x = 0$, then

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & 1_B \end{pmatrix} &= \begin{pmatrix} e_1 & 0 \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_2 & 0 \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} n_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & x \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} n_1 & x \\ 0 & 1_B \end{pmatrix} \end{aligned}$$

are three distinct nil clean expressions of $\begin{pmatrix} a & 0 \\ 0 & 1_B \end{pmatrix}$ in R. This contradiction shows that $\mathrm{Nin}(eRe) = 1$. Similarly, $\mathrm{Nin}((1-e)R(1-e)) = 1$. □

2.5 Rings of nil clean index 3

The next proposition gives a sufficient condition for rings to have nil clean index 3.

Theorem 2.5.1. *If* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* $Nin(A) = Nin(B) = 1$ *and* ${}_AM_B$ *is a bimodule with* $|M| = 3$, *then* $Nin(R) = 3$.

Proof. This is similar to the proof of the implication "(⇐)" of **Proposition 2.4.1**. □

The condition of **Proposition 2.5.1** is a sufficient condition, but not necessary, as shown by the following example.

Example 2.5.2. $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ *is a ring of nil clean index* 3.

We see that, $\mathrm{nil}(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$. Using **Lemma 3.2.1**, we get $\mathrm{Nin}(R) \leq 4$. Also

$$\eta\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\},$$

thus $\mathrm{Nin}(R) \geq 3$. Similarly verifying for each element we see that $\mathrm{Nin}(R) = 3$. But it is not of the form $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. □

Next we have the following proposition for matrix ring.

Proposition 2.5.3. *Let S be a ring with unity and let $n \geq 2$ be an integer. Then*

(i) $Nin(M_n(S)) \geq 3$.

(ii) $Nin(M_n(S)) = 3$ *iff* $n = 2$ *and* $S \cong \mathbb{Z}_2$.

Proof. Let $E_{ij} \in M_n(S)$ such that (i, j)th entry is 1 and rest of the entries are 0. So for $A = E_{11}$,

$$\eta(A) \supseteq \left\{ E_{11} + \sum_{i=2}^{n} r_i E_{1i}, E_{11} + \sum_{i=2}^{n} s_i E_{i1} | \forall r_i, s_i \in S \quad (2 \leq i \leq n) \right\}.$$

So we have

$$\text{Nin}(R) \geq |\eta_R(a)| \geq 2|S|^{n-1} - 1.$$

(i) If $|S| \geq 3$ or $n \geq 3$, then

$$\text{Nin}(R) \geq \min\{2.3^{2-1} - 1,\ 2.3^{3-1} - 1\} = 5.$$

By **Example 2.5.2**, $\text{Nin}(M_2(\mathbb{Z}_2)) = 3$, so in general $\text{Nin}(R) \geq 3$.

(ii) If $\text{Nin}(R) = 3$, the above argument shows

$$3 = \text{Nin}(R) \geq 2|S|^{n-1} - 1$$
$$\Rightarrow \qquad 2 \geq |S|^{n-1}.$$

So we must have $n = 2$ and $|S| = 2$. Therefore $S \cong \mathbb{Z}_2$. The converse part is obviously true as $\text{Nin}(M_2(\mathbb{Z}_2)) = 3$. □

Theorem 2.5.4. *Let R be a ring. If $Nin(R) = 3$ then one of the following holds:*

(i) $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where A and B are rings with $Nin(A) = Nin(B) = 1$ and ${}_AM_B$ is a bimodule with $|M| = 3$.*

(ii) $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$, *where A and B are rings with $Nin(A) = Nin(B) = 1$ and ${}_AM_B$, ${}_BN_A$ are bimodules with $|M| = |N| = 2$.*

Proof. Let $\text{Nin}(R) = 3$. Then R is non abelian. Let $e \in R$ be a noncentral idempotent. Set

$$A = eRe,\ B = (1-e)R(1-e),\ M = eR(1-e),\ N = (1-e)Re.$$

Since e is noncentral, M and N are not both zero, so we have two cases

Case I:

Let $M \neq 0$, $N = 0$ or $M = 0$, $N \neq 0$. Without loss of generality let $M \neq 0$, $N = 0$. Then $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Clearly by **Lemma 2.2.5**,

$$2 \leq |M| \leq \text{Nin}(R) = 3.$$

Also by **Lemma 2.2.7**, we have

$$\text{Nin}(A) < \text{Nin}(R) \text{ and } \text{Nin}(B) < \text{Nin}(R).$$

By **Lemma 2.2.6**, if $|M| = 2$ then

$$3 = \text{Nin}(R) = 2\text{Nin}(A)\text{Nin}(B),$$

which is a contradiction. So $|M| = 3$. Now by **Lemma 2.2.5**, we see that

$$\begin{aligned} 3 = \text{Nin}(R) &\geq \text{Nin}(A)\text{Nin}(B) + |M| - 1 \quad &&\text{or } \text{Nin}(R) \geq 2\text{Nin}(A)\text{Nin}(B) \\ \Rightarrow \quad &\text{Nin}(A)\text{Nin}(B) \leq 1 &&\text{or } \text{Nin}(A)\text{Nin}(B) \leq \frac{3}{2} \\ \Rightarrow \quad &\text{Nin}(A)\text{Nin}(B) = 1; \text{ that is, } \text{Nin}(A) = \text{Nin}(B) = 1. \end{aligned}$$

So we get (i).

Case II:

Let $N \neq 0$ and $M \neq 0$. So $|N| \geq 2$ and $|M| \geq 2$. Now

$$\eta(e) \supseteq \{e + w, e + z; w \in M, 0 \neq z \in N\}.$$

Thus

$$3 = \text{Nin}(R) \geq |\eta(e)| \geq |M| + |N| - 1$$

$$\Rightarrow \quad 4 \leq |M| + |N| \leq 4 \quad \Rightarrow |M| = |N| = 2.$$

Again $C = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \subseteq R$, so $\text{Nin}(C) \leq \text{Nin}(R) = 3$.

But $\text{Nin}(C) = 2\text{Nin}(A)\text{Nin}(B) \leq 3 \Rightarrow \text{Nin}(A) = \text{Nin}(B) = 1$, proving (ii). □

Note: Ring homomorphisms in general do not preserve the nil clean index. For example, if we consider a ring R of nil clean index 2, then R cannot be abelian, so $\text{Nin}(R[[x]])$ can not be finite. But R is a homomorphic image of $R[[x]]$. However in case of $\text{Nin}(R) \quad = 1$, we have the following result.

Theorem 2.5.5. *The homomorphic image of a ring R with $Nin(R) = 1$ is again a ring with $Nin(R) = 1$, provided idempotents of R can be lifted modulo the kernel of the homomorphism.*

Proof. Straightforward. □

2.6 Formal triangular ring with nil clean index 4

Theorem 2.6.1. *Let* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$*, where* A *and* B *are rings,* ${}_AM_B$ *is a non trivial bimodule. Then* $\mathrm{Nin}(R) = 4$ *if and only if one of the following holds:*

(i) $(M, +) \cong C_2$ *and* $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 2$.

(ii) $(M, +) \cong C_4$ *and* $\mathrm{Nin}(A) = \mathrm{Nin}(B) = 1$.

(iii) $(M, +) \cong C_2 \oplus C_2$ *plus one of the following*

(a) $\mathrm{Nin}(A) = \mathrm{Nin}(B) = 1$.

(b) $\mathrm{Nin}(A) = 1$, $B = \begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$, *where* $\mathrm{Nin}(S) = \mathrm{Nin}(T) = 1$ *and* $|W| = 2$, *and* $eM(1_B - f) + (1_A - e)Mf \neq 0$*, for all* $e^2 = e \in A$ *and* $f \in \eta(b)$*, where* $b \in B$ *with* $|\eta(b)| = 2$.

(c) $\mathrm{Nin}(B) = 1$, $A = \begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$, *where* $\mathrm{Nin}(S) = \mathrm{Nin}(T) = 1$ *and* $|W| = 2$, *and* $eM(1_B - f) + (1_A - e)Mf \neq 0$*, for all* $e^2 = e \in B$ *and* $f \in \eta(a)$*, where* $a \in A$ *with* $|\eta(a)| = 2$.

Proof. ($\Leftarrow$) If (i) holds then by **Lemma 2.2.6**, we get $\mathrm{Nin}(R) = 4$.
If (ii) holds then $\mathrm{Nin}(R) \geq |M| = 4$. Now, for any $\alpha = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in R$,

$$\eta(\alpha) = \left\{ \begin{pmatrix} e & w \\ 0 & f \end{pmatrix} \in R : \; e \in \eta(a), \; f \in \eta(b), \; w = ew + fw \right\}.$$

Because $|M| = 4$, $|\eta(a)| \leq 1$ and $|\eta(b)| \leq 1$, it follows that $|\eta(\alpha)| \leq 4$. Hence $\mathrm{Nin}(R) = 4$.
Let (iii) (a) hold. Then $\mathrm{Nin}(R) \geq |M| = 4$. Now, for any $\alpha = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in R$,

$$\eta(\alpha) = \left\{ \begin{pmatrix} e & w \\ 0 & f \end{pmatrix} \in R : \; e \in \eta(a), \; f \in \eta(b), \; w = ew + fw \right\}.$$

Because $|M|= 4$, $|\eta(a)|\leq 1$ and $|\eta(b)|\leq 1$, it follows that $|\eta(\alpha)|\leq 4$. Hence $\mathrm{Nin}(R) = 4$.

Suppose (iii) (c) hold. Then clearly $\mathrm{Nin}(R) \geq |M|= 4$. Let $\alpha = \begin{pmatrix} a & w \\ 0 & b \end{pmatrix} \in R$. We show that $|\eta(\alpha)|\leq 4$ and hence $\mathrm{Nin}(R) = 4$ holds. Since $\mathrm{Nin}(B) = 1$, we can assume that $\eta(b) = \{f_0\}$. Then as above we have

$$\eta(\alpha) = \left\{ \begin{pmatrix} e & z \\ 0 & f_0 \end{pmatrix} \in R : e \in \eta(a), z = ez + zf_0 \right\}$$

If $|\eta(a)|\leq 1$, then $|\eta(\alpha)|\leq |\eta(a)|.|M|\leq 4$. So we can assume that $|\eta(a)|= 2$. Write $\eta(a) = \{e_1,\ e_2\}$. Thus $\eta(\alpha) = T_1 \bigcup T_2$, where

$$T_i = \left\{ \begin{pmatrix} e_i & z \\ 0 & f_0 \end{pmatrix} \in R : (1_A - e_i)z = zf_0 \right\} \qquad (i = 1,\ 2).$$

Since

$$\eta(1_A - a) = \{1_A - e_1,\ 1_A - e_2\},$$

the assumption $(iii)(c)$ shows that

$$\{z \in M : (1_A - e_i)z = zf_0\}$$

is a proper subgroup of $(M, +)$; so $|T_i|\leq 2$ for $i = 1,\ 2$. Hence

$$|\eta(\alpha)|\leq |T_1|+|T_2|\leq 4.$$

($\Rightarrow$) Suppose $\mathrm{Nin}(R) = 4$. Then $2 \leq |M|\leq \mathrm{Nin}(R) = 4$. If $|M|= 2$ then $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 2$ by **Lemma 2.2.6**, so (i) holds.

Suppose $|M|= 3$. By **Lemma 2.2.5**, we have

$$\mathrm{Nin}(A) + |M|\leq \mathrm{Nin}(R),$$

showing $\mathrm{Nin}(A) \leq 2$. Similarly, $\mathrm{Nin}(B) \leq 2$. But $\mathrm{Nin}(A) = 2 = \mathrm{Nin}(B)$ will give

$\mathrm{Nin}(R) \geq 6$ by **Lemma 2.2.5**

and $\mathrm{Nin}(A) = \mathrm{Nin}(B) = 1$ will give

$\mathrm{Nin}(R) = 3$ by **Theorem 2.5.1**.

Hence the only possibility is $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 2$, so without loss of generality we assume that $\mathrm{Nin}(A) = 2$ and $\mathrm{Nin}(B) = 1$. Write

$$M = \{0, x, 2x\}.$$

Now by **Theorem 2.4.1**, we have

$$A = \begin{pmatrix} T & N \\ 0 & S \end{pmatrix},$$

where T & S are rings, ${}_TN_S$ is bimodule with $\mathrm{Nin}(T) = \mathrm{Nin}(S) = 1$ and $|N| = 2$. Note that for $e \in \mathrm{Idem}(A)$, $ex \in \{0,\ x\}$. Indeed, if $ex = 2x$, we have

$$2x = ex = e(ex) = e(2x) = e(x+x) = ex + ex = 2x + 2x = 4x = x$$

which is not possible.

Now let $a = \begin{pmatrix} 1_T & 0 \\ 0 & 0 \end{pmatrix} \in A$, such that

$$\begin{aligned} a &= \begin{pmatrix} 1_T & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1_T & y \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Let us denote, $e_1 = \begin{pmatrix} 1_T & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1_T & y \\ 0 & 0 \end{pmatrix}$, $n_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $n_2 = \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix}$. Clearly $e_1,\ e_2 \in \mathrm{Idem}(A)$ & $n_1,\ n_2 \in \mathrm{Nil}(A)$. Now we have following cases:

Case I:

Let $e_1x = e_2x = 0$. Then we have an element $\beta = \begin{pmatrix} 1_A - a & 0 \\ 0 & 0 \end{pmatrix} \in R$ such that

$$\begin{aligned} \beta &= \begin{pmatrix} 1_A - e_1 & z \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -n_1 & -x \\ 0 & 0 \end{pmatrix} \qquad \forall z \in M \\ &= \begin{pmatrix} 1_A - e_2 & z \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -n_2 & -x \\ 0 & 0 \end{pmatrix} \qquad \forall z \in M \end{aligned}$$

are six nil clean expressions for β, which implies $|\eta(\beta)| \geq 6$. That is $\mathrm{Nin}(R) \geq 6$, which is not possible.

Case II:

Let $e_1x = e_2x = x$. Then we have an element
$\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R$ such that

$$\begin{aligned} \alpha &= \begin{pmatrix} e_1 & z \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_1 & -x \\ 0 & 0 \end{pmatrix} & \forall z \in M \\ &= \begin{pmatrix} e_2 & z \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_2 & -x \\ 0 & 0 \end{pmatrix} & \forall z \in M \end{aligned}$$

are six nil clean expressions for α, which implies $|\eta(\alpha)| \geq 6$. That is $\mathrm{Nin}(R) \geq 6$, which is also not possible.

Case III:

Let $e_1x = x$ and $e_2x = 0$. Then we have $(e_1 - e_2)x = x$. Let $j = e_1 - e_2$. Then clearly $j \in \mathrm{Nil}(A)$ and we have

$$\begin{aligned} & jx = x, \\ \Rightarrow \quad & (1_A - j)x = 0, \\ \Rightarrow \quad & x = 0, \quad (\text{ as } 1 - j \in \mathrm{U}(A)). \end{aligned}$$

This is a contradiction.

Case IV:

Let $e_1x = 0$ and $e_2x = x$. Then as in **Case III**, we get a contradiction. Hence if $M \cong C_3$, $\mathrm{Nin}(R)$ is never 4. Suppose $|M| = 4$. If

$$(M, +) \cong C_4,$$

then $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 1$ by **Lemma 2.2.6**, so (ii) holds. If

$$(M, +) \cong C_2 \oplus C_2,$$

then since $\mathrm{Nin}(R) = 4$, by **Lemma 2.2.5**, we have $\mathrm{Nin}(A)\,\mathrm{Nin}(B) \leq 2$. Now if $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 1$ then $(iii)(a)$ holds. If $\mathrm{Nin}(A)\,\mathrm{Nin}(B) = 2$, without loss of generality we can assume $\mathrm{Nin}(A) = 2$ and $\mathrm{Nin}(B) = 1$. So by **Theorem 2.4.1**, we have

$$A = \begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$$

where $\mathrm{Nin}(S) = \mathrm{Nin}(T) = 1$, and $|W|= 2$. To complete the proof, suppose on contrary that

$$eM(1_B - f) + (1_A - e)Mf = 0$$

for some $f^2 = f \in B$ and $e \in \eta(a)$, where $a \in A$ with $|\eta(a)|= 2$. Then $ew = wf$ for all $w \in M$. It is easy to check that $\eta(a) = \{e,\ e + j\}$ where $j = \begin{pmatrix} 0 & w_0 \\ 0 & 0 \end{pmatrix} \in A$ with $0 \neq w_0 \in W$. Thus, for $\gamma := \begin{pmatrix} 1_A - e & 0 \\ 0 & f \end{pmatrix}$,

$$\eta(\gamma) \supseteq \left\{ \begin{pmatrix} 1_A - e & 0 \\ 0 & f \end{pmatrix}, \begin{pmatrix} 1_A - (e + j) & 0 \\ 0 & f \end{pmatrix}, \begin{pmatrix} 1_A - e & w \\ 0 & f \end{pmatrix} : w \in M \right\},$$

so $|\eta(\gamma)|\geq 5$, a contradiction. Hence $(iii)(c)$ holds. Similarly $(iii)(b)$ can be proved.$\square$