Chapter 3

Weak clean index of a ring

3.1 Introduction

In this chapter we have introduced and studied weak clean index of arbitrary rings and characterized all rings with weak clean index 1, 2 and 3.

Definition 3.1.1. *For any element a of R, we define*

$$\chi(a) = \{e \in R \mid e^2 = e \text{ and } a - e \in \mathrm{U}(R) \text{ or } a + e \in \mathrm{U}(R)\}.$$

The weak clean index of R denoted by $\mathrm{Win}(R)$ is defined as

$$\sup\{|\chi(a)|: a \in R\},$$

where $|\chi(a)|$ denotes the cardinality of the set $\chi(a)$.

3.2 Basic properties

Some basic properties related to weak clean index are presented here as a preparation for the chapter.

Lemma 3.2.1. *Let R be a ring and $e, a, b \in R$., Then the following hold:*

(i) For a central nilpotent $n \in R$, $|\chi(n)| = 1$. Whereas for a central idempotent $e \in R$, $|\chi(e)| \geq 1$. Thus $\mathrm{Win}(R) \geq 1$, for any ring R.

(ii) If $a - b \in \mathrm{J}(R)$ then $|\chi(a)| = |\chi(b)|$.

(iii) If $e \in \chi(a)$ then $1 - e \in \chi(1 - a)$ or $1 - e \in \chi(1 + a)$. The converse holds if $2 \in \mathrm{J}(R)$.

(iv) Let σ be an automorphism or anti-automorphism of R. Then $e \in \chi(a)$ iff $\sigma(e) \in \chi(\sigma(a))$; so $|\chi(a)| = |\chi(\sigma(a))|$. In particular $|\chi(a)| = |\chi(uau^{-1})|$, where u is a unit of R.

(v) *If a ring R has at most n units or at most n idempotents, then* $\mathrm{Win}(R) \leq n$*. In particular, if R is a local ring then* $\mathrm{Win}(R) \leq 2$*.*

(vi) *If R is local, then* $\mathrm{Win}(R) = 1$ *iff* $R/\mathrm{J}(R) \cong \mathbb{Z}_2$*.*

(vii) *Let R be a clean ring with* $2 \in \mathrm{U}(R)$*. Then* $\mathrm{Win}(R) = |\chi(2^{-1})|$*, or in other words* $\mathrm{Idem}(R) = \chi(2^{-1})$*.*

Proof. (*i*) Let a be a central nilpotent such that $a^n = 0$ for some $n \in \mathbb{N}$. Then

$$a = (a+1) - 1$$

is a weak clean expression, hence

$$1 \in \chi(a) \text{ thus } |\chi(a)| \geq 1.$$

If possible let $e(\neq 1) \in \chi(a)$, then there exists a $u \in \mathrm{U}(R)$ such that

$$a = u + e \text{ or } u - e.$$

If $a = u - e$, by using binomial expansion and the fact that $a^n = 0$ we have

$$\begin{aligned} 0 =& (u-e)^n \\ =& u^n - \binom{n}{1} e u^{n-1} + \binom{n}{2} e u^{n-2} - \cdots + (-1)^{n-1} e u + (-1)^n e. \end{aligned}$$

This implies

$$u^n \in eR,$$

contradicting the fact that $e \neq 1$. Next, if $a = u + e$, similarly we get a contradiction. Let e be a central idempotent. Then

$$e = 1 - (1 - e)$$

is a weak clean expression for e. Thus $|\chi(e)| \geq 1$. For example, $\bar{4} \in \mathrm{Idem}(\mathbb{Z}_6)$,

$$\chi(\bar{4}) = \{\bar{1}, \bar{3}\}$$

as

$$\overline{4} = \overline{1} + \overline{3} = \overline{5} - \overline{1}$$

where $\overline{3}, \overline{1} \in \mathrm{Idem}(\mathbb{Z}_6)$. This example shows that for central idempotent e, $|\chi(e)|$ need not be equal to one.

(ii) let $w = a - b \in \mathrm{J}(R)$. If $e \in \chi(a)$, we have

$$a + e \in \mathrm{U}(R) \text{ or } a - e \in \mathrm{U}(R).$$

Case I:

If

$$\begin{aligned} & u = a + e \in \ U(R), \text{ then} \\ & u = b + w + e \\ \Rightarrow \quad & b + e = u - w \in \mathrm{U}(R) \\ \Rightarrow \quad & e \in \chi(b). \end{aligned}$$

Case II:

If $v = a - e \in \mathrm{U}(R)$, similarly we get $b - e = v - w \in \mathrm{U}(R)$, so $e \in \chi(b)$. Therefore

$$\chi(a) \subseteq \chi(b).$$

By symmetry

$$\chi(b) \subseteq \chi(a),$$

hence $\chi(a) = \chi(b)$.

(iii) Let $e \in \chi(a)$. Then we have

$$a + e \in \mathrm{U}(R) \text{ or } a - e \in \mathrm{U}(R).$$

If $a - e \in \mathrm{U}(R)$, then we have

$$(1 - a) - (1 - e) = e - a \in \mathrm{U}(R),$$

so $1 - e \in \chi(1 - a)$. Similarly if $a + e \in \mathrm{U}(R)$, then we have

$$(1 + a) - (1 - e) = a + e \in \mathrm{U}(R).$$

Therefore $1 - e \in \chi(1 + a)$.

Conversely, if $(1 - e) \in \chi(1 - a)$, we have

$$(1 - a) - (1 - e) = u \in \mathrm{U}(R) \text{ or } (a - 1) + (1 - e) = v \in \mathrm{U}(R),$$

that is, $a - e = -u$ or $a - e = v$, so in this case $e \in \chi(a)$. If $(1 - e) \in \chi(1 + a)$,

$$(1 + a) - (1 - e) = u \in \mathrm{U}(R) \text{ or } (a + 1) + (1 - e) = v \in \mathrm{U}(R),$$

implying, $a + e = u$ or $a - e = v - 2 \in \mathrm{U}(R)$, as $2 \in \mathrm{J}(R)$. Hence we get $e \in \chi(a)$.

(iv) and (v) are straightforward.

(vi) R is a local ring, so we have $\mathrm{Win}(R) \leq 2$, as $\mathrm{Idem}(R) = \{0, 1\}$. Let

$$R/\mathrm{J}(R) \cong \mathbb{Z}_2.$$

Then, R is uniquely clean[18]. If possible let $\mathrm{Win}(R) = 2$, that is, there exists an element $a \in R$ such that $\{0, 1\} = \chi(a)$. So $a \in \mathrm{U}(R)$ and $a - 1 \in \mathrm{U}(R)$ or $a + 1 \in \mathrm{U}(R)$. If

$$a \in \mathrm{U}(R) \text{ and } u = a - 1 \in \mathrm{U}(R),$$

then we have two clean expressions for a, which is a contradiction. Similarly if

$$a \in \mathrm{U}(R) \text{ and } u = a + 1 \in \mathrm{U}(R),$$

then we have, two clean expressions for u, which is a contradiction, hence $\mathrm{Win}(R) = 1$. Conversely, let $\mathrm{Win}(R) = 1$. Then $\mathrm{In}(R) = 1$ as $\mathrm{In}(R) \leq \mathrm{Win}(R)$. Hence the result follows by **Theorem** 2.1 of [18].

(vii) Let $e \in \mathrm{Idem}(R)$ and let $2 \in \mathrm{U}(R)$. Now we have $(2^{-1} - e) \in \mathrm{U}(R)$, as $2(1-2e)$ is the inverse of $2^{-1} - e$. Therefore $\mathrm{Idem}(R) \subseteq \chi(2^{-1})$, so $\mathrm{Win}(R) = |\chi(2^{-1})|$. □

In a ring R, $q \in R$ is called quasi-regular element, if there is a $p \in R$, such that

$$q + p + qp = 0 = p + q + pq.$$

The set of all all quasi-regular elements of R is denoted by $Q(R)$.

Lemma 3.2.2. *If S is a subring of a ring R, where R and S may not share same identity, then* $\mathrm{Win}(S) \leq \mathrm{Win}(R)$.

Proof. For $a \in R$, let

$$J(a) = J_1(a) \cup J_2(a),$$

where

$$J_1(a) = \{q \in Q(R) : (a-q)^2 = a - q\} \text{ and}$$
$$J_2(a) = \{q \in Q(R) : (q-a)^2 = q - a\}.$$

Claim:

$$\mathrm{Win}(R) = \sup\{|J(b)| : b \in R\}.$$

Note that

$$\mathrm{U}(R) = \{1 + q : q \in Q(R)\}.$$

For any $a \in R$,

$$\chi(a) = \{(a-1) - j : j \in J_1(a-1)\} \cup \{j - (a-1) : j \in \mathrm{J}_2(a-1)\}.$$

Therefore $|\chi(a)| = |J(a-1)|$. Thus

$$\mathrm{Win}(R) = \sup\{|J(b)| : b \in R\}.$$

Because $Q(S) \subseteq Q(R)$ it follows that $\mathrm{Win}(S) \leq \mathrm{Win}(R)$.

Proof of $|\chi(a)| = |J(a-1)|$:

We have $e \in \chi(a)$

$$\Leftrightarrow \quad a - e = u \text{ or } a + e = u, \text{ for some } u \in \mathrm{U}(R)$$
$$\Leftrightarrow \quad a - u = e \text{ or } u - a = e, \text{ for some } u \in \mathrm{U}(R)$$
$$\Leftrightarrow \quad a - 1 - q = e \text{ or } 1 + q - a = e, \text{ for some } q = 1 + u \text{ as } \mathrm{U}(R) = 1 + Q(R)$$
$$\Leftrightarrow \quad (a-1) - q = e \text{ or } q - (a-1) = e$$
$$\Leftrightarrow \quad e \in J_1(a-1) \text{ or } e \in J_2(a-1).$$

□

Theorem 3.2.3. *Let $k \geq 1$ be an integer. Then the following are equivalent for a ring R :*

(i) $\mathrm{Win}(R[[x]]) = k$.

(ii) $\mathrm{Win}(R[x]) = k$.

(iii) *R is abelian and* $\mathrm{Win}(R) = k$.

Proof. By **Lemma 3.2.2**, we have

$$\mathrm{Win}(R) \leq \mathrm{Win}(R[x]) \leq \mathrm{Win}(R[[x]]).$$

Suppose that R is not abelian and e is a non-central idempotent of R. Let $er \neq re$ for some $r \in R$. So either

$$er(1-e) \neq 0 \text{ or } (1-e)re \neq 0.$$

Without loss of generality we may assume that $er(1-e) \neq 0$. For $i = 1, 2, 3, \ldots$

$$\begin{aligned} a :=& (1 + er(1-e)) - e \\ =& (1 + er(1-e)(1+x^i)) - (e + er(1-e)x^i), \end{aligned}$$

where $(1 + er(1-e)(1+x^i)) \in \mathrm{U}(R[x])$, as

$$(1 + er(1-e)(1+x^i))(1 - er(1-e)(1+x^i)) = 1 - (1 + er(1-e)(1+x^i))^2 = 1$$

and $e + er(1-e)x^i \in \mathrm{Idem}(R[x])$. Thus there are infinitely many distinct weak clean expressions of a in $R[x]$. Now suppose R is abelian. It is easy to see that idempotents of $R[[x]]$ are all in R, and for any

$$\alpha = a_0 + a_1 x + a_2 x^2 + \cdots \in R[[x]]$$

where $a_0, a_1, a_2, \ldots \in R$,

$$\chi_{R[[x]]}(\alpha) \subseteq \chi_R(a_0).$$

Thus $|\chi(\alpha)| \leq |\chi(a_0)|$, so $\mathrm{Win}(R[[x]]) \leq \mathrm{Win}(R)$. Hence the result follows. □

3.3 Rings with weak clean index 1, 2 and 3

Theorem 3.3.1. Win$(R) = 1$ *iff* R *is abelian and for any* $0 \neq e^2 = e \in R$, $e \neq u+v$ *for all* $u, v \in \mathrm{U}(R)$.

Proof ($\Rightarrow$) Let $e^2 = e \in R$. For any $0 \neq r \in R$,

$$1 - e = [1 + er(1-e)] - [e + er(1-e)]$$

are two weak clean expressions of $1 - e$; so $e = [e + er(1-e)]$. That is $re = ere$. Similarly, we have $er = ere$. So R is abelian. Suppose that $0 \neq e^2 = e \in R$, $e = u+v$ for some $u, v \in \mathrm{U}(R)$. Then $v = v + 0 = -u + e$ are two weak clean expressions of v, implying $|\chi(v)| \geq 2$, a contradiction.

($\Leftarrow$) Let $a \in R$ has two weak clean expressions,

$$a = u_1 + e_1 \text{ or } u_1 - e_1 \text{ and}$$

$$a = u_2 + e_2 \text{ or } u_2 - e_2$$

for $e_1, e_2 \in \mathrm{Idem}(R)$, $e_1 \neq e_2$ and $u_1, u_2 \in \mathrm{U}(R)$.

Case I:

If $a = u_1 + e_1 = u_2 + e_2$, we have

$$e_1 - e_2 = u_2 - u_1.$$

Define $f := e_1(1 - e_2)$. Then $f = f^2 \in \mathrm{Idem}(R)$. Now

$$\begin{aligned} f =& [e_2 + (u_2 - u_1)](1 - e_2) \\ =& u_2(1 - e_2) - u_1(1 - e_2) \\ =& [u_2(1 - e_2) + e_2] - [u_1(1 - e_2) + e_2]. \end{aligned}$$

As $u_2(1 - e_2) + e_2, u_1(1 - e_2) + e_2 \in \mathrm{U}(R)$, we have, $f = 0$. Hence $e_1 = e_1 e_2$. By symmetry, we have $e_2 = e_1 e_2$. Hence $e_1 = e_2$, a contradiction. So $\chi(a) \leq 1$.

Case II:

If $a = u_1 + e_1 = u_2 - e_2$, then

$$e_1 + e_2 = u_2 - u_1.$$

Define $f := e_1(1-e_2)$. Then $f = f^2$ and

$$\begin{aligned} f =& [-e_2 + (u_2 - u_1)](1-e_2) \\ =& u_2(1-e_2) - u_1(1-e_2) \\ =& [u_2(1-e_2) + e_2] - [u_1(1-e_2) + e_2]. \end{aligned}$$

As $u_2(1-e_2)+e_2, u_1(1-e_2)+e_2$ are units in R, we have $f = 0$, so $e_1 = e_1e_2$. By symmetry, $e_2 = e_1e_2$. Hence $e_1 = e_2$, a contradiction. So $\chi(a) \leq 1$.

Case III:

If $a = u_1 - e_1 = u_2 - e_2$, we have

$$e_1 - e_2 = u_1 - u_2.$$

Define $f := e_1(1-e_2)$. Then $f = f^2 \in \mathrm{Idem}(R)$ and

$$\begin{aligned} f =& [e_2 + (u_1 - u_2)](1-e_2) \\ =& u_1(1-e_2) - u_2(1-e_2) \\ =& [u_1(1-e_2) + e_2] - [u_2(1-e_2) + e_2]. \end{aligned}$$

Since R is abelian, $u_2(1-e_2)+e_2, u_1(1-e_2)+e_2 \in \mathrm{U}(R)$. This is again a contradiction. As in above case $\chi(a) \leq 1$. Thus combining above cases we conclude that $\mathrm{Win}(R) = 1$.

Lemma 3.3.2. *Let $R = A \times B$ be a direct product of rings A and B, such that* $\mathrm{Win}(A) = 1$. *Then* $\mathrm{Win}(R) = \mathrm{Win}(B)$.

Proof. Since A, B are subrings of R, so by **Lemma 3.2.2**,

$$\mathrm{Win}(B) \leq \mathrm{Win}(R).$$

If $\mathrm{Win}(B) = \infty$, then $\mathrm{Win}(R) = \infty$, thus we have $\mathrm{Win}(R) = \mathrm{Win}(B)$. So let

$$\mathrm{Win}(B) = k < \infty$$

where k is a positive integer. So there is a $b \in B$, such that $|\chi(b)| = k$. Now for $(0, b) \in R$, $|\chi(0, b)| = k$, hence $\mathrm{Win}(R) \geq k$. Suppose that $\mathrm{Win}(R) > k$. Then there exists $(a, b) \in R$ that has at least $k+1$ weak clean expressions in R. Let g be an integer such that $1 \leq g \leq k$ and let

$$(a,b) = \begin{cases} (u_i, v_i) + (e_i, f_i), & \text{i = 1, 2, 3, \dots, g} \\ (u_j, v_j) - (e_j, f_j), & \text{j = g+1, g+2, \dots, k, k+1.} \end{cases}$$

are $k+1$ distinct weak clean expressions for (a, b), such that no two (e, f)'s are equal. Now,

$$\begin{aligned} a &= u_i + e_i \qquad (i = 1, 2, 3, \dots, g,) \\ &= u_j - e_j, \qquad (j = g+1, g+2, \dots, k+1) \end{aligned}$$

are weak clean expressions of a in S. Since $|\chi(a)| \leq 1$, all $e_i's$ and $e_j's$ are equal. So

$$\begin{aligned} k+1 =& |\chi((a, b))| \\ =& |\{(e_i, f_i), (e_j, f_j) | i = 1, 2, 3, \dots, g; j = g+1, g+2, \dots, k+1\}| \\ =& |\{e_i, e_j | i = 1, 2, 3, \dots, g, \}| \times |\{f_i, f_j | j = g+1, g+2, \dots, k\}| \\ =& |\chi(a)| \times |\chi(b)| \\ =& |\chi(b)|, \end{aligned}$$

which is a contradiction. This proves the result. □

Definition 3.3.3. *Lee and Zhou [30], called a ring R, a elemental ring. If idempotents of R are trivial and $1 = u + v$, for some $u, v \in \mathrm{U}(R)$.*

Theorem 3.3.4. *For a ring R, $\mathrm{Win}(R) = 2$ iff one of the following holds:*

(i) *R is elemental.*

(ii) *$R = A \times B$, where A is elemental ring and $\mathrm{Win}(B) = 1$.*

(iii) *$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where $\mathrm{Win}(A) = \mathrm{Win}(B) = 1$ and ${}_AM_B$ is a bimodule with $|M| = 2$.*

Proof (⇐) If (i) holds then by the definition of elemental ring, we have $1 = u + v$ for some $u, v \in \mathrm{U}(R)$. Therefore by **Theorem 3.3.1**, $\mathrm{Win}(R) > 1$. Also by **Lemma 3.2.2(v)**, $\mathrm{Win}(R) \leq |\mathrm{Idem}(R)| = 2$. So $\mathrm{Win}(R) = 2$.

If (ii) holds then $\mathrm{Win}(R) = 2$ by (i) and **Lemma 3.3.2**.

If (iii) holds, for $\alpha_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\left\{ \begin{pmatrix} 1 & w \\ 0 & 0 \end{pmatrix} : w \in M \right\} \subseteq \chi(\alpha_0).$$

So,

$$\mathrm{Win}(R) \geq |\chi(\alpha_0)| \geq |M| = 2.$$

For any $\alpha = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in R$,

$$|\chi(\alpha)| = \left| \left\{ \begin{pmatrix} e & w \\ 0 & f \end{pmatrix} \in R : e \in \chi(a), f \in \chi(b), w = ew + wf \right\} \right|.$$

As $|M| = 2$, $|\chi(a)| \leq 1$ and $|\chi(b)| \leq 1$, it follows $|\chi(\alpha)| \leq 2$. Hence $\mathrm{Win}(R) = 2$.

$(\Rightarrow)$ Suppose R is abelian. As $\mathrm{Win}(R) \neq 1$, there exists $(0 \neq) e = e^2 \in R$ such that

$$e = u + v, \text{ where } u, v \in \mathrm{U}(R).$$

So we have $e = eu + ev$, where $eu, ev \in \mathrm{U}(eR)$. Hence

$$\mathrm{Win}(eR) \geq 2.$$

But $\mathrm{Win}(eR) \leq \mathrm{Win}(R) = 2$ by **Lemma 3.2.2**. So $\mathrm{Win}(eR) = 2$. Now $R = A \times B$, where $A = eR$ and $B = (1-e)R$, so it follows that $\mathrm{Win}(B) = 1$. If A has a non trivial idempotent f then

$$A = fA + (e - f)A$$

where

$$f = fu + fv \text{ and } e - f = (e-f)u + (e-f)v.$$

Now $fu, fv \in \mathrm{U}(fA)$ and $(e-f)u, (e-f)v \in \mathrm{U}((e-f)A)$, so by **Theorem 5** of [30] we have

$$\mathrm{In}(fA) \geq 2 \text{ and } \mathrm{In}((1-f)A) \geq 2,$$

so

$$\mathrm{In}(A) \geq 2 \times 2 = 4.$$

As $\text{In}(R) \leq \text{Win}(R)$, this is a contradiction. Thus (i) holds if $e = 1$ and (ii) holds if $e \neq 1$. Suppose R is not abelian and let $e^2 = e \in R$ be a non central idempotent. If

$$eR(1-e) \neq 0 \text{ and } (1-e)Re \neq 0,$$

then for

$$0 \neq x \in eR(1-e) \text{ and } 0 \neq y \in (1-e)Re$$

we have

$$\begin{aligned} 1 - e &= (1 + x) - (x + e) \\ &= (1 + y) - (y + e). \end{aligned}$$

Therefore $|\chi(1-e)| \geq 3$, which is a contradiction. So without loss of generality we can assume that

$$eR(1-e) \neq 0 \text{ and } (1-e)Re = 0.$$

The Peirce decomposition of R gives

$$R = \begin{pmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{pmatrix}.$$

As above $2 = \text{Win}(R) \geq |eR(1-e)|$; so $|eR(1-e)| = 2$. Write

$$eR(1-e) = \{0, x\}.$$

Suppose $\text{Win}(eRe) = 2$. Then there exists $a \in R$ such that $|\chi(a)| = 2$. Thus we have the following cases.

Case I:

Let $a = u_1 + e_1 = u_2 + e_2$, where $u_1, u_2 \in \mathrm{U}(eRe)$ and $e_1, e_2 \in \mathrm{Idem}(eRe)$. If $e_1 x = 0$, we have for $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R$,

$$\begin{aligned} A &= \begin{pmatrix} u_1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} u_2 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} e_2 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} u_1 & x \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} e_1 & x \\ 0 & 1 \end{pmatrix} \end{aligned}$$

are three distinct weak clean expressions of A in R, which implies $|\chi(A)| \geq 3$, a contradiction. If $e_1 x = x$, then for $B = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$,

$$\begin{aligned} B &= \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} u_2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} u_1 & x \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e_1 & x \\ 0 & 0 \end{pmatrix} \end{aligned}$$

are three distinct weak clean expressions of B in R, which implies $|\chi(B)| \geq 3$, a contradiction.

Case II:

Let $a = u_1 - e_1 = u_2 + e_2$, where $u_1, u_2 \in \mathrm{U}(eRe)$ and $e_1, e_2 \in \mathrm{Idem}(eRe)$. So if $e_1 x = 0$, we have for $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R$

$$\begin{aligned} A &= \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} e_1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} u_2 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} e_2 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} u_1 & x \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} e_1 & x \\ 0 & 1 \end{pmatrix} \end{aligned}$$

are three distinct weak clean expressions of A in R, which implies $|\chi(A)| \geq 3$, a contradiction.

If $e_1 x = x$ then for $B = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\begin{aligned} B &= \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} u_2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} u_1 & x \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} e_1 & x \\ 0 & 0 \end{pmatrix} \end{aligned}$$

are three distinct weak clean expressions of B in R, which implies $|\chi(B)| \geq 3$, again a contradiction.

Case III:

Let $a = u_1 - e_1 = u_2 - e_2$, where $u_1, u_2 \in \mathrm{U}(eRe)$ and $e_1, e_2 \in \mathrm{Idem}(eRe)$. Then we get a contradiction similar to **Case I**.

This shows that $\mathrm{Win}(eRe) = 1$. Similarly $\mathrm{Win}((1-e)R(1-e)) = 1$. □

Theorem 3.3.5. Win$(R) = 3$ *iff* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *where* Win$(A) =$ Win$(B) = 1$ *and* ${}_AM_B$ *is a bimodule with* $|M| = 3$.

Proof. ($\Leftarrow$) For $\alpha_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\left\{ \begin{pmatrix} 1 & w \\ 0 & 0 \end{pmatrix} : w \in M \right\} \subseteq \chi(\alpha_0).$$

So,

$$\text{Win}(R) \geq |\chi(\alpha_0)| \geq |M| = 3.$$

For any $\alpha = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in R$,

$$|\chi(\alpha)| = \left| \left\{ \begin{pmatrix} e & w \\ 0 & f \end{pmatrix} \in R : e \in \chi(a), f \in \chi(b), w = ew + wf \right\} \right|.$$

As $|M| = 3$, $|\chi(a)| \leq 1$ and $|\chi(b)| \leq 1$ it follows $|\chi(\alpha)| \leq 3$, hence Win$(R) = 3$.

($\Rightarrow$) Suppose Win$(R) = 3$. From the proof of **Theorem 3.3.4**, we see that an abelian ring not satisfying condition (i) and (ii), contains a subring whose weak clean index is greater than 4. Therefore R must be non abelian.

Let e be a non central idempotent in the ring R. Then the Peirce decomposition of R gives

$$R = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

Let $A = eRe$, $B = (1-e)R(1-e)$, $M = eR(1-e)$, $N = (1-e)Re$. Suppose $|M| \neq 0$ and $|N| \neq 0$. As

$$\chi(1-e) \supseteq \{e - x, e - y : x \in M, 0 \neq y \in N\},$$

it follows that

$$3 = \text{Win}(R) \geq |\chi(1-e)| > |M| + |N| - 1.$$

Therefore $|M| = |N| = 2$. Write

$$M = \{0, x\} \text{ and } N = \{0, y\}.$$

Note that

$$2x = 0 = 2y.$$

If $xyx = 0$, then $(x + y + xy + yx)^4 = 0$ and

$$\chi(1 - e) \supseteq \{e, e - x, e - y, e + x + y + xy + yx\},$$

so $\mathrm{Win}(R) \geq 4$, a contradiction.
If $yxy = 0$, then $(x + y + xy + yx)^4 = 0$ and

$$\chi(2 - e) \supseteq \{1 - e, 1 - e + x, 1 - e + y, 1 - e + x + y + xy + yx\},$$

therefore $\mathrm{Win}(R) \geq 4$, a contradiction. Hence $xyx \neq 0$ and $yxy \neq 0$. It follows that

$$xyx = x \text{ and } yxy = 0.$$

Let $f = xy$ and $g = yx$. Clearly f, g are idempotents. So we have

$$R \supseteq L := \begin{pmatrix} fRf & M \\ N & gRg \end{pmatrix}.$$

By **Lemma 3.2.2**, $\mathrm{Win}(L) \leq 3$, but for $\alpha = \begin{pmatrix} 0 & x \\ y & g \end{pmatrix}$ we have

$$\begin{aligned} \alpha &= \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix} \\ &= \begin{pmatrix} 0 & x \\ y & g \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} f & x \\ y & 0 \end{pmatrix} + \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \\ &= \begin{pmatrix} f & 0 \\ y & g \end{pmatrix} + \begin{pmatrix} f & x \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} f & x \\ 0 & g \end{pmatrix} + \begin{pmatrix} f & 0 \\ y & 0 \end{pmatrix}. \end{aligned}$$

That is $|\chi(\alpha)| \geq 5$ in L, which is a contradiction. So either $|M| = 0$ or $|N| = 0$.

Without loss of generality we may assume that $|N|= 0$. So

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}.$$

Clearly

$$2 \leq |M| \leq 3 = \text{Win}(R).$$

By **Lemma 3.2.2**, $\text{Win}(A) \leq 3$. To prove that $|M|= 3$, on contrary let $M = \{0, x\}$. Assume $\text{Win}(A) = 2$. Then there exists at least one $a \in A$ such that $|\chi(a)| \geq 2$.

Case I:

Let $a = u_1 + e_1 = u_2 - e_2$ be two distinct weak clean expressions of a in A, where $u_1, u_2 \in \text{U}(A)$ and $e_1, e_2 \in \text{Idem}(A)$. Then $e_1x = u_2x - u_1x - e_2x = -e_2x + x - x = -e_2x = e_2x$. If $e_1x = 0$, then for $\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, we have $\chi(\alpha) \supseteq \left\{ \begin{pmatrix} e_i & w \\ 0 & 1 \end{pmatrix} : i = 1, 2; w \in M \right\}$, showing that $\text{Win}(R) \geq 4$, which is not possible. If $e_1x = x$, then for $\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, we have $\chi(\alpha) \supseteq \left\{ \begin{pmatrix} e_i & w \\ 0 & 0 \end{pmatrix} : i = 1, 2; w \in M \right\}$, showing that $\text{Win}(R) \geq 4$, which is a contradiction.

Similarly in **Case II** letting $a = u_1 + e_1 = u_2 + e_2$ be two distinct weak clean expressions and in **Case III** letting $a = u_1 - e_1 = u_2 - e_2$ be two distinct weak clean expressions of a in A, where $u_1, u_2 \in \text{U}(A)$ and $e_1, e_2 \in \text{Idem}(A)$, we get contradictions. Therefore $\text{Win}(A) = 1$, similarly $\text{Win}(B) = 1$. Now by **Theorem 3.3.4**, we have $\text{Win}(R) = 2$, a contradiction, hence $|M|= 3$.

Now it remains to show that $\text{Win}(A) = \text{Win}(B) = 1$. For $e^2 = e \in A$, we have

$$M = eM \oplus (1 - e)M.$$

Without loss of generality, let $|eM| \neq 0$. On contrary let us assume $\text{Win}(A) > 1$. So we have $a \in A$ such that $|\chi(a)| \geq 2$, i.e., we have at least two distinct weak clean expressions of a in A.

Case I:

If $a = u_1 + e_1 = u_2 - e_2$, where $u_1, u_2 \in \mathrm{U}(A)$ and $e_1, e_2 \in \mathrm{Idem}(A)$ such that $e_1 \neq e_2$. Let $M = e_1 M$. Then for $w \in M$ and for $\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$\alpha = \begin{pmatrix} u_2 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} e_2 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} u_1 & -w \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e_1 & w \\ 0 & 0 \end{pmatrix},$$

implying $\chi(\alpha) \geq 4$, a contradiction. If $e_1 M = 0$, for $\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ we have

$$\alpha = \begin{pmatrix} u_2 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} e_2 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} u_1 & -w \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e_1 & w \\ 0 & 1 \end{pmatrix},$$

implies $\chi(\alpha) \geq 4$, thus a contradiction.

Similarly in **Case II**, letting $a = u_1 + e_1 = u_2 + e_2$ be two distinct weak clean expressions and in **Case III**, letting $a = u_1 - e_1 = u_2 - e_2$ be two distinct weak clean expressions of a in A, where $u_1, u_2 \in \mathrm{U}(A)$ and $e_1, e_2 \in \mathrm{Idem}(A)$, we get contradictions. Therefore we have $\mathrm{Win}(A) = 1$. Similarly $\mathrm{Win}(B) = 1$. $\square$