Chapter 5: A Study of Mortality Models for Oldest-Old Force of Mortality in Assam

5.1 Introduction

A very extensive literature, for the most part, created in the most recent decades, concentrates on mortality at old and very old ages and relevant possible causes. The related research work includes demography, actuarial sciences, gerontology, biology, biostatistics, epidemiology, etc. Exact evaluations of mortality at advanced ages are essential to improving forecasts of mortality and the population size of the oldest-old age group. The estimates of mortality force at extreme ages are difficult because of small numbers of survivors to these ages in most countries. The relation between mortality and age is the oldest topic in demography. The first explanatory model and the most powerful parametric mortality modelling the literature is that proposed by Benjamin Gompertz [23]. He recognized that an exponential pattern in age captured the behavior of human mortality for large portions of the life table [30]. Gompertz [23] assumed that the force of mortality μ_x at age $x,$ was an exponential function of age, $\mu_x = ae^{bx}$. Over much of the age range, this model still gives an excellent approximation at higher ages, be that as it may, the law does not work so well. The paper Bongaarts [7] talked about that for some reasons the Gompertz model gives a satisfactory fit to adult mortality rates, yet this model underestimates of actual mortality at youngest adult ages (under 40) and overestimates at the oldest ages (over 80). As far back as Gompertz, many models have been recommended to mathematically describe survival and mortality curves [56], of which the Gompertz model and the Weibull model [62] are the most generally used at present [21], [22]. Interestingly, the Gompertz model is more commonly used to describe biological systems, whereas the Weibull model is more commonly applicable to technical devices [21], [22]. Makeham [46] added an additional parameter to this model to consider the force of accidental death, thought to be a constant independent of age, and obtained the model $\mu_x = ae^{bx} + c$. The paper Bongaarts [7] examined that the Makeham model represents a clear improvement over the Gompertz model at younger ages, but it still overestimates mortality at the oldest ages. Another model developed by an actuary, Perks [49], which has not gotten as much consideration as the over two models is the logistic model, where the force of mortality at age x is given by the 4-parameter function $\mu_x = c + \frac{ae^{bx}}{1+de^{bx}}$. Beard [4] obtained the 3-parameter model by assuming that the parameter $c = 0$ in the logistic model, $\mu_x = \frac{ae^{bx}}{1+de^{bx}}$. Kannisto [34], a demographer, used the simple 3-parameter model $\mu_x = \frac{ae^{bx}}{1+ae^{bx}} + c$.

It is noted that the Gompertz ($c = 0$; $d = 0$), Makeham ($d = 0$), Beard ($c = 0$) and Kannisto ($c = 0$; $d = a$) models are all special cases of the logistic model. The paper by Doray [19] discussed that logistic type models for the force of mortality provide a better fit to mortality data of people aged over 85 than Makeham's models where the force of mortality increases exponentially with age. Thatcher et al. [56] fit the Gompertz, logistic, Kannisto, and Weibull models also the Heligman and Pollard [29] model and the quadratic model to mortality data of aged people in 13 industrialized countries for the periods 1960-70, 1970-80, 1980-90 and for the cohort born in 1871-80 by using maximum likelihood method. The data utilized were deaths at ages 85 and over for the quadratic model and ages 80 and over for all the other models. The best fit was reliably given by the Kannisto and logistic models for all countries in every period and for the cohort data. All the models mentioned above create close estimations of μ_x at ages 80 to 95. After age 95, the Gompertz and Makeham forces of mortality continue to increase exponentially with age, while for the Kannisto, Beard and logistic models, μ_x tends asymptotically to a constant as x increases.

A descriptive model used by Coale & Kisker [14] in a limited range of ages. They fitted $\ln(\mu_x)$ by a quadratic function of over x for the purpose of interpolating in the range of ages from 85 to 110. The model is given by,

$$\mu_x = ae^{bx}k^{x^2},$$

$$\Rightarrow \ln \mu_x \;= A + bx + Cx^2, C < 0,$$

where $A = \ln a$ and $C = \ln k$

This model is also known as quadratic model. Wilmoth [64] used the model for estimating μ_x at age 110 from data which extended above age 85.

In this chapter, the following models namely, Gompertz, Makeham, Logistic, Beard, and Kannisto models are considered to describe the force of human mortality at oldest old ages for Assam.

5.2 Objectives

The main focus of this chapter is to select the best fit model for projection of oldest-old mortality rates based on the complete life table of Assam for the period 2009-13. In this chapter, the six mortality models: Gompertz, Makeham, Logistic, Kannisto, Beard and Coale-Kisker models have been analysed based on reliable mortality data of Assam for the period 2009-13.

5.3 Materials and Methods

5.3.1 Data source

The original data of abridged life tables of Assam for total, rural and urban area for both the gender are taken from sample registration system (SRS). Then the complete life tables are constructed from abridged life table by using the methodology used by Saikia and Borah [1].

5.3.2 Levenverg - Marquardt iteration method

The four mortality models i.e. Gompertz, Makeham, Logistic, Kannisto, Beard and Coale-Kisker models are considered here. The parameters of the models are estimated using the Levenverg - Marquardt iteration method as discussed in the previous chapter based on reliable mortality data of Assam for the period 2009-13. The Levenverg - Marquardt iteration method is also known as the damped least-squares method provides a numerical solution to the problem of minimizing a function, generally nonlinear, over a space of parameters of the function. Matlab version 7.11.0 is used for the estimation of the parameters.

5.4 Results and Discussion

The six mortality models, Gompertz, Makeham, Logistic, Kannisto, Beard and Coale-Kisker have been fitted to force of mortality from ages 80 to 100, by estimating the parameters of the models using Levenverg-Marquardt iteration method. The investigation consists of total, rural and urban area for both the gender of Assam. The force of mortality rate for all is increasing except female in the rural area. To fit the various model, the age x has been normalized by mean 90 and standard deviation 6.205. After fitting the models for three cases total, rural and urban area of Assam for both the gender, the goodness-of-fit statistics is evaluated for choosing the best fit model. There are two goodness-of-fit measures are considered. The first measure is the root mean square error (RMSE) and the other is the sum squared error (SSE). The coefficient of determination (R^2) is also evaluated. The estimated goodness-of-fit measures for total, rural and urban area population of Assam are presented from **Table 5.1** to **Table 5.3** for both male and female. **Table 5.4** shows the estimated parameters of the best fit model for the three cases total, rural and urban for both the gender. The observed values of total, rural and urban area along with the best fit estimated values are given by **Table 5.5** to **Table 5.7**. The best fitting curve for the three cases are represented from **Figure 5.1** to **Figure 5.6** respectively for male and female separately.

In view of our outcomes, it is observed that all six models are comparatively robust in adapting themselves to various and changing mortality levels and patterns. The evaluated

R^2 values indicate that all the models are efficient. Based on our selection criteria as discussed in the previous chapter, the six mortality models are arranged for their validity in the following order, beginning with the least SSE and RMSE. However, between the first and second positions is difficult to establish, as the respective values of RMSE and SSE of these models are roughly equal.

5.4.1 For Total Male Population:

1. Beard: It is seen that among the six models, the evaluated values of SSE and RMSE are least for the Beard model. The fit is great and the deviations little. This model is in every way the best model of the six in describing the mortality for a total male population of Assam.

2. Coale and Kisker: This quadratic model gives likewise acceptable outcomes for the total male population in Assam, yet not exactly comparable to the one by the Beard model. A different pattern of old-age central rates of mortality is provided by the Coale-Kisker model. Here, the exponential rate of mortality increase at very old ages is not constant, as in the classical Gompertz model, but declines linearly.

3. Logistic: The performance of this model is good for ages 80-94 and from ages 95 it is bad as by the over two models.
4. Gompertz: It is notable that this traditional model overestimates the rise of mortality with age and it is therefore not surprising that the other models, grew substantially later, are developed on it.
5. Makeham: The fitted value of the parameter c in the Makeham model ($\mu_x = ae^{bx} + c$) is little, so the contrast between the Gompertz and Makeham models is unimportant.
6. Kannisto: The assessed SSE and RMSE values are most noticeably unfavorable given by three parameter Kannisto model. The fit is not good at all and hence it is not useful for a total male in Assam.

5.4.2 For Total Female Population:

1. Gompertz, Makeham, and Beard: The three models in particular Gompertz, Makeham and Beard models give similar SSE, RMSE values for the total female population in

Assam and are all far closer to the observed values. Hence, all the three models are acceptable for describing the mortality for a total female in Assam. In Gompertz and Makeham, the force of mortality increases exponentially with age and is an unbounded function.

2. Coale-Kisker: The estimated value given by this model is closely similar to the observed values however not exactly great as given by the over three models.
3. Kannisto: The description of mortality given by this simple model is closely similar to that of the Kannisto model.
4. Logistic: This model is not all accounts the best. The performance of the logistic model is not satisfactory for a total female in Assam. The fit is bad, deviations higher.

5.4.3 For Male Population in Rural Area:

1. Beard: Beard model is the first choice for describing mortality pattern of the rural male population like the total male population of Assam.
2. Makeham: The performance of this model is not so bad for describing mortality pattern for the rural male population in Assam.
3. Gompertz: This model gives a very good estimate value from ages 80-88 and from ages 89 onward it overestimates the mortality.
4. Coale-Kisker: Though this model does not give so good fit but fitted the data decisively better than logistic type model.
5. Logistic: This model impressively overestimates the mortality, the error, with watched information widening progressively.
6. Kannisto: Like total male population, Kannisto model does not fit the male data in the rural area nearly so well.

5.4.4 For Female Population in Rural Area:

1. Kannisto: The Kannisto model is simply a special case of the logistic model, but it is useful only for the rural female population. This three-parameter model is the first choice as it gives least SSE and RMSE than the other five models. This model gives a very good fit at ages 80-100.

2. Beard, Coale-Kisker: As SSE and RMSE values are similar for both the model Beard and Coale-Kisker so it is difficult to a choice between them. The fitting of both the models to data at ages 80-100 is almost identical.

3. Makeham: Makeham model fits the data much better than the exponential Gompertz model Logistic model.

4. Logistic: The four parameter logistic model does not give satisfactory results as the estimated values are found to be far to observed data.

5. Gompertz: This model is found to give the worst fit for a male in an urban area and hence not suitable for describing the force of mortality.

5.4.5 For Male Population in Urban Male:

1. Beard: For urban area male population, this model is the first choice as it is given by least SSE and RMSE values.

2. Coale-Kisker: The quadratic model gives a very good fit at ages 85, but it fails to describe even approximately the values below 85 years.

3. Logistic: This model fit the data much better than the exponential Gompertz model and Makeham model but not so well as given by the above two.

4. Makeham: The estimated values of the force of mortality given by this model are higher than the observed values at all the ages 80-100 hence not suitable for projecting force of mortality for the urban male population.

5. Gompertz: The Gompertz model seems to be not suitable for prediction of old age mortality.

6. Kannisto: For a male in an urban zone, the Kannisto model is the only case where the observed value is significantly different from the prediction.

5.4.6 For Female Population in Urban Male:

1. Makeham: It is observed from our result that the Makeham model gives least RMSE, SSE among the six models.

2. Logistic: Though This is the second choice for the urban population but it gives similar estimations as given by theMakeham model. So it can also be useful for the female in the urban area.
3. Gompertz: For an urban female, this classical model gives a good fit from ages 80-85 and then it diverges.
4. Coale-Kisker: This model gives a very good fit at ages 85-90 and, from that point forward, a basically unbiased dispersing as far as age 103; but it fails to describe even approximately the values below 85 years.
5. Beard: The estimated values given by the Beard model are far closer to the Gompertz and it is not easy to rank them.
6. Kannisto: Moreover, the fitting of the Kannisto model seems to be not satisfactory and hence it is not acceptable for urban female population also.

Table 5.1: Estimated Values of SSE, RMSE, $\boldsymbol{R^2}$in Total Area.

Model	Male			Female		
	SSE ($\times 10^{-5}$)	RMSE ($\times 10^{-5}$)	R^2	SSE ($\times 10^{-10}$)	RMSE ($\times 10^{-6}$)	R^2
Gompertz	0.1698	0.0897	.9997	0.0001	0.0021	.9999
Makeham	0.1858	0.0941	.9997	0.0001	0.0021	.9999
Logistic	0.0054	0.0161	.9997	0.0240	0.0338	.9899
Beard	0.0011	0.0071	.9999	0.0001	0.0021	.9999
Kannisto	0.1864	0.0942	.9549	0.0005	0.0048	.9998
Coale –kisker	0.0020	0.0097	.9998	0.0002	0.0034	.9998

Table 5.2: Estimated Values of SSE, RMSE, $\boldsymbol{R^2}$ in Rural Area.

Model	Male			Female		
	SSE ($\times 10^{-5}$)	RMSE ($\times 10^{-5}$)	R^2	SSE ($\times 10^{-10}$)	RMSE ($\times 10^{-6}$)	R^2
Gompertz	0.0077	0.0191	.9997	0.0858	0.0639	.9997
Makeham	0.0054	0.0160	.9998	0.0262	0.0353	.9998
Logistic	0.0537	0.0537	.9996	0.0755	0.0600	.9898
Beard	0.0016	0.0087	.9999	0.0009	0.0064	.9999
Kannisto	0.0880	0.0650	.9849	0.0002	0.0028	.9999

Coale -kisker	0.0110	0.0228	.9997	0.0009	0.0067	.9999

Table 5.3: Estimated Values of SSE, RMSE, R^2 in Urban Area.

Model	Male			Female		
	SSE ($\times 10^{-5}$)	RMSE ($\times 10^{-5}$)	R^2	SSE ($\times 10^{-5}$)	RMSE ($\times 10^{-5}$)	R^2
Gompertz	0.0018	0.0092	.9997	0.0161	0.0191	.9997
Makeham	0.0014	0.0080	.9997	0.0129	0.0160	.9997
Logistic	0.0035	0.0130	.9998	0.0144	0.0537	.9998
Beard	0.0030	0.0119	.9999	0.0186	0.0087	.9999
Kannisto	9.1119	0.6587	.9996	0.0144	0.0262	.9996
Coale -kisker	0.0010	0.0214	.9999	0.0164	0.0228	.9999

Table 5.4: Estimated parameters of the best fit models for total, rural and urban area for both male and female.

Parameters	Confidence bounds of parameters (95%)	Total		Rural		Urban	
		Male	Female	Male	Female	Male	Female
		Beard (a, b, d)	Beard (a, b, d)	Beard (a, b, d)	Kannisto $(a, b. c)$	Beard (a, b, d)	Logistic $(a, b. c, d)$
a	$\hat{a}$	0.1911	0.1035	0.1927	0.0865	0.1988	0.3098
	L.B	0.1911	0.1029	0.1927	0.0852	0.1988	0.3098
	U.B	0.1912	0.1041	0.1928	0.0878	0.1988	0.3099
b	$\hat{b}$	0.2688	0.0653	0.2455	-0.0551	0.4941	0.8136
	L.B	0.2687	0.0649	0.2454	-0.0559	0.1940	0.8136
	U.B	0.2688	0.0657	0.2455	-0.0545	0.1941	0.8137
c	$\hat{c}$				0.0069		6.081×10^{-6}
	L.B				0.0058		-9.701×10^{-6}
	U.B				0.0080		2.186×10^{-6}
d	$\hat{d}$	-6.187×10^{-5}	0.0049	-.00012		2.432×10^{-5}	4.831×10^{-6}
	L.B	-0.0002784	-0.0009	-0.0003		-2.751×10^{-5}	-2.264×10^{-5}
	U.B	0.0001547	0.0106	0.00016		7.615×10^{-5}	3.231×10^{-5}

Table 5.5: Observed and Estimated Force of Mortality Given by the Best Fit Model in Total Area.

Age	Male		Female			
	Observed	Estimated Beard	Observed	Estimated		
				Gompertz	Makeham	Beard
80	0.12394	0.12392	0.09275	0.09276	0.09275	0.09276
81	0.12942	0.12941	0.09373	0.09374	0.09372	0.09374
82	0.13515	0.13514	0.09472	0.09472	0.09471	0.09472
83	0.14114	0.14112	0.09571	0.09572	0.09571	0.09572
84	0.14738	0.14737	0.09672	0.09673	0.09672	0.09673
85	0.15391	0.15389	0.09774	0.09775	0.09773	0.09775
86	0.16072	0.16070	0.09877	0.09878	0.09876	0.09877
87	0.16784	0.16782	0.09981	0.09981	0.09980	0.09981
88	0.17527	0.17525	0.10086	0.10087	0.10085	0.10087
89	0.18303	0.18301	0.10192	0.10193	0.10191	0.10193
90	0.19113	0.19111	0.10299	0.10300	0.10299	0.10300
91	0.19960	0.19957	0.10408	0.10408	0.10407	0.10408
92	0.20843	0.20841	0.10517	0.10518	0.10517	0.10518
93	0.21766	0.21764	0.10628	0.10629	0.10627	0.10629
94	0.22730	0.22727	0.10740	0.10741	0.10739	0.10740
95	0.23736	0.23734	0.10853	0.10854	0.10852	0.10853
96	0.24787	0.24784	0.10967	0.10968	0.10966	0.10968
97	0.25884	0.25882	0.11082	0.11083	0.11082	0.11083
98	0.27030	0.27028	0.11199	0.11200	0.11198	0.11200
99	0.28227	0.28224	0.11317	0.11318	0.11316	0.11317
100	0.29477	0.29474	0.11436	0.11437	0.11435	0.11437

Table 5.6: Observed and Estimated Force of Mortality Given by the Best Fit Model in Rural Area.

Age	Male		Female	
	Observed	Estimated Beard	Observed	Estimated Kannisto
80	0.12977	0.12974	0.09330	0.09328
81	0.13500	0.13498	0.09260	0.09259

82	0.14045	0.14043	0.09191	0.09189
83	0.14612	0.14610	0.09122	0.09120
84	0.15202	0.15199	0.09053	0.09052
85	0.15815	0.15813	0.08985	0.08984
86	0.16454	0.16451	0.08918	0.08917
87	0.17118	0.17115	0.08851	0.08850
88	0.17809	0.17806	0.08785	0.08784
89	0.18527	0.18525	0.08719	0.08718
90	0.19275	0.19272	0.08653	0.08653
91	0.20053	0.20050	0.08588	0.08588
92	0.20862	0.20859	0.08524	0.08524
93	0.21704	0.21701	0.08460	0.08460
94	0.22580	0.22577	0.08396	0.08397
95	0.23492	0.23489	0.08333	0.08334
96	0.24440	0.24437	0.08271	0.08271
97	0.25426	0.25423	0.08209	0.08209
98	0.26452	0.26449	0.08147	0.08148
99	0.27520	0.27517	0.08086	0.08087
100	0.28631	0.28628	0.08025	0.08026

Table 5.7: Observed and Estimated Force of Mortality Given by the Best Fit Model in Urban Area.

Age	Male		Female	
	Observed	Estimated Beard	Observed	Estimated Logistic
80	0.08965	0.08966	0.08351	0.08349
81	0.09708	0.09709	0.09521	0.09519
82	0.10513	0.10514	0.10854	0.10853
83	0.11384	0.11385	0.12375	0.12373
84	0.12327	0.12329	0.14109	0.14107
85	0.13349	0.13350	0.16086	0.16083
86	0.14455	0.14457	0.18339	0.18336
87	0.15653	0.15655	0.20909	0.20905

88	0.16951	0.16953	0.23838	0.23834
89	0.18356	0.18358	0.27178	0.27173
90	0.19877	0.19880	0.30985	0.30980
91	0.21524	0.21527	0.35326	0.35321
92	0.23308	0.23311	0.40276	0.40269
93	0.25240	0.25244	0.45918	0.45911
94	0.27332	0.27336	0.52352	0.52344
95	0.29597	0.29601	0.59686	0.59677
96	0.32050	0.32055	0.68048	0.68038
97	0.34706	0.34712	0.77582	0.77570
98	0.37583	0.37589	0.88452	0.88438
99	0.40697	0.40704	1.00844	1.00829
100	0.44070	0.44078	1.14972	1.14956

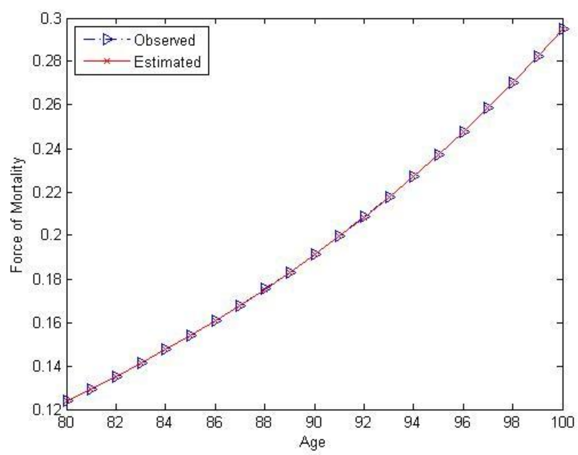

Figure 5.1: Observed and estimated force of mortality at ages 80-100 given by best fit Beard model for male (total).

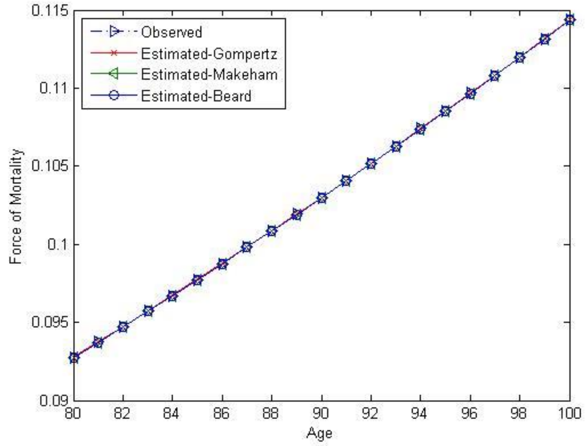

Figure 5.2: Observed and estimated force of mortality at ages 80-100 given by best fit three models Gompertz, Makeham and Beard for female (total).

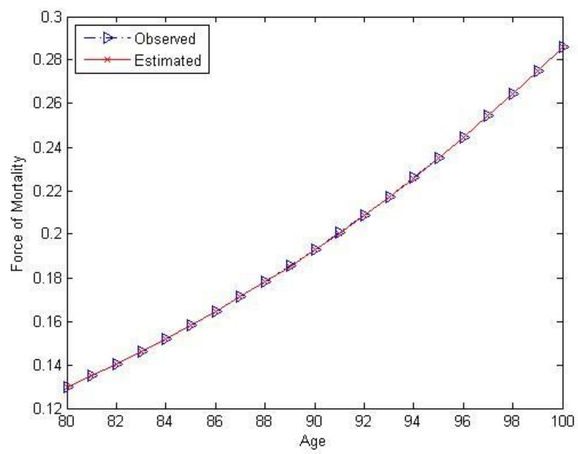

Figure 5.3: Observed and estimated force of mortality at ages 80-100 given by best fit Beard model for male (rural).

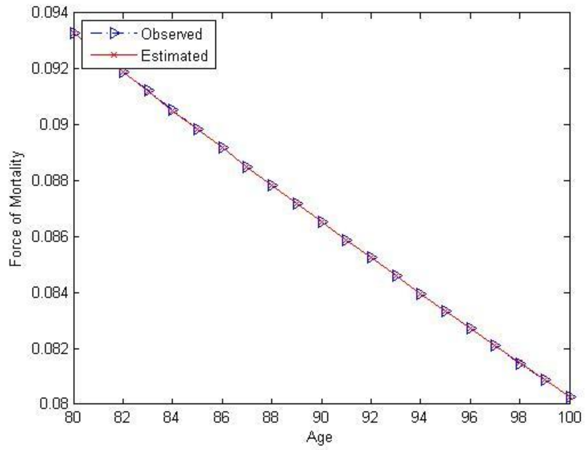

Figure 5.4: Observed and estimated force of mortality at ages 80-100 given by best fit Kannisto model for female (rural).

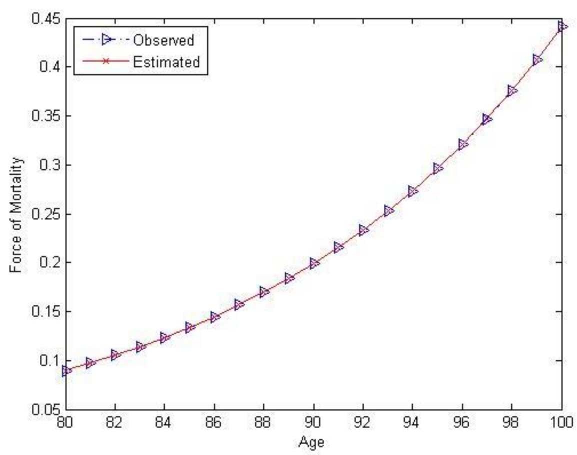

Figure 5.5: Observed and estimated force of mortality at ages 80-100 given by best fit Beard model for male (urban).

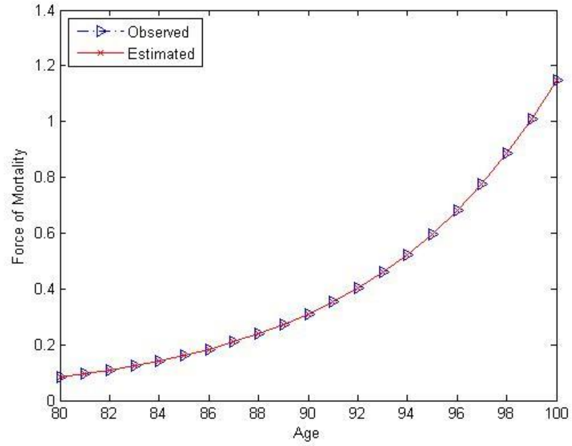

Figure 5.6: Observed and estimated force of mortality at ages 80-100 given by best fit Logistic model for female (urban).

5.5 Conclusion

In this chapter, six mortality models Gompertz, Makeham, Logistic, Beard, Kannisto and Coale-Kisker have been fitted for total, rural and urban area for both male and female in Assam by using Levenberg Marquardt iteration method. No method, however, can guarantee best results in all circumstances. In the case of mathematical functions, the results will be satisfactory to the extent that the actual force of mortality conforms to the functional form used. Based on our results, the best-fit mortality model is selected for describing the force of mortality of Assam for total, rural and urban area for both the gender. In view of our results, it is said that the three models specifically Gompertz, Makeham and Beard models can be used for describing the mortality for the female (total area) population in Assam. On the other hand for the male population, Beard model is a more reasonable model in describing the mortality. In the case of rural area, Beard and Kannisto models give preferred outcome over the others for male and female separately. As the fitted value of the parameter c in the Makeham model is very small so the difference between Gompertz and Makeham model is negligible. So the four parameter Logistic model is the first preference for a female in an urban area in Assam. It is noted that the Coale-Kisker model is more appropriate for a male in urban region and From our results, it is seen that rate of increase in mortality with age and level of mortality is highest for urban area female population. The Coale-Kisker model which is also known as the quadratic model is pragmatic, however it has minimal hypothetical support. (On the contrary: if extended indefinitely it would imply that the force of mortality will eventually reach zero, and this can only happen if immortality is possible). Basically, the quadratic model uses a parabola as an approximation to a more general curve. The Beard model is a special case of the logistic model, yet it is a valuable one. It will only work if the parameter c in the logistic model is small. A decreasing rate of increase in the force of mortality and, in particular, a mortality leveling-off is the features of mortality laws belonging to the logistic class. At long last, we may conclude that the Beard model approximation is the best of the six models for projection of oldest old force of mortality for both male and female population in total area of Assam. In case of rural area, Beard and Kannisto models are appropriate for male and female respectively. For urban area, Makeham model approximation is the best of the six models for male. As a result of the comparison we conclude that the the four parameter Logistic model lead to the best results for female population in projecting the force of mortality in Assam.

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