Chapter 3: Parameter estimation of Weibull growth models in forestry

3.1 Introduction

The Weibull model was first introduced by Ernst Hjalmar Waloddi Weibull in 1951. Initially, it was described as a statistical distribution. It has many applications in population growth, agricultural growth, height growth and is also used to describe survival in cases of injury or disease or in population dynamic studies [30]. In 1997, Lianjun Zhang [93] used this model to describe tree height-diameter data of ten conifer species. In the paper by Fekedulegn et al. [22] used this model for the top height data of Norway spruce from the Bowmont Norway spruce Thinning Experiment. Colbert et al. [15] tried to define some character developments such as forest trees height growth and diameter development by using the model. The paper by Karadavut et al. [41] used this model to evaluate the relative growth rate of silage corn. Ozel and ertekin in 2011 [60] studied the Weibull growth model and applied it to the oriental beach Juvenilities growth. Weibull model was also used to study the height growth of Pinus radiate by Colff and Kimberley in 2013 [16]. Lumbres et al. [53] used this model to describe the diameter at breast height of Pinus Kesiya.

The Weibull growth model can be derived from the one parameter Weibull distribution function, given by

$$W(t) = 1 - exp\left(-t^{\delta}\right) \; ; t > 0, \tag{3.1}$$

where $\delta > 0$ is a shape parameter. The distribution function has the point of inflection at $t = [(\delta - 1)/\delta]^{\frac{1}{\delta}}$ and $W = 1 - exp(-(1 - \delta^{-1}))$. Then the eq. **(3.2**) can be used to get a sigmoidal growth curve for empirical use [75].

$$w(t) = b + (A - b)W(ct; \theta) \tag{3.2}$$

For the Weibull distribution,

$$w(t) = A - (A - b)exp\{-(ct)^{\delta}\} \tag{3.3}$$

The form of the model **(3.3**) is the model that David Ratkowsky [64] calls the Weibull model. The equation **(3.3**) can also be written as,

$$w(t) = A - \beta\, exp(-kt^{m}). \tag{3.4}$$

Where $A = A$, $\beta = A - b$, $k = c^{\delta}$ and $m = \delta$. Philippe Grosjean [30] used a three parameter Weibull model with considering $b = 0$ in eq. **(3.3**), which is given in eq. **(3.5**). This eq. gives the Weibull model with three parameters.

$$w(t) = A(1 - exp(-kt^m)) \qquad (3.5)$$

In 2000, Alistair Duncan Macgregor Dove [19] used a two parameters Weibull model for investigating parasite richness of nine species of fishes. He used

$$w(t) = A(1 - exp(-kt)). \qquad (3.6)$$

This model can be derived from the Weibull model with three parameters by considering $m = 1$ to the eq. **(3.5)**. Dove also defined the parameters of the model **(3.6)** for parasite richness of fishes as A is the maximum regional fauna richness and k is an index of the mean infracommunity richness.

In this chapter, the model forms **(3.4)**, **(3.5)** and **(3.6)** are considered as the Weibull model with four, three and two parameters respectively. The properties and the derivations of the Weibull models play a crucial rule for estimating the parameters. This Chapter provides the most fundamental properties and some useful definition of the parameters for initial estimates of the Weibull models, which is also a major requirement for estimating the parameters using any iteration method. Five well-known Forestry data sets are considered to estimate the parameters of these models.

3.2 Objective

The main objective of this chapter is to discuss certain properties of three Weibull growth models in forestry viewpoint, estimate the parameters of these models and provide initial value specifications of the parameters.

3.3 Methods and materials

The growth models considered for this study are a Weibull model with two parameters **(3.6**), Weibull model with three parameters **(3.5**) and Weibull model with four parameters **(3.4**). For these three models consider, w is the dependent variable, t is the independent variable, A, β, k and m are parameters to be estimated, log is the natural logarithms and $\exp(e)$ is the base of the natural logarithms. The parameters are estimated using the Newton –Raphson method of nonlinear regression. The maximum diameter data and top height growth of babul *(Acacia Nilotica)* tree are used to fit the growth model. These two sets of data are presented in **Table 2.1**. The data are based on the analysis of sample plot data of Uttar Pradesh, Maharashtra and Madhya Pradesh [37]. The top height age, the cumulative basal area production and the mean diameter at breast height data, originated from the Bowmont Norway spruce thinning experiment, sample plot 3661 [21], [22] are also used. These data sets are repeatedly measured on a five year cycle from age 20 to 64 and are presented in **Table 2.2**.

The Weibull growth models can be written as

$$w_i = f(t_i, \mathbf{B}) + \varepsilon_i, \tag{3.7}$$

$i = 1,2, \cdots, n,$ **B** is the vector of parameters b_j $(b_1, b_2, \cdots, b_q)$ to be estimated. Where q is the number of parameter, n is the number of observations and $\varepsilon_i's$ are random errors in the models have mean zero and constant variance σ^2. The model selection process described in the chapter 1 is used to analyze and select the best fit growth model. A software package has been developed in FORTRAN 77 for fitting of the models.

3.4.1 Method of estimation

For estimating the parameters of these models, the sum of square residue $S(\boldsymbol{B})$ is minimized under the assumption that the $\varepsilon_i's$ are independent $N(0,\sigma^2)$ random variable.

$$S(\boldsymbol{B}) = \sum_{i=1}^{n} [w_i - f(t_i, \mathbf{B})]^2 \tag{3.8}$$

Since w_i and t_i are fixed observations, so $S(\boldsymbol{B})$ is a function of $\boldsymbol{B}$. Now the eq. **(3.8)** is differentiated with respect to $\boldsymbol{B}$ and equating the result to zero,

$$f_j = \sum_{i=1}^{n} [w_i - f(t_i, \mathbf{B})] \left[\frac{\partial f(t_i, \boldsymbol{B})}{\partial b_j}\right] = 0 \tag{3.9}$$

for $j = 1,2,\cdots,q$. This provides a system of q nonlinear equation with q unknown parameters and that must be solved for $\boldsymbol{B}$ using any iteration method. In this Chapter, the Newton-Raphson iteration method is used to solve the equations. The general Newton-Raphson method for system of nonlinear equation is given by

$$\mathbf{B}^{(r+1)} = \mathbf{B}^{(r)} - {J_r}^{-1} F\left(\mathbf{B}^{(r)}\right);\ r = 0, 1, 2, 3 \cdots \tag{3.10}$$

Where $J_r = \begin{bmatrix} \frac{\partial f_1}{\partial b_1} & \frac{\partial f_1}{\partial b_2} \cdots & \frac{\partial f_1}{\partial b_q} \\ \frac{\partial f_2}{\partial b_1} & \frac{\partial f_2}{\partial b_2} \cdots & \frac{\partial f_2}{\partial b_q} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_q}{\partial b_1} & \frac{\partial f_q}{\partial b_2} \cdots & \frac{\partial f_q}{\partial b_q} \end{bmatrix}_{[\mathbf{B}^{(r)}]}$, $\mathbf{B}^{(r)} = \left[{b_1}^{(r)}, {b_2}^{(r)}, \cdots, {b_q}^{(r)}\right]^T$ and

$$F\left(\mathbf{B}^{(r)}\right) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_q \end{bmatrix}_{[\mathbf{B}^{(r)}]} .$$

The eq. **(3.10)** is the $(r+1)th$ iteration of the parameters in Newton-Raphson method. Since this is an iteration method, so the process may be repeated using a pre-defined stopping criterion. This method requires specifications of the starting values of the parameters to be estimated. The initial value specifications of the parameters are given in the next section.

3.5 Results and discussion

3.5.1 Properties of Weibull growth model

In this Chapter, an attempt has been made to discuss the different fundamental properties of the Weibull growth models. There is a clear relationship between the properties of various mathematical models and the estimation of their respective parameters.

3.5.1.1 Properties of four Parameters Weibull Model

In the form of the model **(3.4)**, A, β, k and m are the parameters and are defined correspondingly as: A is the asymptote or the limiting value of the response variable, β is the biological constant, k is the parameter governing the rate at which the response variable approaches its potential maximum and m is the allometric constant [22].

The Weibull growth model rises from a point b, at the starting growth setting $t = 0; w = A - \beta = A - A + b = b$; to the limiting value of A, which is the maximum possible value of $w(t)$; that is when $t \to \infty \Rightarrow w \to A$. Examining the model at the starting of the growth, which is most preferably when the independent variable (t) is zero, the only way to understand and make clear the meaning and possible range of the parameter b that is defined as a biological constant. In other word from the expression $w(0) = A - \beta$, it is logical to define the parameter β as a constant that should make the expression $A - \beta$ reasonably small enough to at least consider it as a possible value of the model parameter estimate at the starting growth. Based on the model assumption and evaluation of the model at the start of growth, it is evident that:

- $A > 0$, since a is the limiting value.
- The parameter β is always positive $(\beta > 0)$ and its size depends on the size of the parameter A. Since if $\beta = 0$ then $w = A$ at the starting of the growth and if $\beta < A$ then $w > A$ at the start of the growth and both cases violate the model assumption concerning the parameter a which states that when $t \to \infty \Rightarrow w \to A$.
- For biological growth analysis, the parameter k and m must be positive.
- The Weibull model is sigmoidal when $m > 1$, otherwise it has no inflexion point [30].

The Weibull model does not pass through the origin (when $t = 0; w \nrightarrow 0$) and that is the main limitation of the model for some biological growth.

Now,

$$\frac{dw}{dt} = \beta kmt^{m-1}e^{-kt^m}. \tag{3.11}$$

The first derivative of the model describes the slope of the curve or the rate of change of the dependent variable with respect to the independent variable and is positive. That indicates that the model is an increasing function of the independent variable.

To investigate some important properties of the model, the second derivative of the model is derived and given by

$$\frac{d^2w}{dt^2} = \beta km(m-1)t^{m-2}e^{-kt^m} - \beta k^2m^2t^{2m-2}e^{-kt^m},$$

$$\frac{d^2w}{dt^2} = kmt^{m-2}(A-w)(m-1-kmt^m). \tag{3.12}$$

It is seen that the second derivative of the model is positive $\left(\frac{d^2w}{dt^2} > 0\right)$ for $w < A - \beta e^{\frac{1-m}{m}}$; zero $\left(\frac{d^2w}{dt^2} = 0\right)$ for $w = A - \beta e^{\frac{1-m}{m}}$ and negative $\left(\frac{d^2w}{dt^2} < 0\right)$ for w $> A - \beta e^{\frac{1-m}{m}}$. In terms of the predictor variable; the second derivative is positive for $0 \leq t < \left(\frac{m-1}{km}\right)^{\frac{1}{m}}$; zero for $t = \left(\frac{m-1}{km}\right)^{\frac{1}{m}}$ and negative for $\left(\frac{m-1}{km}\right)^{\frac{1}{m}} < t < \infty$.

So the model approaches the asymptote at an increasing rate for w $< A - \beta e^{\frac{1-m}{m}}$ and at a decreasing rate for w $< A - \beta e^{\frac{1-m}{m}}$. The point where the model makes a transition from an increasing to a decreasing slope, the second derivative of the model is zero $\left(\frac{d^2w}{dt^2} = 0\right)$ and at the same point the growth function has a constant slope. The point is known as the point of inflection and it occurs at $w = A - \beta e^{\frac{1-m}{m}}$. This is one of the most important properties of the model.

3.5.1.2 Properties of three Parameters Weibull Model

From the equation **(3.5)**, the Weibull model with three parameters is given by:

$$w(t) = A(1 - exp(-kt^m)),$$

here A, k and m are the parameters and they have the similar properties and definitions with the Weibull model with four parameters, t is the independent variable and w is the response variable.

The Weibull model with three parameters passes through the origin that is when $t = 0; w = 0$. This is a major advantage of this model in case of some biological growth. Now the derivatives of the model with respect to the independent variable are given bellow:

$$\frac{dw}{dt} = Akmt^{m-1}e^{-kt^m}, \tag{3.13}$$

$$\frac{d^2w}{dt^2} = kmt^{m-2}(A - w)(m - 1 - kmt^m). \tag{3.14}$$

For the Weibull model with three parameters, it is seen that the second derivative of the model is positive $\left(\frac{d^2w}{dt^2} > 0\right)$ for w $< A - Ae^{\frac{1-m}{m}}$; zero $\left(\frac{d^2w}{dt^2} = 0\right)$ for $w = A - Ae^{\frac{1-m}{m}}$ and negative $\left(\frac{d^2w}{dt^2} < 0\right)$ for $w > A - Ae^{\frac{1-m}{m}}$. In terms of the predictor variable; the second derivative is positive for $0 \leq t < \left(\frac{m-1}{km}\right)^{\frac{1}{m}}$ zero for $t = \left(\frac{m-1}{km}\right)^{\frac{1}{m}}$ and negative for $\left(\frac{m-1}{km}\right)^{\frac{1}{m}} < t < \infty$. Again from **(3.13)**, it is seen that the first derivative of the Weibull model $\left(\frac{dw}{dt}\right)$ is positive. This means that the yield model is an increasing function of the independent variable.

From the above discussion, it is achieved that the three-parameter Weibull model approaches the asymptote at an increasing rate for values of the dependent variable less than $A - Ae^{\frac{1-m}{m}}$. The differential form of the growth model is a decreasing function of the independent variable for w $> A - Ae^{\frac{1-m}{m}}$. This implies that the yield function of the three parameter Weibull model is approached the asymptote at a decreasing rate for values of the dependent variable greater than $A - Ae^{\frac{1-m}{m}}$. The point where the growth function of the Weibull model makes a transition from an increasing to a decreasing slop, the growth function has a constant slop and the point is termed as the point of inflection of the model which occurs at $-Ae^{\frac{1-m}{m}}$.

3.5.1.3 Properties of two Parameters Weibull Model

The two parameters Weibull growth model is given by the equation **(3.6**) and which can be written as

$$w(t) = A(1 - exp(-kt)).$$

The parameters of this model A and k also have the similar properties and definitions with the Weibull model with four parameters.

The Weibull model is started from 0 (as when $t = 0; w = 0$) to the limiting value of A (as $t \rightarrow \infty; w = A$), which is also known as the upper asymptote of this model. Also since this model passes through the origin, so for some biological growth it may be an advantage of this model. The first and second derivatives of the model with respect to the independent variable are given by:

$$\frac{dw}{dt} = Ake^{-kt}, \tag{3.15}$$

$$\frac{d^2w}{dt^2} = -Ak^2e^{-kt}. \tag{3.16}$$

This describes that the first derivative of the model is always positive, which indicates that the model parameter estimates increase monotonically as the independent variable increases.

Table 3.1: Summary of the properties of the Weibull growth models.

	Weibull 4 parameter	**Weibull 3 parameter**	**Weibull 2 parameter**
Integral form	$A - \beta\ exp(-kt^m)$	$A(1 - exp(-kt^m))$	$A(1 -\ exp\ (-kt))$
Lower asymptote	$-\infty$	$-\infty$	$-\infty$
Upper asymptote	A	A	A
Starting point of the growth function	$A - \beta$	0	0
Growth rate $\left(\frac{dw}{dt}\right)$	$\beta kmt^{m-1}e^{-kt^m}$	$Akmt^{m-1}e^{-kt^m}$	Ake^{-kt}
Maximum growth rate	$\beta mk^{\frac{1}{m}}e^{\frac{1-m}{m^2}}$	$Amk^{\frac{1}{m}}e^{\frac{1-m}{m^2}}$	$Undefined$
Relative growth rate as function of time	$\beta kmt^{m-1}\left(Ae^{kt^m} - \beta\right)^{-1}$	$A^2kmt^{m-1}\left(e^{kt^m} - 1\right)^{-1}$	$ke^{-kt}(1 - e^{-kt})^{-1}$
Relative growth rate as function	$km\left(\frac{1}{k}\ln\frac{\beta}{A-w}\right)^{\frac{m-1}{m}}\frac{A-w}{w}$	$km\left(\frac{1}{k}\ln\frac{A}{A-w}\right)^{\frac{m-1}{m}}\frac{A-w}{w}$	$k(A-w)/_{w}$

of biomass			
Second derivative of the growth function $\left(\frac{d^2w}{dt^2}\right)$	$kmt^{m-2}(A-w)(m-1-kmt^m)$	$kmt^{m-2}(A-w)(m-1-kmt^m)$	$-Ak^2e^{-kt}$
Point of inflection $w(t)=$	$A-\beta e^{\frac{1-m}{m}}$	$A-Ae^{\frac{1-m}{m}}$	$No\ point\ of\ inflection$
Domain of the independent variable	$(0,\infty)$	$(0,\infty)$	$(0,\infty)$
Domain of the dependent variable	$(A-\beta, A)$	$(0, A)$	$(0, A)$

Also the second derivative of the model reveals that it is always negative, that is, the growth curve is a decreasing function of t and this implies that increment of a stand parameter, decreases over the entire range of the independent variable. The slop of the yield curve decreases as the independent variable increases. Since the second derivative does not show a change in sign so the model has no point of inflection and this is the major drawback of this model for forestry growth and yield modeling. The **Table 3.1** illustrate the fundamental mathematical properties of the Weibull growth model to proper understand the mathematics of the models and avoids problems encountered in the method of parameter estimation of the nonlinear Weibull growth models.

3.5.2 Starting value specification

All iteration procedures require initial values of the parameters to be estimated and the better these initial estimates are the faster will be the convergence to the fitted value. There is no any general method for obtaining initial estimates of the parameters. Based on available information of the parameters the initial values of the parameters have been found which are given below

Starting value of A: From the earlier discussion, it is noticed that for Weibull growth model, the parameter A is defined as the limiting value of the dependent variable. Therefore for the biological growth, the parameter A is specified as the maximum value of the dependent variable in the data.

Starting value of β: The starting value for the biological constant, β, is specified by evaluating the model at the start of the growth when the predictor variable is zero. To specify the starting value of the parameter β for the Weibull model is given below:

$$w_0 = A_0 - \beta_0,$$

$$\beta_0 = A_0 - w_0. \tag{3.17}$$

where A_0 is the starting value of the parameter A, w_0 is the value of the response variable at time $t = 0$.

Starting value of k: The parameter k is defined as the rate constant at which the response variable approaches its maximum possible value A. On the basis of this definition the parameter c can be written as, $c = \frac{(w_n - w_1)}{A_0(t_n - t_1)}$; where w_1 and w_n are the value of the response variable corresponding to the first (t_1) and the last observations (t_n) respectively. A_0 is the starting value specified for the parameter A.

Starting value of m: The equation **(3.4**) can be written as

$$m = \frac{1}{\log t}\left\{\log\left(\frac{1}{k}\left(\log\frac{\beta}{A-w}\right)\right)\right\}.$$

Now at time $t = h/_2$, where h is the last value of the independent variable in the data set,

$$m = \frac{1}{log(h/2)}\left\{log\left(\frac{1}{k_0}\left(log\frac{\beta_0}{A_0 - w_{h/_2}}\right)\right)\right\}. \tag{3.18}$$

From the equation **(3.18**), the starting value can be estimated for the parameter m. Here A_0, β_0, k_0 are the starting values for the respective parameters and $w_{h/_2}$ is the value of the response variable at time $t = \frac{h}{2}$.

Similarly procedure can be used for finding the initial values of three parameters and two parameters Weibull model.

3.5.3 Parameter estimations

Three Weibull growth models are fitted to the maximum diameter data, top height growth of babul *(Acacia Nilotica)* tree in India and the top height age, the cumulative basal area production and the mean diameter at breast height data originated from the Bowmont Norway spruce thinning experiment. The parameters of these models are estimated using Newton-Raphson method of estimation. To analyze the fit, the selection criteria are used from chapter 1.

3.5.3.1 For top height growth of babul tree

Parameter estimates for the Weibull models with the corresponding observed, predicted value along with statistical analysis to top height data of babul tree are presented in **Table 3.2**. In the case of top height data, all estimated parameters are logically consistent and biologically significant. The chi-square test is not applicable for Weibull growth model with four parameters as the degree of freedom is zero. This is due to, the available data set with five observations and the model has four parameters. In that case, the degree of freedom becomes zero and there is no any chi-square value for zero degree of freedom. Based on step II, Weibull growth model with two parameters is rejected due to having less than 95% level of significance. In the fourth step, no growth results are eliminated as both results have R_a^2 value 0.99. The 95% confidence levels of all surviving results are demonstrated in **Table 3.7**. All estimated parameters of Weibull three and four parameters model are significantly different from zero at 95% confidence level. Finally, in case of top height growth of Babul tree it is observed that, the Weibull growth model with four parameters provides better fit with the value of $R^2_{prediction}$ and R^2 are 99.99 and 99.99 respectively.

Table 3.2: Fitting of Weibull growth models for top height growth of Babul tree.

Age(Year)		Observed Data	Weibull model		
			Four parameters	Three parameters	Two parameters
5		8.14	8.145	8.055	5.868
10		12.19	12.181	12.297	10.582
15		14.93	14.914	14.963	14.368
20		16.70	16.743	16.718	17.409
25		17.98	17.957	17.902	19.852
Parameters	A		20.2807	20.6852	29.8276
	β		17.8422	0.4933	---

	k	0.3854	0.8715	0.2191
	m	1.0349	---	---
χ^2		Not applicable	0.0022	1.3516
$RMSE$		0.02	0.07	1.55
R^2 $(in\,\%)$		99.99	99.96	80.40
R_a^2		0.99	0.99	0.22
$R_{prediction}^2$ $(in\,\%)$		99.99	99.93	64.94

3.5.3.2 For maximum diameter growth of babul tree

The estimation of parameters for the Weibull growth models and the summary of statistical analysis to maximum diameter growth data of babul tree are presented in **Table 3.3**. In this case, no result is rejected in step I. For maximum diameter growth data of babul tree, the chi-square test is also not applicable for Weibull growth model with four parameters as it has zero degree of freedom. Weibull growth model with three parameters is rejected due to having less than 95% level of significance. In step III, comparing the value of RMSE, Von Weibull growth model with two parameters and Weibull growth model with four parameters are promoted to the next level. No results are eliminated in step V, as all parameters of the surviving results are significantly different from zero at 95% confidence level (**Table 3.7**). The eliminated results (if any) in each step are also highlighted accordingly in **Table 3.3**. Finally, Weibull growth model with four parameters with the $R_{prediction}^2$ and R^2 values 99.98 and 99.99 respectively is selected as the best fit model for maximum diameter growth data of babul tree then others.

3.5.3.3 For top height growth from the Bowmont Norway spruce Thinning Experiment

The estimated parameters along with the observed and predicted values for top height growth data from Bowmont Norway spruce thinning experiment along with the statistical analysis are presented in **Table 3.4**. All parameters of the candidate models are logically consistent and biologically significant. The table values of χ^2 for 95%

level of significance is found to be higher than the calculated χ^2 values for Weibull growth model with four and three parameters. The RMSE values are given by 0.110m and 0.486m for Weibull models with four parameters and three parameters respectively. The 95% confidence levels of all surviving results are also demonstrated in **Table 3.7**; where all surviving models are significantly different from zero. The Weibull growth model with three parameters are rejected in the step IV as the adjusted determination coefficient values of the model is found to be 0.98. After eliminating all the results, it is find that the Weibull growth model with four parameters is more appropriate for the top height growth data from Bowmont Norway spruce thinning experiment with $R^2_{prediction}$ and R^2 value 99.91 and 99.93 respectively.

Table 3.3: Fitting of Weibull growth models for maximum diameter growth of Babul tree.

Age(Year)		Observed Data	Weibull model		
			Four parameters	Three parameters	Two parameters
5		12.19	12.22	12.02	12.20
10		20.83	20.72	20.95	20.85
15		26.92	27.06	27.15	26.97
20		31.49	31.43	31.37	31.32
25		34.29	34.29	34.21	34.39
Parameters	A		38.7870	39.7578	41.8641
	β		35.4811	0.3602	---
	k		0.2894	1.0554	0.3446
	m		1.2213	---	---
χ^2			Not applicable	0.006	0.001
$RMSE$			0.09	0.15	0.09
R^2 $(in\ \%)$			99.99	99.96	99.99
R_a^2			0.99	0.99	0.99
$R^2_{prediction}$ $(in\ \%)$			99.98	99.95	99.97

Table 3.4: Fitting of Weibull growth models for top height growth data from Bowmont Norway spruce thinning experiment.

Age(Year)		Observed Data	Weibull model		
			Four parameters	Three parameters	Two parameters
20		7.30	7.36	6.30	3.33
25		9.00	9.02	9.20	6.33
30		10.90	10.76	11.34	9.02
35		12.60	12.47	13.06	11.43
40		13.90	14.07	14.51	13.60
45		15.40	15.54	15.77	15.55
50		16.90	16.87	16.87	17.30
55		18.20	18.05	17.85	18.87
60		19.00	19.08	18.73	20.28
65		20.00	19.98	19.53	21.54
Parameters	A		24.6476	33.5743	32.6944
	β		18.5303	---	---
	k		0.0693	0.2080	0.1076
	m		1.2992	0 .6223	---
χ^2			0.009	0.252	6.589
$RMSE$			0.109	0.486	1.803
R^2 $(in\,\%)$			99.93	98.60	80.82
R_a^2			0.99	0.98	0.71
$R^2_{prediction}$ $(in\,\%)$			99.91	98.37	78.62

3.5.3.4 For mean diameter at breast height growth from the Bowmont Norway spruce Thinning Experiment

Mean diameter at breast height and Cumulative basal area production data from Bowmont Norway spruce thinning experiment are also considered in this study. Analysis of Mean diameter at breast height growth data is presented in **Table 3.5**. Weibull growth model with two parameters are rejected in step II due to table values of χ^2 for 95% level of significance is found to be less than the calculated χ^2 values for that model.

Table 3.5: Fitting of Weibull growth models for mean diameter at breast height from Bowmont Norway spruce thinning experiment.

Age(Year)		Observed Data	Weibull model		
			Four parameters	Three parameters	Two parameters
20		8.40	8.42	6.54	4.95
25		10.40	10.24	10.32	9.13
30		12.35	12.44	13.31	12.66
35		14.74	14.81	15.83	15.64
40		17.13	17.20	18.02	18.16
45		19.50	19.50	19.96	20.28
50		21.49	21.64	21.70	22.07
55		23.82	23.58	23.27	23.59
60		25.55	25.29	24.70	24.87
65		26.50	26.78	26.02	25.94
Parameters	A		33.15763	49.02721	31.7848
	β		25.74324	---	---
	k		0.03980	0.14311	0.1694
	m		1.54465	0.72304	---
χ^2			0.013	0.783	2.771
$RMSE$			0.165	0.888	1.313
R^2 $(in\,\%)$			99.93	97.87	95.35
R_a^2			0.99	0.97	0.93
$R^2_{prediction}$ $(in\,\%)$			99.89	97.57	95.05

It is observed that the RMSE values for Weibull models for Mean diameter at breast height are given by 0.165m and 0.888m respectively for Weibull model with four parameters and three parameters respectively. The 95% confidence levels of all surviving results are demonstrated in **Table 3.7**; where Weibull growth model with three parameters is eliminated as some of its parameter are not significantly different from zero. Finally, the Weibull growth model with four parameters is found to be more appropriate for the mean diameter at breast height data from Bowmont Norway spruce thinning experiment with $R^2_{prediction}$ and R^2 value 99.89 and 99.93 respectively.

3.5.3.5 For cumulative basal area production from the Bowmont Norway spruce Thinning Experiment

In the case of Cumulative basal area production data, the Weibull model with four parameters does not give fit due to having a singular matrix during computation (

Table 3.6). The Weibull model with three parameters is also rejected as its asymptotic parameter estimates less value than the dominated value of the observed data. The Weibull growth model with two parameters provides better results for cumulative basal area production with $R^2_{prediction}$ and R^2 value 95.23 and 95.61 respectively but still the result is not acceptable as its adjusted determination coefficient value is found to be 0.93. The 95% confidence levels of the parameters of Weibull growth model with two parameters are also demonstrated in **Table 3.7**.

From the results, it is observed that for top height growth and maximum diameter growth data of babul tree, top height age data and the mean diameter at breast height data from the Bowmont Norway spruce thinning; Weibull growth model with four parameters is found to be more suitable than the remaining growth models. Whereas for cumulative basal area production from the Bowmont Norway spruce thinning experiment, the Weibull growth model with four parameters failed to provide fit due to having a singular matrix during computation. In that case, no forms of Weibull model provide a good fit.

Table 3.6: Fitting of Weibull growth models for Cumulative basal area production from Bowmont Norway spruce thinning experiment.

Age(Year)		Observed Data	Weibull model		
			Four parameters	Three parameters	Two parameters
20		37.99	Not fitted due to having a singular matrix during computation	13.77	24.03
25		49.00		35.47	43.78
30		60.41		56.94	60.01
35		68.91		74.87	73.36
40		78.73		88.32	84.33
45		89.83		97.61	93.35
50		98.60		103.61	100.76
55		107.00		107.26	106.85
60		114.80		109.37	111.86
65		119.54		110.53	115.98
Parameters	A			111.7254	134.9999
	β			0.1315	---
	k			1.5379	0.1960
	m			---	---
χ^2				51.35	9.75
$RMSE$				10.514	5.585
R^2 $(in\ \%)$				84.47	95.61
R_a^2				0.80	0.93
$R_{prediction}^2$ $(in\ \%)$				82.64	95.23

Table 3.7: 95% Confidence intervals of the parameters of Weibull growth models.

Data	Models	A		β		k		m	
		Lower limit	Upper limit	Lower limit	Lower limit	Upper limit	Upper limit	Lower limit	Upper limit
Top height growth of babul trees.	Weibull four Parameters	15.949	24.612	6.619	29.065	0.098	0.673	0.248	1.822
	Weibull three Parameters	18.663	22.708	0.438	0.549	0.757	0.986	---	---
Maximum diameter growth of	Weibull four Parameters	27.668	49.906	11.522	59.440	0.014	0.564	0.178	2.264

babul trees	Weibull two Parameters	41.666	42.063	---	---	0.341	0.348	---	---
Top height growth data from Bowmont	Weibull four Parameters	21.127	28.168	14.430	22.631	0.052	0.086	1.059	1.539
	Weibull three Parameters	1.599	65.549	0.010	0.406	0.384	0.861	---	---
mean diameter at breast height	Weibull four Parameters	28.542	37.773	20.576	30.910	0.029	0.051	1.299	1.791
	Weibull three Parameters	-22.455	120.509	-0.055	0.341	0.385	1.060	---	---
Cumulative basal area	Weibull two Parameters	130.007	139.993	---	---	0.181	0.211	---	---

3.6 Conclusion

In this Chapter, some fundamental properties of the Weibull models have been discussed. The parameters are estimated using Newton-Raphson iteration method and construe some of the appropriate statistical outputs from forestry viewpoint. Good initial estimates are required to estimate the parameters from any iteration method. This study develops some expressions to specify the initial values of the parameters based on the definitions and properties of the parameter of the Weibull models. These expressions are very useful to specify the initial values of the parameters for Weibull models in the forestry data. From the results, it is noticed that the Newton-Raphson algorithm is a useful method in case of Weibull models. It is observed from the above results that the Weibull model with four parameters produces the best fit for four forestry data sets. The Weibull model with two parameters provides a satisfactory result for cumulative basal area production. The Properties of the Weibull functions will help to select the Weibull models for appropriate field of forestry and use these models to predict and control a forestry system in a more mathematical manner.

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