Chapter 4: A Study on von Bertalanffy Growth Model

4.1 Introduction

Forest growth models expect to depict the progression of the woods nearly and resolutely enough to address the problems of the forester or biological science researcher. Among those growth models, some models strongly established themselves in between the researcher for their assimilation quality, accuracy or for the significant biological properties of the parameters. The von Bertalanffy model is also one of them.

Karl Ludwig von Bertalanffy (19th September 1901 – 12th June 1972) was an Austrian-born life scientist referred to as one amongst the founders of general systems theory (GST). His mathematical model of associate degree organism's growth over time, published in 1934, continues to be in use these days [[69], [83]]. In its simplest version, the supposed von Bertalanffy growth equation is expressed as an equation of length over time. The model is from studies of so-called allometric relations in organisms and based on a hypothesis which is that the rate of change of size of an organism is the difference between anabolic rate and the catabolic rate. The mathematical formulation of the hypothesis is,

$$\frac{dw}{dt} = \eta w^m - Kw, \tag{4.1}$$

where w is a measure of plant size, t is plant age and η, K, m are parameters. He assumed that the parameter m is in the range of $0 \leq m < 1$. $\eta > 0$ and $K > 0$ are constant of anabolism and catabolism respectively. The von Bertalanffy model permits fast initial growth to slow without imposing strict asymptotic conditions. This model is likewise esteemed on the grounds that its parameters are biologically interpretable [82]. Moreover, the von Bertalanffy model collapses with most of the growth models used in forestry, depending upon the values of its parameters [86]. In principle, in this way, a von Bertalanffy model could fit a wide variety of growth data sets. However, it is flexible to the point that it can be very difficult to fit to data, because different combinations of parameter values can produce very similar growth curves [61].

The von Bertalanffy growth model was used in fitting forestry growth data. Fekedulegn et al., [22] used this model for Bowmen Norway Spruce thinning experiment using the Marquardt iteration method along with eight other models. The model was also used along with four other models to examine and compare for five height data by Colbert et al. [15]. They used NLIN procedure in SAS to fit the growth models. Shi et al., [77] has been used the von Bertalanffy growth model along with three other models for six species of crops. They used a special case of von Bertalanffy model known as ontogenetic growth model. They developed a list of R functions for fitting the models.

The von Bertalanffy growth model can also be used in another field of sciences. Ramirez-Bautista et al., [65] used this model along with two other models to study on tree lizards. They used a non-linear regression technique to estimate the parameters. The model was also used to study on piglets by Renner-Martin et al., [67]. They used an add-in of Excel and the least square method to fit the growth model. The von Bertalanffy growth model was widely used also in the study of fish [[43], [51], [47], [10], [80]].

4.1.1 Formulations and re-parameterizations of von Bertalanffy model:

The von Bertalanffy growth model is based on a first order ordinary differential equation (4.1). Integrating the equation; accomplished by employing Bernoulli's equation for integration of differential equation; lead to growth model presented in the equation (4.2).

$$w(t) = \left\{\frac{\eta}{K} - \frac{\eta}{K} e^{-K(1-m)(t-t_0)}\right\}^{\frac{1}{1-m}}, \tag{4.2}$$

where w is the total plant size at age t; with the given initial condition at $t = t_o, w(t_o) = 0$; t_o defines as the theoretical time at zero height. When $t \to \infty$

$$w \to \left(\frac{\eta}{K}\right)^{\frac{1}{1-m}}.$$

If the asymptote is taken as A; then $\left(\frac{\eta}{K}\right)^{\frac{1}{1-m}} = A$; the equation (4.2) will be becomes

$$w(t) = \left\{A^{1-m} - A^{1-m}e^{-k(t-t_0)}\right\}^{\frac{1}{1-m}}, \tag{4.3}$$

where $k = K(1-m)$; by considering $\beta = A^{1-m}e^{Kt_0}$, the equation (4.3) can be written as

$$w(t) = \{A^{1-m} - \beta e^{-kt}\}^{\frac{1}{1-m}}, \tag{4.4}$$

here, A, β, k and m are the parameters and can characterize as: A is the asymptote; β is the biological constant; k is the parameter governing the rate at which the regressand approaches its potential maximum and m is the allometric constant. The form (4.4) is considered as a von Bertalanffy growth model [[14], [15], [22]]. The equation (4.3) can also be written as:

$$w = A\left\{1 - e^{-k(t-t_0)}\right\}^{\frac{1}{1-m}}. \tag{4.5}$$

The form of the model (4.5) was used by Roman-Roman et al., [70]. Most of the literature used an another version of the von Bertalanffy model [[25], [10], [50], [56], [55]];

$$w = A\left\{1 - e^{-k(t-t_0)}\right\}, \tag{4.6}$$

which is obtained by considering $m = 0$ in the equation (4.5).

Again solving the ordinary differentiation equation (4.1) with the initial condition $y(t_0) = b$;

$$w(t) = \left\{\frac{\eta}{K} - \left(\frac{\eta}{K} - b^{1-m}\right) e^{-K(1-m)(t-t_0)}\right\}^{\frac{1}{1-m}}, \quad (4.7)$$

$$= \left\{A^{1-m} - \left(A^{1-m} - b^{1-m}\right)e^{-k(t-t_0)}\right\}^{\frac{1}{1-m}}. \quad (4.8)$$

Cloern and Nichols [11] considered the von Bertalanffy growth model by assuming $m = 0$ in the equation (4.8),

$$w(t) = A - (A - b)e^{-k(t-t_0)}. \quad (4.9)$$

The equation (4.9) can also be written as equation (4.10) by considering the initial time $t_0 = 0$. This form of model was used as von Bertalanffy growth model in the study by Lv and Pitchford [54] and Cailliet et al., [10].

$$w(t) = A - (A - b)e^{-kt}. \quad (4.10)$$

Again if one assume that at $t = 0; y = 0$; that means at the start of the growth the height is zero; then from the equation (4.9), one can have an another form of von Bertalanffy growth model, which is

$$w(t) = A(1 - e^{-kt}). \quad (4.11)$$

The equation (4.11) was considered as a von Bertalanffy growth model with two parameters by James [38].

From the above discussion, it is clear all the form of von Bertalanffy growth model are integral forms of the ordinary differential equation (4.1) with different assumptions and consideration.

4.2 Objective

In this Chapter, an attempt has been made to discuss about the von Bertalanffy model, by studying their various re-parameterization and properties of the parameters. The main focus of this Chapter is to introduce some new method of estimations to fit the candidate models. The methods of estimation present in this study demands less computation and can use any growth data. The performances of these methods are also demonstrated using five well-known published forestry data.

4.3 Methods and materials

The growth models considered for this study are von Bertalanffy model with four parameters (4.4), von Bertalanffy with three parameters (4.10) and von Bertalanffy growth model with two parameters (4.11). The properties of the parameters of those models are discussed in this Chapter. The parameters are estimated using the methods described here. The maximum diameter data and top height growth of babul *(Acacia Nilotica)* tree are used to fit the growth model. These two sets of data are presented in **Table 2.1**. The data are based on the analysis of sample plot data of Uttar Pradesh, Maharashtra and Madhya Pradesh [37]. The top height age, the cumulative basal area production and the mean diameter at breast height data, originated from the Bowmont Norway spruce thinning experiment, sample plot 3661 [[21], [22]] are also used and presented in **Table 2.2**.

4.3.1 Method of estimation

Most of the literature discussed in this study used to fit the von Bertalanffy model by using some well-known algorithm. This study is trying to introduce some new methods of estimation by which one can easily fit the model without using any dearly-won software.

4.3.1.1 Estimation of two parameters von Bertalanffy growth model (4.11).

4.3.1.1.1 Method A: Composite two points method

We consider, total number of observation is n. Let w_1 is the total plant size at age $t = t_1$ and w_2 is the total plant size at age $t = t_n$. By changing of scale and origin of the independent variables, we can consider, $t_1 = 1$ and $t_n = n$. Then the equation (4.11) can be written as

$$w_1 = A(1 - e^{-k}), \tag{4.12}$$

$$w_2 = A(1 - e^{-kn}). \tag{4.13}$$

Now by solving equation (4.12) and (4.13),

$$w_1 x^n - w_2 x + (w_2 - w_1) = 0. \tag{4.14}$$

Where $x = e^{-k}$. The equation (4.14) can be solved using any iteration method. After finding the value of x, the parameter k can be estimated using $\hat{k} = -\ln x$. Then the parameter A can be estimated using any one of the equations (4.12) and (4.13).

Now since the iteration method need an initial value. For the initial value specification, the following procedure has been developed:

For the first and second data of the data set, the equation (4.14) can be written as

$$w_1 x^2 - w_2 x + (w_2 - w_1) = 0. \tag{4.15}$$

Which is a quadratic equation in x and by solving it one have two values of x. The non-negative value(s) of x can be used as starting value.

4.3.1.1.2 Method B: Composite two sums method

In this method, assume that the parameter A is known from the previous method. Then rewriting the model form (4.11) in terms of k and considering the sum of all observations,

$$\hat{k} = \frac{1}{n}\ln\left[\prod_{i=1}^{n}\frac{1}{\left(1-\frac{w_i}{A}\right)^{\frac{1}{t_i}}}\right]; i = 1, 2, \cdots, n. \tag{4.16}$$

After estimating the parameter k, again rewriting the model in terms of A and adding for the entire observations, the parameter A is estimated as-

$$\hat{A} = \frac{1}{n}\sum_{i=1}^{n}\frac{w_i}{1-e^{-kt_i}}; \; i = 1, 2, \cdots, n. \tag{4.17}$$

4.3.1.2 Estimation of three parameters von Bertalanffy growth model (4.10).

The Von Bertalanffy growth model with three parameters (4.10) can also be written as

$$w(t) = A(1 - b_1 e^{-kt}), \tag{4.18}$$

where $b_1 = \frac{A-b}{A}$. To estimate the parameters of the model form (4.18), the following procedure has been followed.

4.3.1.2.1 Method A: Composite three points method

Suppose n be the total number of observation and let t_a, t_b and t_c are any three observations from the set of data. Then for $i = a, b, c$; the equation (4.18) can be written as

$$\ln w_i = \ln A + \ln(1 - b_1 e^{-kt_i}). \tag{4.19}$$

Now,

$$\ln w_a - \ln w_b = \ln \left\{ \frac{1 - b_1 e^{-kt_a}}{1 - b_1 e^{-kt_b}} \right\}, \tag{4.20}$$

and

$$\ln w_b - \ln w_c = \ln \left\{ \frac{1 - b_1 e^{-kt_b}}{1 - b_1 e^{-kt_c}} \right\}. \tag{4.21}$$

From the equation (5.18) and (5.19), one obtain an equation of the form

$$(A_1 A_2 - A_2) x^{t_c} + (1 - A_1 A_2) x^{t_b} + (A_2 - 1) x^{t_a} = 0, \tag{4.22}$$

where $A_1 = \exp(\ln w_a - \ln w_b)$ and $A_2 = \exp(\ln w_b - \ln w_c)$.

The equation (5.20) can be solved using any iteration method, then the parameter k can be estimated as,

$$\hat{k} = \ln \frac{1}{x}.$$

After estimating the parameter k; the parameters b_1 can be estimated using the equation (4.21) and then from the relation $\hat{b} = A - Ab_1$, the parameter b can be estimated. To estimate the parameter A the following relation is used

$$\hat{A} = \exp\{\ln w_c - \ln(1 - b_1 e^{-kt_c})\}.$$

For some equidistant data set, consider $r = \left[\frac{n}{3}\right]$, $t_a = r, t_b = 2r$ and $t_c = 3r$. In this case the equation obtained by solving equations (4.20) and (4.21) is in a quadratic form with $x = e^{-rk}$.

4.3.1.2.2 Method B: Composite two partial sums method

Let n be the number of observation and $r = \left[\frac{n}{2}\right]$. Assume that the parameter A is known. Then for $i = 1,2,\cdots,n$; the model form (4.18) can be written as:

$$\ln b_1 = kt_i + \ln\left\{1 - \frac{w_i}{A}\right\}. \tag{4.23}$$

For the first and second d observations, the sum can be expressed as;

$$r \ln b_1 = k\sum_{i=1}^{r} t_i + \ln\left\{\prod_{i=1}^{r}\left\{1 - \frac{w_i}{A}\right\}\right\}, \tag{4.24}$$

$$r \ln b_1 = k\sum_{i=r+1}^{2r} t_i + \ln\left\{\prod_{i=r+1}^{2r}\left\{1 - \frac{w_i}{A}\right\}\right\}. \tag{4.25}$$

From the equation (4.24) and (4.25); the parameters k can be estimated as

$$\hat{k} = \frac{1}{\sum_{i=r+1}^{2r} t_i - \sum_{i=1}^{r} t_i} \ln \frac{\prod_{i=1}^{r} \left\{1 - \frac{w_i}{A}\right\}}{\prod_{i=r+1}^{2r} \left\{1 - \frac{w_i}{A}\right\}}. \tag{4.26}$$

After estimating k, the parameter b_1 can be estimated by considering sum of (4.23) for all $i = 1, 2, \cdots, n$; that is

$$\widehat{b_1} = \exp\left[\frac{k}{n}\sum_{i=1}^{n} t_i + \frac{1}{\mathrm{n}}\ln\left\{\prod_{i=1}^{n}\left\{1 - \frac{w_i}{A}\right\}\right\}\right]. \tag{4.27}$$

Then the parameter b can be estimated using its definition with the parameter b_1. Again to estimate the parameter A, the estimated value of k and b_1 are used, which is given by

$$\hat{A} = \left\{\prod_{i=1}^{n} \frac{w_i}{1 - b_1 e^{-Kt_i}}\right\}^{\frac{1}{n}}. \tag{4.28}$$

4.3.1.3 Estimation of four parameters von Bertalanffy growth model (4.4).

From the von Bertalanffy growth model with four parameters (4.4),

$$w = (A^{1-m} - \beta e^{-kt})^{\frac{1}{1-m}},$$

$$w = A(1 - b_2 e^{-kt})^{D}. \tag{4.29}$$

where $b_2 = \frac{\beta}{A^{1-m}}$ and $D = \frac{1}{1-m}$.

4.3.1.3.1 Method A: (composite method using three points)

In this method, assume that the parameter D is known from its definition. Then let n be the total number of observation, t_a, t_b and t_c are any three observations from the set of data. Then for $i = a, b, c$; the equation (4.29) can be written as

$$\ln w_i = \ln A + D \ln(1 - b_2 e^{-kt_i}). \tag{4.30}$$

Now,

$$\ln w_a - \ln w_b = D \ln \left\{ \frac{1 - b_2 e^{-kt_a}}{1 - b_2 e^{-kt_b}} \right\}, \tag{4.31}$$

and

$$\ln y_b - \ln y_c = D \ln \left\{ \frac{1 - b_2 e^{-kt_b}}{1 - b_2 e^{-kt_c}} \right\}. \tag{4.32}$$

From the equation (4.31), (4.30) and (4.32), one obtain an equation of the form

$$(A_1 A_2 - A_2) x^{t_c} + (1 - A_1 A_2) x^{t_b} + (A_2 - 1) x^{t_a} = 0, \tag{4.33}$$

where $A_1 = \exp \frac{\ln w_a - \ln w_b}{D}$ and $A_2 = \exp \frac{\ln w_b - \ln w_c}{D}$.

The equation (4.33) can be solved using any iteration method, then the parameter k can be estimated as,

$$\hat{k} = \ln\frac{1}{x}.$$

After estimating the parameter k; the parameters b_2, A and D can be estimated using the equations (4.31), (4.32) and (4.33). The estimated parameters are given by

$$\widehat{b_2} = \frac{1 - \exp\frac{\ln w_b - \ln w_c}{D}}{\exp(-kt_b) - \exp\frac{\ln w_b - \ln w_c}{D}\exp(-kt_c)},$$

$$\hat{A} = \exp\{\ln w_c - D\ln(1 - b_2 e^{-kt_c})\},$$

$$\widehat{D} = \frac{\ln\, w_a - \ln A}{\ln(1 - b_2 e^{-kt_a})}.$$

After estimating the parameters b_2 and D, the parameters β and m can be estimated using the relations $\beta = b_2 A^{1-m}$ and $m = \frac{D-1}{D}$. For some equidistant data set, consider $r = \left[\frac{n}{3}\right]$, $t_a = r, t_b = 2r$ and $t_c = 3r$. In this case, the equation obtained by solving equations (4.31) and (4.32) is in quadratic form with $x = e^{-rk}$.

4.3.1.3.2 Method B: (composite method using two partial sums)

In this method, assume that the parameter b_2 and K are known. Then let n be the total number of observation. The equation (4.29) can be written as

$$\ln w = \ln A + D\ln(1 - b_2 e^{-kt}). \tag{4.34}$$

Let $r = \left[\frac{n}{2}\right]$, $S_1 = \sum_{i=1}^{r} \ln w_i$ and $S_2 = \sum_{i=r+1}^{2r} \ln w_i$. Then,

$$S_1 = r \ln A + D \ln\left\{\prod_{i=1}^{r}(1 - b_2 e^{-ki})\right\}, \tag{4.35}$$

$$S_2 = r \ln A + D \ln\left\{\prod_{i=r+1}^{2r}(1 - b_2 e^{-ki})\right\}. \tag{4.36}$$

From the equation (4.35) and (4.36) the estimated parameters A and D are given by

$$\widehat{D} = \frac{S_2 - S_1}{\{\ln(\prod_{i=r+1}^{2r}(1 - b_2 e^{-kt_i})) - \ln(\prod_{i=1}^{r}(1 - b_2 e^{-kt_i}))\}},$$

$$\hat{A} = \exp\left\{\frac{S_1 - D\ln(\prod_{i=1}^{r}(1 - b_2 e^{-kt_i}))}{r}\right\}.$$

After estimating the parameters A and D; the parameter k and b_2 can be estimated as follows:

For $i = 1, 2, \ldots, n$; the equation (4.29) can be written as

$$w_i = A(1 - b_2 e^{-kt_i})^D,$$

$$\Rightarrow \left(\frac{w_i}{A}\right)^{\frac{1}{D}} = 1 - b_2 e^{-kt_i},$$

$$\Rightarrow b_2 = e^{kt_i}\left\{1 - \left(\frac{w_i}{A}\right)^{\frac{1}{D}}\right\},$$

$$\Rightarrow \ln b_2 = kt_i + \ln\left\{1 - \left(\frac{w_i}{A}\right)^{\frac{1}{D}}\right\}.$$

Now for first r observations,

$$r \ln b_2 = k \sum_{i=1}^{r} t_i + \ln\left\{\prod_{i=1}^{r}\left\{1-\left(\frac{w_i}{A}\right)^{\frac{1}{D}}\right\}\right\}. \tag{4.37}$$

For the second r observations,

$$r \ln b_2 = k \sum_{i=r+1}^{2r} t_i + \ln\left\{\prod_{i=r+1}^{2r}\left\{1-\left(\frac{w_i}{A}\right)^{\frac{1}{D}}\right\}\right\}. \tag{4.38}$$

Now, Solving (4.37) and (4.38); the required parameters can be estimated as

$$\hat{k} = \frac{1}{\sum_{i=r+1}^{2r} t_i - \sum_{i=1}^{r} t_i} \ln\left[\frac{\prod_{i=1}^{r}\left(1-\left(\frac{w_i}{A}\right)^{\frac{1}{D}}\right)}{\prod_{i=r+1}^{2r}\left(1-\left(\frac{w_i}{A}\right)^{\frac{1}{D}}\right)}\right],$$

$$\widehat{b_2} = \exp\left\{\frac{k}{n}\sum_{i=1}^{n} t_i + \frac{1}{n}\ln\left\{\prod_{i=1}^{n}\left(1-\left(\frac{w_i}{A}\right)^{\frac{1}{D}}\right)\right\}\right\},$$

After estimating the parameters b_2 and D, the required parameters β and m can be estimated using relations $\beta = b_2 A^{1-m}$ and $m = \frac{D-1}{D}$.

4.4 Results and discussion

4.4.1 Properties of von Bertalanffy growth model

The von Bertalanffy growth model (4.4) rises from a point $\{A^{1-m} - \beta\}^{\frac{1}{1-m}}$ to the limiting value A. Based on the model assumption and evaluation of the model at the start of growth, it is evident that:

a) $A > 0$, since A is the limiting value.

b) From the von Bertalanffy growth model, the parameter m lies in $[0, 1)$. The Figure 4.1 demonstrate the shapes of the model for different values of the parameter m.

c) The parameter β is always positive ($\beta > 0$) and its size depends on the size of the parameter A. Since if $\beta = 0$ then $w = A$ at the starting of the growth and if $\beta < 0$ then $w > A$ at the start of the growth and both cases violate the model assumption concerning the parameter A which states that when $t \to \infty \Rightarrow w \to A$.

d) For biological growth analysis, the parameter k must be positive.

The model is an increasing function of the independent variable as its first derivative is always positive. The model approaches the asymptote at an increasing rate for $w < Ae^{\frac{\ln m}{1-m}}$ and at a decreasing rate for $w > Ae^{\frac{\ln m}{1-m}}$. The model has a sigmoidal shape and its point of inflection occurs at $w = Ae^{\frac{\ln m}{1-m}}$. In terms of the predictor variable; the second derivative is positive for $0 \le t < -\frac{\ln\frac{A^{1-m}(1-m)}{\beta}}{k}$; zero for $t = -\frac{\ln\frac{A^{1-m}(1-m)}{\beta}}{k}$ and negative for $-\frac{\ln\frac{A^{1-m}(1-m)}{\beta}}{k} < t < \infty$.

For the von Bertalanffy model with three parameters (4.10) and von Bertalanffy model with two parameters (4.11), the parameters A and k have same biological significance as the model (4.4). The parameter b represents the height at time $t = 0$. This properties can also be analysed by considering the parameter $\beta = A - b$ and $m = 0$ in the model form (4.4).

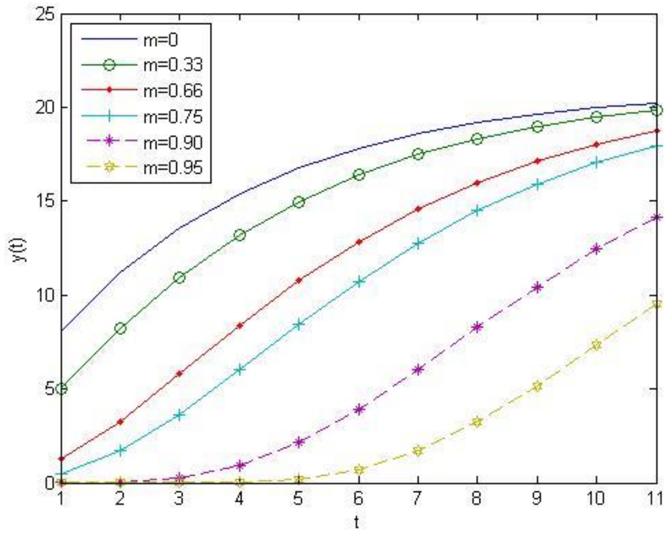

Figure 4.1: The shapes of von Bertalanffy model for different values of the allometric parameter.

Table 4.1: Summary of some basic properties of von Bertalanffy model.

	von Bertalanffy **model with four parameters**	von Bertalanffy **model with three parameters**	von Bertalanffy **model with two parameters**
Integral form of the growth function $w(t)$	$\{A^{1-m} - \beta e^{-kt}\}^{\frac{1}{1-m}}$	$A-(A-b)e^{-kt}$	$A(1-e^{-kt})$.
Upper asymptote	A	A	A
Starting point of the growth function	$\{A^{1-m} - \beta\}^{\frac{1}{1-m}}$	b	0
Growth rate $\left(\frac{dw}{dt}\right)$	$\frac{\beta k}{1-m} e^{-kt} w^m$	$(A-b)ke^{-kt}$	Ake^{-kt}
Relative growth rate as function of time	$\frac{Bk}{(1-m)(e^{kt}A^{1-m}-\beta)}$	$\frac{k(A-b)}{(e^{kt}-1)A+b}$	$\frac{k}{e^{kt}-1}$
Relative growth rate as function of response variable	$\frac{k(A^{1-m}-w^{1-m})}{w^{(1-m)}(1-m)}$	$\frac{k(A-w)}{w}$	$\frac{k(A-w)}{w}$
Second derivative of the growth function		$-w(A-b)k^2e^{-kt}$	$-wAk^2e^{-kt}$

$\left(\frac{d^2w}{dt^2}\right)$	$\frac{\beta k^2}{1-m}e^{-kt}\left\{-w^{1-m} + \frac{(w^{1-m}-A^{1-m})w^{2m-1}m}{1-m}\right\}$		
Point of inflection $w(t) =$	$Ae^{\frac{\ln m}{1-m}}$	Does not exist	Does not exist
Domain of the independent variable	$[0, \infty)$	$[0, \infty)$	$[0, \infty)$
Domain of the dependent variable	$\left[\{A^{1-m}-\beta\}^{\frac{1}{1-m}}, A\right]$	$[b, A]$	$[0, A]$

The properties of von Bertalanffy growth models can be analysed by considering the parameter $\beta = A$ and $m = 0$ in the model form (4.4). Some basic properties of four parameters von Bertalanffy model in equation (4.4), the three parameters von Bertalanffy model in equation (4.10) and the von Bertalanffy growth model with two parameters (4.11) are summarized in **Table 4.1**. From the **Table 4.1**, it is observed that the von Bertalanffy model with three and two parameters never have any point of inflection. The second derivatives of the models are always negative; which indicates that the models are decreasing function of the independent variable. The slope of the yield curve decreases as the independent variable increases.

4.4.2 Parameter estimation

Von Bertalanffy growth models are fitted to top height and maximum diameter growth data of babul trees compiled from Uttar Pradesh, Maharashtra and Madhya Pradesh of India. The parameters of these models are estimated using the methods of estimation describe above.

4.4.2.1 For top height growth of babul tree

The estimation of parameters for the growth models along with the summary of statistical analysis to top height growth data of babul tree are presented in **Table 4.2**.

Step I: The estimated parameters of all growth models are logically consistent and biologically significant.

Step II: The chi-square test is not applicable for von Bertalanffy growth model with four parameters as the degree of freedom is zero. No results are rejected in step II.

Step III: Considering the relative value of RMSE, the two best results are selected in this step. Comparing the values of RMSE, Von Bertalanffy growth model with four parameters (method A and B) and Von Bertalanffy growth model with three parameters (method A and B) are promoted to the next level.

Step IV: In the fourth step, no growth results are eliminated as all surviving results have R_a^2 value 0.99.

Step V: All surviving results along with the 95% confidence level are demonstrated in **Table 4.7**. It is observed that the Von Bertalanffy growth model with four parameters (method A and B) is removed as some of the parameters of this model are not significantly different from zero at 95% confidence level.

Step VI: The sixth and final selection criterion is based on $R^2$and $R^2_{prediction}$, as this statistic gives some indication of the predictive capability of the growth models. From the final step, the best growth model is selected. In case of top height growth data of babul tree, the Von Bertalanffy growth model with three parameters with method B is found to be more suitable as the value of $R^2_{prediction}$ and R^2 (99.99 and 99.99 respectively) are more than the remaining surviving growth models. The eliminated results in each step are highlighted accordingly in the **Table 4.2**.

Table 4.2: Fitting of von Bertalanffy growth models for top height growth of Babul tree in India.

Age	Observed data	Estimated value					
		von Bertalanffy with two parameter		von Bertalanffy with three parameter		von Bertalanffy with four parameter	
		Method A	Method B	Method A	Method B	Method A	Method B
5	8.14	8.14	7.83	8.14	8.17	8.14	8.21
10	12.19	12.83	12.47	12.19	12.16	12.19	12.21
15	14.93	15.53	15.21	14.93	14.88	14.93	14.92
20	16.70	17.08	16.84	16.78	16.73	16.76	16.73
25	17.98	17.98	17.81	18.04	17.99	17.98	17.94
Parameters	A	19.1968	19.2059	20.6610	20.6625	20.3454	20.2883
	β or b	---	---	2.1537	2.2901	13.2735	13.3014
	k	0.5517	0.5237	0.3908	0.3854	0.4175	0.4165
	m	---	---	---	---	0.0909	0.0873
χ^2		0.063	0.026	0.000	0.000	Not applicable	
RMSE		0.43	0.25	0.04	0.03	0.03	0.04
R^2(in %)		98.52	99.51	99.98	99.99	99.99	99.98
R_a^2		0.98	0.99	0.99	0.99	0.99	0.99
$R^2_{prediction}$ (in %)		97.92	99.27	99.96	99.99	99.98	99.98

4.4.2.2 For maximum diameter growth of babul tree

The estimation of parameters for the growth models and the summary of statistical analysis to maximum diameter growth data of babul tree are presented in **Table 4.3**.

Table 4.3: Fitting of von Bertalanffy growth models for maximum diameter growth of Babul tree in India.

Age	Observed data	Estimated value					
		von Bertalanffy with two parameter		von Bertalanffy with three parameter		von Bertalanffy with four parameter	
		Method A	Method B	Method A	Method B	Method A	Method B
5	12.19	12.19	12.22	12.19	12.02	12.19	12.07
10	20.83	20.81	20.86	20.83	20.89	20.83	20.86
15	26.92	26.92	26.96	26.92	27.08	26.92	27.06
20	31.49	31.24	31.28	31.21	31.44	31.14	31.36
25	34.29	34.29	34.32	34.24	34.48	34.05	34.32

Parameters	A	41.6807	41.6625	41.4644	41.5830	40.3274	40.6768
	β or b	---	---	-0.0677	-0.6404	28.0537	27.5570
	k	0.3460	0.3473	0.3498	0.3566	0.3842	0.3863
	m	---	---	---	---	0.0909	0.0993
χ^2		0.002	0.002	0.003	0.005	Not applicable	
RMSE		0.114	0.100	0.13	0.14	0.189	0.101
R^2(in %)		99.98	99.98	99.97	99.97	99.94	99.98
R_a^2		0.99	0.99	0.99	0.99	0.99	0.99
$R^2_{prediction}$ (in %)		99.96	99.97	99.94	99.93	99.86	99.97

In this case, Von Bertalanffy growth model with three parameters (method A and B) are rejected due to the non-logical estimation of the parameters. In both the cases, the parameter b estimate negative value (parameter b can't be negative as it represents the height at time $t = 0$.). The eliminated results in each step are also highlighted accordingly in the **Table 4.3**. For maximum diameter growth data of babul tree, the chi square test is also not applicable for Von Bertalanffy growth model with four parameters as it has zero degree of freedom. No results are eliminated in that step. In the third step, comparing the values of RMSE, Von Bertalanffy growth model with two parameters (Method B) and Von Bertalanffy growth model with four parameters (Method B) are promoted for the next level. The Von Bertalanffy model with four parameters (method B) is eliminated in the step V, as some of the parameters of this model are not significantly different from zero at 95% confidence level (**Table 4.7**). After eliminating the results in each step, finally it is found that Von Bertalanffy growth model with two parameters (Method B) with the $R^2_{prediction}$ and R^2 values 99.97 and 99.98 respectively as the best fit model.

4.4.2.3 *For top height growth from the Bowmont Norway spruce Thinning Experiment*

The parameter estimates for the von Bertalanffy models with the corresponding observed and predicted data to top height growth are presented in **Table 4.4** along with the summary of the statistical analysis. It is observed that for method A, the

RMSE values for top height growth are given by $2.85m, 0.13m$ and $0.13m$ for the candidate models. The table values of χ^2 for 99.5% level of significance is found to be higher than the computed χ^2 values for von Bertalanffy models with four parameters and with three parameters; whereas it is 75% higher in the case of von Bertalanffy models with two parameters at $(n-1-p)$ degree of freedom. The adjusted coefficients of correlation, R_a^2, shows above 0.99 for both models except the von Bertalanffy model with two parameters, which is found to be 0.46. Again from the value of $R^2_{prediction}$, it is clear that, one can expect from the model with two parameters, to explain about 45.44% of the variability in predicting new observations, as compared to the approximately 51.9% of the variability in the original data whereas the model with three parameters and four parameters able to explain about 99.85% of the variability in predicting new observations, as compared to the approximately 95.89% of the variability in the original data.

Table 4.4: Fitting of von Bertalanffy growth models for Top height growth data from Bowmont Norway spruce Thinning Experiment.

Age	Observed data	Estimated data					
		von Bertalanffy with two parameter		von Bertalanffy with three parameter		von Bertalanffy with four parameter	
		Method A	Method B	Method A	Method B	Method A	Method B
20	7.3	7.3	5.24	7.29	7.18	7.34	7.23
25	9	11.97	9.11	9.16	9.1	9.18	9.1
30	10.9	14.95	11.97	10.90	10.88	10.9	10.85
35	12.6	16.85	14.09	12.51	12.53	12.51	12.48
40	13.9	18.07	15.66	14.01	14.06	14.01	14.01
45	15.4	18.85	16.81	15.40	15.47	15.4	15.43
50	16.9	19.35	17.67	16.69	16.79	16.69	16.75
55	18.2	19.67	18.30	17.89	18	17.89	17.98
60	19	19.87	18.77	19	19.13	19	19.12
64	20	20	19.12	20.03	20.18	20.03	20.17
a m e t	A	20.2301	20.0982	33.4000	33.4387	32.2587	32.8397

	β or b	---	---	5.2750	5.1069	18.9000	18.0243
	K	0.4476	0.3019	0.0744	0.0759	0.0826	0.0825
	m	---	---	---	---	0.0909	0.1091
χ^2		4.95	1.46	0.01	0.01	0.01	0.01
RMSE		2.85	1.18	0.13	0.12	0.13	0.12
R^2(in %)		51.90	91.64	99.89	99.91	99.89	99.91
R_a^2		0.46	0.90	0.99	0.99	0.99	0.99
$R_{prediction}^2$ (in %)		45.44	90.46	99.85	99.87	99.85	99.87

For method B, The tabulated values of χ^2 for 99.5% level of significance is found to be higher than the calculated χ^2 values for both von Bertalanffy models with four and three parameters whereas it is 97.5% higher in case of von Bertalanffy models with two parameters. The RMSE values are given by 0.12m, 0.12m and 1.18m for von Bertalanffy models with four, three and two parameters respectively. The adjusted coefficients of correlation, R_a^2, shows that all models have value above 0.90.

Table 4.5: Fitting of von Bertalanffy growth models for mean diameter breast height data from Bowmont Norway spruce Thinning Experiment.

Age	Observed data	Estimated value					
		von Bertalanffy with two parameter		von Bertalanffy with three parameter		von Bertalanffy with four parameter	
		Method A	Method B	Method A	Method B	Method A	Method B
20	8.40	8.40	6.08	6.87	7.59	6.98	7.34
25	10.40	14.20	10.77	9.67	10.26	9.72	10.12
30	12.35	18.21	14.39	12.35	12.79	12.35	12.73
35	14.74	20.98	17.19	14.87	15.19	14.86	15.17
40	17.13	22.89	19.35	17.25	17.47	17.24	17.46
45	19.50	24.22	21.01	19.50	19.63	19.50	19.60
50	21.49	25.13	22.30	21.63	21.68	21.63	21.61
55	23.82	25.76	23.30	23.64	23.62	23.65	23.49
60	25.55	26.20	24.06	25.55	25.46	25.55	25.25
64	26.50	26.50	24.66	27.35	27.21	27.34	26.89
Parameters	A	27.1731	26.6638	58.8250	59.2153	54.4230	50.6681
	β or b	$---$	$---$	3.9000	4.7722	34.1885	43.5185

	k	0.3698	0.2586	0.0557	0.0531	0.0664	0.0673
	m	$---$	$---$	$---$	$---$	0.0909	0.0148
χ^2		7.825	2.173	0.421	0.147	0.366	0.204
RMSE		4.009	1.716	0.603	0.424	0.571	0.448
R^2(in %)		56.64	92.06	99.02	99.52	99.12	99.46
R_a^2		0.51	0.91	0.99	0.99	0.98	0.99
$R_{prediction}^2$ (in %)		50.95	90.32	98.85	99.38	98.95	99.38

Again from the values of $R_{prediction}^2$, it is clear that, the model with two parameters explain about 94.46% of the variability in predicting new observations, as compared to the approximately 91.65% of the variability in the original data whereas the models with three and four parameters able to explain about 99.87% of the variability in predicting new observations, as compared to the approximately 99.91% of the variability in the original data.

From the results of top height growth, it is found that, no results are eliminated in step I. Von Bertalanffy with two parameters (method A) is rejected in step II for having less than 95% level of significance. In the third step, comparing the values of RMSE, Von Bertalanffy growth model with three parameters (Method A and B) and Von Bertalanffy growth model with four parameters (Method A and B) are promoted to the next level. But the Von Bertalanffy growth model with four parameters (method A and B) is removed at step V, as some of the parameters of this model are not significantly different from zero at 95% confidence level, which is presented in **Table 4.7**. And finally, in case of top height growth data, the Von Bertalanffy growth model with three parameters with method B is found to be more suitable with the value of $R_{prediction}^2$ and R^2 99.87 and 99.9 respectively.

4.4.2.4 For mean diameter at breast height growth from the Bowmont Norway spruce Thinning Experiment

The estimation of parameters for the growth models along with the summary of statistical analysis to mean diameter at breast height are presented in **Table 4.5** .The eliminated results in each step are also highlighted accordingly. Here, all estimated parameters are biologically realistic. For mean diameter at breast height data, it is also observed that von Bertalanffy with two parameters (method A and method B) are eliminated as both have the calculated χ^2 value less then tabulated value at 95% level of significance. Von Bertalanffy with three parameters (method B) and von Bertalanffy with four parameters (method B) are promoted to the next step with less value of RMSE in step III. In step IV, von Bertalanffy with four parameters (method B) is eliminated as some of their parameters are not significantly different from zero (**Table 4.7**). Finally, based on R^2 and $R^2_{prediction}$, the better result is chosen and it is found as von Bertalanffy with three parameters (method B) with the $R^2_{prediction}$ and R^2 values 99.38 and 99.52 respectively.

Table 4.6: Fitting of von Bertalanffy growth models for cumulative basal area production data from Bowmont Norway spruce Thinning Experiment.

Age	Observed data	Estimated value					
		von Bertalanffy with two parameter		von Bertalanffy with three parameter		von Bertalanffy with four parameter	
		Method A	Method B	Method A	Method B	Method A	Method B
20	37.99	37.99	28.22	37.94	37.03	38.27	37.31
25	49.00	64.20	49.87	49.48	48.68	49.58	48.60
30	60.41	82.29	66.46	60.41	59.72	60.41	59.44
35	68.91	94.77	79.19	70.76	70.16	70.73	69.81
40	78.73	103.38	88.95	80.55	80.05	80.53	79.67
45	89.83	109.32	96.43	89.83	89.42	89.83	89.05
50	98.60	113.41	102.17	98.61	98.28	98.63	97.93
55	107.00	116.24	106.57	106.93	106.67	106.95	106.35

60	114.80	118.19	109.95	114.80	114.61	114.80	114.30
64	119.54	119.54	112.53	122.25	122.13	122.21	121.81
Parameters	A	122.536	121.0449	254.9126	255.7066	238.1906	240.1057
	β or b	---	---	25.7469	24.7172	125.1226	117.0685
	k	0.3711	0.2655	0.0547	0.0548	0.0642	0.0642
	m	---	---	---	---	0.0909	0.1046
χ^2		28.587	7.682	0.155	0.139	0.154	0.113
RMSE		16.376	6.870	1.198	1.094	1.188	1.007
R^2(in %)		62.33	93.37	99.80	99.83	99.80	99.86
R_a^2		0.58	0.93	0.99	0.99	0.99	0.99
$R^2_{prediction}$ (in %)		57.13	92.04	99.10	99.75	99.71	99.79

4.4.2.5 For cumulative basal area production from the Bowmont Norway spruce Thinning Experiment

The estimation of parameters for the growth models and the summary of statistical analysis to cumulative basal area production are presented in **Table 4.6**. The best result is selected and found as von Bertalanffy with three parameters (method B) with the $R^2_{prediction}$ and R^2 values 99.75 and 99.83 respectively. The eliminated results in each step are highlighted accordingly in the **Table 4.6**.

Table 4.7: 95% Confidence intervals of the parameters of the candidate models.

Data	Models	Method	A		β or b		K		m	
			Lower limit	Upper limit	Lower limit	Upper limit	Lower limit	Upper limit	Lower limit	Upper limit
Top Height of Babul	VB 4	A	19.93	20.75	1.01	25.54	0.36	0.47	−0.16	0.34
		B	19.97	20.61	0.38	23.11	0.38	0.46	−0.12	0.29
	VB 3	A	19.85	21.47	1.39	2.91	0.341	0.441	– –	– –
		B	20.09	21.23	0.78	2.80	0.35	0.42	– –	– –
Maximum diameter of Babul	VB 4	B	39.73	41.62	8.53	46.58	0.34	0.43	−0.06	0.26
	VB 2	B	40.79	42.53	– –	– –	0.33	0.36	– –	– –

Top Height	VB 4	A	21.11	43.40	−42.38	80.18	0.003	0.16	−0.63	0.82
		B	20.85	44.83	−41.85	77.899	0.001	0.16	−0.62	0.85
	VB 3	A	28.44	38.36	4.86	5.69	0.06	0.09	− −	− −
		B	29.05	37.83	4.72	5.49	0.06	0.09	− −	− −
Mean Diameter Breast Height	VB 4	B	30.02	71.32	−58.34	145.38	0.005	0.13	−0.44	0.47
	VB 3	B	45.113	73.317	4.187	5.36	0.035	0.071	− −	− −
cumulative basal area production	VB 4	B	176.77	303.44	−115.53	349.66	0.027	0.101	−0.19	0.39
	VB 3	B	221.46	289.96	23.20	26.23	0.04	0.066	− −	− −

Von Bertalanffy with two parameters (method A and method B) is eliminated as both have the calculated χ^2 value less then tabulated value at 95% level of significance. Von Bertalanffy with three parameters (method B) and von Bertalanffy with four parameters (method B) are promoted to the next step with less value of RMSE in step III. In step IV, von Bertalanffy with four parameters (method B) is eliminated as some of their parameters are not significantly different from zero (**Table 4.7**).

From the obtained results, it is observed that for top height growth data of babul tree, top height age data, the mean diameter at breast height data and for cumulative basal area production from the Bowmont Norway spruce thinning; von Bertalanffy with three parameters (method B) is found to be more suitable than the remaining growth models. Whereas for maximum diameter growth data of babul tree, the von Bertalanffy with two parameters (method B) provides a good fit.

4.5 Conclusion

The basic focus of this Chapter is to account on the von Bertalanffy growth model and introduce few methods of estimation to fit the model which are easy to use and free from any cost. The brief discussions on the evolution of the von Bertalanffy model may help to understand the model intimately. This Chapter also discuss about various parameterizations of the model. The newly introduced methods of estimations are easy to understand. Especially, the method A is very helpful when a few numbers of observations are available. Finally, the methods of estimation developed in this study are very well-fitted and fit pleasantly.

*****