Chapter 5: A Study on Chapman Richards Growth Model

5.1 Introduction

In 1938, Karl Ludwing Von Bertalanffy proposed a mechanistic model. His model was

$$\frac{dw}{dt} = \eta w^m - Kw. \tag{5.1}$$

where w is a measure of plant size, t is plant age and η, K, m are parameters. He assumed that the parameter m in the range of $0 < m < 1$. $\eta > 0$ and $K > 0$ are constant of anabolism and catabolism respectively. Unfortunately, the biological basis of this model has been taken too seriously and has led to the use of ill-fitting growth curve [75].

The Chapman-Richards equation was derived from the Von Bertalanffy model when limited imposed by its theoretical background are discarded [92]. In 1959, Dr. F. J. Richards suspected about the theoretical strength and functionality of Von Bertalanffy models as a description of the growth mechanism. He noticed that the freeness of the parameter m in the equation (5.1) provides a flexible family of curves with an arbitrary placed point of inflection. By studying plant growth, Richards [68] proposed to extend the rank of the parameter m of Von Bertalanffy growth model to $m > 0$ rather than $0 < m < 1$. For the positive growth rate with a finite limited size, the parameter η and K must be negative when $m > 1$. Now it may be assumed that $\eta < 0$ and $K < 0$ for $m > 0$, where η and K are constant of catabolism and anabolism respectively. The generalized form of the equation is called Chapman Richards equation because of studies by Richards in 1959 on plant growth and D. G. Chapman in 1961 with fish population [89].

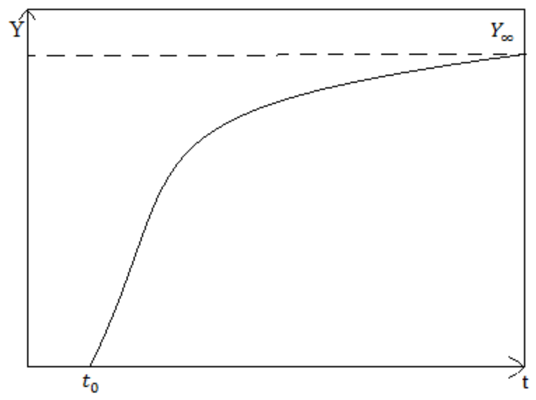

Figure 5.1: Chapman Richards curve for $\boldsymbol{m < 0}$ and $\boldsymbol{\eta, K > 0}$.

Although in the Chapman Richards model, the estimated value of the parameter m assumed to be positive but sometime in some practical application in forestry it may be negative [90] and in that case the inflection points does not exist [[75], [94]]. Also it is convex shaped as shown in Figure 5.1. In that case, the original assumption fails to be valid anymore and then two exceptions could occur in applying Chapman Richards function to tree and stand growth. One occur when $m < 0$ but $\eta > 0$ and $K > 0$ and other occur when $m < 0$ but $\eta > 0$ and $K < 0$. Yuancai et al [90], Fengri et al [23] and Zhao-gang and Feng-ri [94] discussed briefly about these two cases in their respective papers.

5.1.1 Formulation of Chapman-Richards growth functions

5.1.1.1 Integral form

The chapman-Richards growth model is based on the first order ordinary differential equation given by equation (5.1),

$$\frac{dw}{dt} = \eta w^m - Kw,$$

$$\Rightarrow \frac{dw}{dt} + Kw = \eta w^m. \tag{5.2}$$

In the equation (5.2); $\eta < 0$ and $K > 0$ can't be true simultaneously for any m. Here w is the Size of organism, t is the time, $\frac{dw}{dt}$ is the growth rate of the organism, ηw^m is the anabolic growth and Kw is the catabolic growth.

The equation (5.2) is a Bernoulli equation that can be solved with traditional methods. The solution of the differential equation, with the initial condition $w(t_0) = w_0$ is given by,

$$t + c = \frac{m \log w(t) - \log\{Kw(t) - \eta(w(t))^m\}}{K(1-m)}. \tag{5.3}$$

At the initial condition,

$$c = \frac{m\log w_0 - \log\{Kw_0 - \eta(w_0)^m\}}{K(1-m)} - t_0.$$

Putting the value in equation (5.3),

$$t = \frac{m\log\frac{w(t)}{w_0} - \log\frac{\{Kw(t) - \eta\big(w(t)\big)^m\}}{\{Kw_0 - \eta(w_0)^m\}} + t_0K(1-m)}{K(1-m)},$$

$$\Rightarrow \frac{\{Kw(t) - \eta\big(w(t)\big)^m\}}{\{Kw_0 - \eta(w_0)^m\}}.\left(\frac{w(t)}{w_0}\right)^{m-1} = e^{K(t-t_0)(1-m)},$$

$$\Rightarrow w(t) = \left\{\frac{\eta}{K} - \left(\frac{\eta}{K} - (w_0)^{1-m}\right)e^{-K(t-t_0)(1-m)}\right\}^{\frac{1}{1-m}},$$

$$\Rightarrow w(t) = \left|\frac{\eta}{K}\right|^{\frac{1}{1-m}}\left[\pm\left\{1 - \left(1 - \frac{(w_0)^{1-m}}{\eta/K}\right)e^{Kt_0(1-m)}e^{-Kt(1-m)}\right\}\right]^{\frac{1}{1-m}},$$

$$\Rightarrow w(t) = A[\pm\{1 - Be^{-kt}\}]^{\frac{1}{1-m}}, \tag{5.4}$$

where $A = \left|\frac{\eta}{K}\right|^{\frac{1}{1-m}}$, $B = \left(1 - \frac{(w_0)^{1-m}}{\eta/K}\right)e^{Kt_0(1-m)}$ and $k = K(1-m)$.

For $\frac{\eta}{K} > 0$, the equation (5.4) has the $(+)ve$ sign. Otherwise it has the $(-)ve$ sign. The first form of the model (with $(+)ve$ sign) has a wide application in the field of forestry [[23], [90], [22], [94], [24]].

5.1.2 Re-parameterization of the Chapman-Richard model

The growth models can be re-write in some other forms. Some forms of these re-parameterization are more useful due to the specific characteristics of the parameters. But sometime over-parameterization, that is, one has more parameters than are needed, may occur because of the wrong parameterization. Ill-Conditioning can indicate that a model is over-parameterized. Richards's models can also be re-parameterized in several ways. In this study, some useful reparameterization have been discussed. They are very common in the literatures.

From the equation (5.4), the general form of the Chapman Richard model can be written as,

$$w(t) = A\{1 - Be^{-kt}\}^{\frac{1}{1-m}}. \tag{5.5}$$

Here, A, B, k and m are the parameters and are defined correspondingly as: A is the asymptote or the limiting value of the response variable; B is the biological constant, which depends on the size at the beginning; k is the parameter governing the rate at which the response variable approaches its potential maximum and m is the allometric constant, which determine the shape and inflection point of the growth function. If $\frac{1}{1-m} = d$ and $B = \beta$, in the above model (5.5), the new model (5.6) can also be considered as a Chapman Richards model [[59], [60]] with d as the instant rate of growth in the inflection point.

$$w(t) = A\{1 - \beta e^{-kt}\}^{d}. \tag{5.6}$$

In their paper by Tjorve and Tjorve [86] introduced two generalized forms of the Richards model, which are

$$w(t) = A\left(1 + (d-1)\exp\left(-k_1(t-T_i)/d^{d/(1-d)}\right)\right)^{1/(1-d)}, \qquad (5.7)$$

$$w(t) = A\left(1 + \left((y_0/A)^{(1-d)} - 1\right)\exp\left(-k_1 t/d^{d/(1-d)}\right)\right)^{1/(1-d)}, \qquad (5.8)$$

where T_i and y_0 are defined as the time at the inflection (Age at maximum growth) and the mass or the length at age zero (Initial value) respectively and k_1 is the maximum growth rate or the slop at the point of inflection. These two forms can also be expressed as the reparameterization of the form (5.5), by considering $k = \frac{k_1}{d^{d/(1-d)}}, \frac{1}{1-d} = \frac{1}{1-m}$ and $B = (1-d)\exp\left(k_1 T_i/d^{d/(1-d)}\right)$ and $B = 1 - (y_0/A)^{(1-d)}$ respectively in the equations (5.7) and (5.8) respectively. These forms are very needful as each of the parameters controls a separate shape characteristic [86].

The paper by Couble and Lee [13], applied an another form of the Chapman Richards model given in equation (5.9), which is also a reparameterization of the model in the equation (5.5) by considering the size of the organism is zero ($w(t) = 0$) at the start of the growth ($t = t_0$) and with $d = \frac{1}{1-m}$. It is a three parameter growth model with the same definitions of the parameters.

$$w(t) = A\left\{1 - e^{-k(t-t_0)}\right\}^d. \qquad (5.9)$$

An another parameterization was used in lots of literature [[64], [84], [85], [39], [87]] given in equation (5.10), which is just a simplified form of the equation (5.9), by considering the starting time is zero ($t_0 = 0$).

$$w(t) = A\{1 - e^{-kt}\}^d. \qquad (5.10)$$

The Chapman Richards growth model was first used for forest growth modeling in studies report by Turnbull in 1963 and Pienaar in 1965 [90]. This model is universally adequate to forestry application due to its flexibility, accuracy and meaningful analytical properties [[17], [7], [63], [58], [12], [36], [8], [90], [23], [94]]. It accommodates a wide range of growth curves typical of empirical data associated with forest research [2].

In 1983, Oscar Garcia [26] used this model for the height growth of forest stands. He estimated the parameters using a maximum likelihood procedure followed by a modified Newton method. He discussed with two sets of data. One was of Radiate Pine in Kaingoroa forest, New Zealand and the another set was of Radiate Pine of Southland. The dbh growth for 12 stand densities of *Eucalyptus grandis* in South Africa was analyzed through this model by Yuancai et al [90]. They consider the two special cases for this practical application and preferred the case where $m < 0$ and $K < 0$ for some forestry applications. This data set was also used to fit by Fengri et al [23] using the Chapman Richards model. Fengri et al [23] also used to study the volume growth of Permanent sample plot of *Cryptomeria Spp* with 5 initial densities. They compared the model with the Schnute model for both the data sets and finally they concluded that the Chapman Richards model is more simple and convenient and the parameters of this model have a clear biological interpretation. Zhang [93] used Gompertz, Kerf, Landquist, Richards, Weibull and Schnute growth models for examined the changes in stem diameter and height growth in 10 different oak kinds. In 1999, Fekedulegn et al [22] derived the partial derivatives of the Chapman Richards model along with eight other growth models and estimate the parameters using the Marquardt iteration method for top height growth data of Bowment Norway Spruce thining experiment. They determined that though Gompertz and Richards models are not the most efficient models, they play very crucial rules in determining the growth due to meaningful properties of the parameters. They also provide some good initial values for the parameters but missed one thing which was that they told that Chapman Richards growth model has a point of inflection and are sigmoidal. They did not mention anything about the relation between the allometric parameter (m) and the point of inflection, which was indicated by Lei and Zhang [48] in their paper. Lei and Zhang [48] also discussed about the features of the Chapman Richards growth model for forest growth and yield modelling. Zhao-gang and Feng-ri [94] derived a generalized Chapman Richards function and they classified the model into three categories based on the structure of solutions and biological interpretations. They also fitted the generalized model to a group of data set consisting the dbh growth of *Cryptomeria* plantations and the dbh and height growth of Korean Pine tree using the Marquardt method. They also found that the parameters and the expressions of the generalized model were interchangeable in theory and the fitting results were explicitly identical in empirical applications with the Schnute model. The study by Fontes et al. [24] used a dominant height growth of Douglas fir using the Mc-Dill-Amateis, Chapman-Richards and Lundqvist-Korf growth functions. For the selection of the candidate growth model, they adopted three steps. The Chapman Richards and Lundqvist-Korf growth functions were used in their integral and difference forms. The candidate models were fitted by nonlinear regression using the PROC NLIN function based on the Gauss-Newton procedure. Colbert et al. [15] tried to define some characters developments such as forest trees height growth and diameter development by using Chapman Richards, Richards, Von- Bertalanffy and Weibull growth model. They used the Marquardt method in the NLIN procedure to estimate the parameters for each model. In terms of examined features, Chapman Richards model to give the best result. Fang and Gertner [20] studied on white pine height growth in their studies in which they analysed Richards and Morgan-Mercer-Flodin (MMF) growth models. The Bayesian rejection method and nonlinear regression were used to estimate the growth models. Ozel et al. [59] studied the Richards, Gompertz and Weibull growth model to investigate the root collar diameter measurement in beech natural juvenitilies. As a result they found that the Richards growth models provided the best fit. Tjorve and Tjorve [86] introduced two generalized forms of the Richards model family. They mainly discussed various re-parameterization of the four parameter Richards model and represent the negative exponential, Logistic, Von-Bertalanffy and Gompertz models as special cases of the Richards model. Finally they reduced all of these models to only two unified forms of Richard model, which has flexible inflection point and greater degree of freedom compared to the other models (Logistic, Von-Bertalanffy and Gompertz models). They also fitted the models with five other models to six artificial sets of data by using nonlinear regression using Graph Pad Prizm and evaluated the performance based on the corrected Aikaike’s Information Criterion (AIC). As a result they found that only the unified Richard models performed consistently well for all data sets. In his article “additive and multiplicative heat load models comparison”, Erik Kral used the first form of the unified Richards model. Richards, Weibull, Gompertz and Logistic models were used to study the Orient beech Juvenility height and root collar diameter development by Ozel and Ertekin [60]. Results show that the Richards model gave a good fit, while the Gompertz model was quite better for both the height growth and root collar diameter growth. They used the Statistica 6.0 package for analyzed the data. The four parameter Chapman- Richards growth model and three parameter growth model were used to fit the top height age Sitka Spruce national dataset by Lekwadi et al. [49]. They used Gauss- Newton iteration method to estimate the parameters. As a result, they found that the Gauss-Newton method failed to converge for the Chapman-Richards model with four parameters; however the three parameter model estimates all the parameter very logically and highly significantly.

5.1.3 Limiting cases of the Chapman Richards model

From the above discussion, it is clear that each form of the Chapman Richards model can be represented as a re-parameterization of the form in equation (5.5). The Chapman Richards growth model embodies such commonly used growth function as negative exponential, monomolecular, Gompertz, Logistic and the Von Bertalanffy growth models. Special cases of Chapman Richards model are given in **Table 5.1**. The limiting cases of the model also play a key role in forestry study. Lots of author used these models to study the forest system in various time [[52], [55], [21], [22], [57]].

These models can also use in the other field of sciences. Saikia and Borah [[71], [72]] used these models to study oldest-old mortality rates, Kucuk and Eyduran [45] to study Akkaraman and German Blackheaded Mutton X Akkaraman B_1 crosbreed lambs. Kum et al. [46] also used these models for study the weight of Norduz female lambs.

Table 5.1: Limiting cases of Chapmen Richards model with their integral form.

S/N	Common Growth Models	Integral form of the models ($w(t)$)	Derivation from Chapman Richards model
1	Negative Exponential	$A(1-e^{-kt})$	$B=1$ and $m=0$
2	Monomolecular	$A(1-\beta e^{-kt})$	$m=0$ and $\beta=B$
3	Gompertz	$Ae^{-\beta e^{-kt}}$	$m\to 1$ and $\beta=B$
4	Logistic	$\frac{A}{1+\beta e^{-kt}}$	$m=2$ and $\beta=-B$
5	Von Bertalanffy	$\{A^{1-m}-\beta e^{-kt}\}^{\frac{1}{1-m}}$	$\beta=A^{1-m}B$
6	Chapman Richards	$A\{1-\beta e^{-kt}\}^{\frac{1}{1-m}}$	$\beta=B$

5.2 Objective

The main objective of this chapter is to discuss the Chapman Richard growth models in forestry viewpoint and to introduce some new method of estimations to fit the candidate models. Integral form and the limiting case of the Chapman Richard are also discussed along with their various re-parameterizations. The properties of the parameters are studied by observing its nature on forestry.

5.3 Methods and materials

The growth models considered for this study are Chapman Richards model with four parameters (5.6) and Chapman Richards model with three parameters (5.10). For these three models considering, w is the dependent growth variable, t is the independent variable, A, β, K and d are parameters to be estimated and $\exp(e)$ is the base of the natural logarithms. The parameters are estimated using the new methods introduced in this chapter. The maximum diameter data and top height growth of babul *(Acacia Nilotica)* tree are used to fit the growth model. These two sets of data are presented in **Table 2.1**. The data are based on the analysis of sample plot data of Uttar Pradesh, Maharashtra and Madhya Pradesh [37]. The top height age, the cumulative basal area production and the mean diameter at breast height data, originated from the Bowmont Norway spruce thinning experiment, sample plot 3661 [[21], [22]] are also used and presented in **Table 2.2**. After fitting the growth models, the criteria for comparing models are considered from chapter I.

5.3.1 Method of estimation

Most of the literature discussed in this study used to fit the Chapmen Richards model by using some well-known algorithm. This study is trying to introduce some new methods of estimation by which one can easily fit the model without using any dearly-won software.

5.3.1.1 Fitting of the Three Parameter Model in equation (5.10)

5.1.3.1.1 Method A: (method using three equidistance points)

In this method, let n be the total number of observation and let $r = \left[\frac{n}{3}\right]$[1], $t_a = r, t_b = 2r$ and $t_c = 3r$. Then for $i = a, b, c$; the equation (5.10) can be written as

$$\ln w_i = \ln A + d \ln(1 - e^{-kt_i}). \tag{5.11}$$

Now

$$\ln w_a - \ln w_b = d \ln\left\{\frac{1 - e^{-kt_a}}{1 - e^{-kt_b}}\right\}. \tag{5.12}$$

Similarly

$$\ln w_c - \ln w_b = d \ln\left\{\frac{1 - e^{-kt_c}}{1 - e^{-kt_b}}\right\}. \tag{5.13}$$

Now by solving the equation (5.11), (5.12) and (5.13); the parameters can be estimated and then the estimated parameters are given by

$$\hat{k} = \frac{\ln w_c - \ln w_b}{r(\ln w_b - \ln w_a)}, \quad \hat{d} = \frac{\ln w_a - \ln w_b}{\ln \frac{1 - e^{-kt_a}}{1 - e^{-kt_b}}},$$

$$\hat{A} = \exp\{\ln w_c - r \ln(1 - e^{-kt_c})\}.$$

[1] Greatest integer function.

Where w_a, w_b and w_c are respective observations at time t_a, t_b and t_c respectively.

5.1.3.1.2 Method B: (method using two partial sums)

In this method, assume that the parameter k is known from the method I. Then let n be the total number of observation and let $r = \left[\frac{n}{2}\right]^1$, $S_1 = \sum_{i=1}^{r} \ln w_i$ and $S_2 = \sum_{i=r+1}^{2r} \ln w_i$. Then the equation (5.10) can be written as

$$\ln w = \ln A + d \ln(1 - e^{-kt}). \tag{5.14}$$

Now,

$$S_1 = r \ln A + d \ln \left\{ \prod_{i=1}^{r} (1 - e^{-ki}) \right\}, \tag{5.15}$$

and

$$S_2 = r \ln A + d \ln \left\{ \prod_{i=r+1}^{2r} (1 - e^{-ki}) \right\}. \tag{5.16}$$

Now by solving the equations (5.14), (5.15) and (5.16); the estimated parameters are given by

$$\hat{d} = \frac{S_2 - S_1}{\{\ln(\prod_{i=r+1}^{2r}(1 - e^{-kt_i})) - \ln(\prod_{i=1}^{r}(1 - e^{-kt_i}))\}},$$

$$\hat{A} = \exp\left\{ \frac{S_1 - d \ln(\prod_{i=1}^{r}(1 - e^{-kt_i}))}{r} \right\}.$$

After estimating the parameters A and d, the parameter k can be estimated as

$$\hat{k} = \frac{1}{n} \ln \left[\prod_{i=1}^{n} \frac{1}{\left\{1 - \left(\frac{w_i}{A}\right)^{\frac{1}{d}}\right\}^{\frac{1}{t_i}}} \right].$$

5.3.1.2 Fitting of the Four Parameter Model in equation (5.6)

5.1.3.1.3 Method A: (composite method using three equidistance points)

In this method, first, assume that the parameter d is known from its definition. Then let n be the total number of observation and let $r = \left[\frac{n}{3}\right]^1$, $t_a = r, t_b = 2r$ and $t_c = 3r$. Then for $i = a, b, c$; the equation (5.6) can be written as

$$\ln w_i = \ln A + d \ln(1 - \beta e^{-kt_i}). \tag{5.17}$$

Now,

$$\ln w_a - \ln w_b = d \ln \left\{ \frac{1 - \beta e^{-kt_a}}{1 - \beta e^{-kt_b}} \right\}, \tag{5.18}$$

and

$$\ln w_b - \ln w_c = d \ln \left\{ \frac{1 - \beta e^{-kt_b}}{1 - \beta e^{-kt_c}} \right\}. \tag{5.19}$$

From the equation (5.18) and (5.19), an equation is obtained as

$$(A_1 A_1 - A_1)x^2 + (1 - A_1 A_1)x + (A_2 - 1) = 0, \tag{5.20}$$

where $A_1 = \exp \frac{\ln w_a - \ln w_b}{d}$ and $A_2 = \exp \frac{\ln w_b - \ln w_c}{d}$.

Let x is the positive root of the equation (5.20), the parameter k can be estimated as,

$$\hat{k} = -\frac{1}{r}\ln x.$$

After finding the parameter k; the parameters β, A and d can be estimated using the equations (5.17), (5.18) and (5.19). And the parameters are given by

$$\hat{\beta} = \frac{1 - \exp\frac{\ln w_b - \ln w_c}{d}}{\exp(-kt_b) - \exp\frac{\ln w_b - \ln w_c}{d}\exp(-kt_c)},$$

$$\hat{A} = \exp\{\ln w_c - r\ln(1 - \beta e^{-kt_c})\},$$

$$\hat{d} = \frac{\ln\ w_a - \ln A}{\ln(1 - \beta e^{-kt_a})}.$$

5.1.3.1.4 Method B: (composite method using two partial sums)

In this method, assume that the parameter β and k are known. Then let n be the total number of observation and let $r = \left[\frac{n}{2}\right]$, $S_1 = \sum_{i=1}^{r}\ln w_i$ and $S_2 = \sum_{i=r+1}^{2r}\ln w_i$. Then the estimated parameters are given by

$$\hat{d} = \frac{S_2 - S_1}{\{\ln(\prod_{i=r+1}^{2r}(1 - \beta e^{-kt_i})) - \ln(\prod_{i=1}^{r}(1 - \beta e^{-kt_i}))\}},$$

$$\hat{A} = \exp\left\{\frac{S_1 - d\ln(\prod_{i=1}^{r}(1 - \beta e^{-kt_i}))}{r}\right\}.$$

After estimating the parameters A and d; the parameter k and β can be estimated as

$$\hat{k} = \frac{1}{\sum_{i=r+1}^{2r} t_i - \sum_{i=1}^{r} t_i} \ln\left[\frac{\prod_{i=1}^{r}\left(1 - \left(\frac{w_i}{A}\right)^{\frac{1}{d}}\right)}{\prod_{i=r+1}^{2r}\left(1 - \left(\frac{w_i}{A}\right)^{\frac{1}{d}}\right)}\right],$$

$$\hat{\beta} = \exp\left\{\frac{k}{n}\sum_{i=1}^{n} t_i + \frac{1}{n}\ln\left\{\prod_{i=1}^{n}\left(1 - \left(\frac{w_i}{A}\right)^{\frac{1}{d}}\right)\right\}\right\}.$$

5.4 Results and discussion

5.4.1 Properties of Chapman Richards model

In this Chapter, an attempt has been made to discuss the different fundamental properties of the Chapman Richards growth models.

In the form of the model (5.6), A, β, k and d are the parameters and are defined correspondingly as: A is the asymptote or the limiting value of the response variable, β is the biological constant, k is the parameter governing the rate at which the response variable approaches its potential maximum and d is the allometric constant [22] and t is the independent variable w is the response variable of t.

The Chapman Richards growth model rises from a point $A\{1-\beta\}^d$ to the limiting value of A, which is the maximum possible value of $w(t)$. Examining the model at the starting of the growth, which is most preferably when the independent variable (t) is zero, the only way to understand and make clear the meaning and possible range of the parameter β that is defined as a biological constant. In other word from the expression $w(0) = A\{1-\beta\}^d$, it is logical to define the parameter β as a constant that should make the expression $A\{1-\beta\}^d$ reasonably small enough to at least consider it as a possible value of the model parameter estimate at the starting growth.

Based on the model assumption and evaluation of the model at the start of growth, it is evident that:

a) $A > 0$, since A is the limiting value.
b) The parameter β is always positive ($\beta > 0$) and its size depends on the size of the parameter A. Since if $\beta = 0$ then $y = A$ at the starting of the growth and if $\beta < 0$ then $y > A$ at the start of the growth and both cases violate the model assumption concerning the parameter A which states that when $t \to \infty \Rightarrow y \to A$.
c) For biological growth analysis, the parameter k must be positive.
d) From the discussion of the parameter m in the previous sections, it is clear that the parameter d is also positive.

The Chapman Richards model may pass through the origin only when the value parameter $\beta = 1$. It gives another form of the Chapman Richards model presented in equation (5.10). Now,

$$\frac{dw}{dt} = A\beta kd(1 - \beta e^{-kt})^{d-1}e^{-kt}. \tag{5.21}$$

The first derivative of the model describes the slope of the curve or the rate of change of the dependent variable with respect to the independent variable and is positive. This indicates that the model is an increasing function of the independent variable.

Now to investigate some important properties of the model, the second derivative of the model is derived and is given by

$$\frac{d^2w}{dt^2} = A\beta^2k^2d(d - 1)(1 - \beta e^{-kt})^{d-2}e^{-2kt} - A\beta k^2d(1 - \beta e^{-kt})^{d-1}e^{-kt},$$

$$\frac{d^2w}{dt^2} = A\beta k^2 d(\beta d e^{-kt} - 1)(1 - \beta e^{-kt})^{d-2} e^{-kt}. \qquad (5.22)$$

It is seen that the second order derivative of the model is positive $\left(\frac{d^2w}{dt^2} > 0\right)$ or the model approaches the asymptote at an increasing rate for $w < A\left(\frac{d-1}{d}\right)^d$ and negative $\left(\frac{d^2w}{dt^2} < 0\right)$ or the model approaches the asymptote at an decreasing rate for $w > A\left(\frac{d-1}{d}\right)^d$. The point where the model makes a transition from an increasing to a decreasing slope, the second derivative of the model is zero $\left(\frac{d^2w}{dt^2} = 0\right)$ and at that point the growth function has a constant slope. The point is known as the point of inflection and it occurs at $w = A\left(\frac{d-1}{d}\right)^d$. This is the most important property of the model. In terms of the predictor variable; the second derivative is positive for $0 \leq t < \log(\beta d)^{\frac{1}{k}}$; zero for $t = \log(\beta d)^{\frac{1}{k}}$ and negative for $\log(\beta d)^{\frac{1}{k}} < t < \infty$.

Table 5.2: Summary of some basic properties of Chapman Richards model.

	Chapman Richards model with four parameter	**Chapman Richards model with three parameter**
Integral form of the growth function	$w(t) = A\{1 - \beta e^{-kt}\}^d$	$w(t) = A\{1 - e^{-kt}\}^d$
Upper asymptote	A	A
Starting point of the growth function	$A\{1 - \beta\}^d$	0
Growth rate $\left(\frac{dw}{dt}\right)$	$A\beta kd\left(1 - \beta e^{-kt}\right)^{d-1} e^{-kt}$	$Akd\left(1 - e^{-kt}\right)^{d-1} e^{-kt}$
Maximum growth rate	$Ak\left(\frac{d-1}{d}\right)^{d-1}$	$Ak\left(\frac{d-1}{d}\right)^{d-1}$

Relative growth rate as function of time	$\frac{\beta k d}{e^{kt} - \beta}$	$\frac{kd}{e^{kt} - 1}$
Relative growth rate as function of response variable	$Kd\left\{(A/w)^{\frac{1}{d}} - 1\right\}$	$Kd\left\{(A/w)^{\frac{1}{d}} - 1\right\}$
Second derivative of the growth function $\left(\frac{d^2w}{dt^2}\right)$	$A\beta k^2 d(\beta d e^{-kt} - 1)\left(1 - \beta e^{-kt}\right)^{d-2} e^{-kt}$	$Ak^2 d(d e^{-kt} - 1)\left(1 - e^{-kt}\right)^{d-2} e^{-kt}$
Point of inflection $w(t) =$	$A\left(\frac{d-1}{d}\right)^d$	$A\left(\frac{d-1}{d}\right)^d$
Domain of the independent variable	$[0, \infty)$	$[0, \infty)$
Domain of the dependent variable	$(A\{1-\beta\}^d, A)$	$(0, A)$

For the Chapmen Richards model with three parameters (5.10), the parameters A, k and d have same biological significance as the model (5.6). The properties can also be analysed by considering the parameter $\beta = 1$ in the model form (5.6). Some basic properties of four parameter Chapman Richards model in equation (5.6) and the three parameter Chapman Richards model in equation (5.10) are summarised in the **Table 5.2**.

5.4.2 Parameter Estimation

Two Chapman Richards models are fitted to the maximum diameter data, top height growth of babul *(Acacia Nilotica)* tree in India and the top height age, the cumulative basal area production and the mean diameter at breast height data originated from the

Bowmont Norway. The different methods of estimation used for the Chapman Richards models are described above.

5.4.2.1 For top height growth of babul tree

The parameter estimates for the Weibull models with the corresponding observed, predicted value along with statistical analysis to top height data of babul tree are presented in **Table 5.3**.

Table 5.3: Fitting of Chapman Richard growth models for top height growth data of Babul tree.

Age	Observed data	Estimated data			
		Chapman Richard with four parameter		Chapman Richard with three parameter	
		Method A	Method B	Method A	Method B
5	8.14	8.14	8.13	9.03	8.16
10	12.19	12.19	12.15	12.49	12.61
15	14.93	14.93	14.86	14.85	15.20
20	16.70	16.81	16.72	16.61	16.73
25	17.98	18.11	18.01	17.98	17.64
Parameters	A	21.0944	20.9811	24.1618	18.97
	β	0.9341	0.9344	--	--
	k	0.3582	0.3569	0.1703	0.5169
	d	0.90	0.8935	0.5312	0.9298
χ^2		Not Applicable		0.095	0.025
$RMSE$		0.078	0.042	0.421	0.271
R^2 $(in\ \%)$		99.95	99.99	98.56	99.41
R_a^2		0.99	0.99	0.97	0.99
$R^2_{prediction}$ $(in\ \%)$		99.86	99.98	98.47	98.86

To analyze the fit, the selection criteria are used from chapter 1. In case of top height data, all estimated parameters are logically consistent and biologically significant. The chi-square test is not applicable for Chapman Richards model with four parameters as this study used a data set with five observations and the model has four parameters and the resulting degree of freedom becomes zero. Based on step II, Chapman

Richards model with three parameters (method A and method B) are rejected due to having less than 95% level of significance. In the fourth step, no growth results are eliminated as both results have R_a^2 value 0.99.

The 95% confidence levels of all surviving results are demonstrated in **Table 5.8**. All estimated parameters of Chapman Richards model with four parameters (method A and method B) are significantly different from zero at 95% confidence level. Finally, in case of top height growth of Babul tree it is observed that, the Chapman Richards model with four parameters (method B) provides better fit with the value of $R^2_{prediction}$ and R^2 are 99.98 and 99.99 respectively.

5.4.2.2 For maximum diameter growth of babul tree

The estimation of parameters for the Chapman Richard models and the summary of statistical analysis to maximum diameter growth data of babul tree are presented in Table 5.4. For Chapman Richard models, no result is rejected in step I. The eliminated results in each step are also highlighted accordingly in Table 5.4. For maximum diameter growth data of babul tree, the chi-square test is also not applicable for Chapman Richard model with four parameters as it has zero degree of freedom. Chapman Richard model with three parameters (method A and method B) are rejected due to having less than 95% level of significance. In the third step, comparing the value of RMSE, Chapman Richard model with four parameters (method A and method B) are promoted to the next level. No results are eliminated in step V, as all parameters of the surviving results are significantly different from zero at 95% confidence level (**Table 5.8**). At last, the best fit model is chosen and find that Chapman Richard model with four parameters (method A) with the $R^2_{prediction}$ and R^2 values 99.94 and 99.98 respectively is the best among all the models.

Table 5.4: Fitting of Chapman Richard growth models for maximum diameter growth data of Babul tree.

Age	Observed data	Estimated value			
		Chapman Richard with four parameter		Chapman Richard with three parameter	
		Method A	Method B	Method A	Method B
5	12.19	12.19	11.89	14.20	12.18
10	20.83	20.83	20.68	20.80	21.46
15	26.92	26.92	26.88	25.97	27.51
20	31.49	31.30	31.33	30.38	31.33
25	34.29	34.47	34.56	34.29	33.71
Parameters	A	43.1172	43.3367	179.8659	37.5642
	β	1.0263	1.0358	--	--
	k	0.3079	0.3087	0.0104	0.4874
	d	0.90	0.9046	0.5550	1.1821
χ^2		Not Applicable		0.359	0.042
$RMSE$		0.118	0.207	1.109	0.469
R^2 $(in\ \%)$		99.98	99.93	98.04	99.65
R_a^2		0.99	0.99	0.96	0.99
$R^2_{prediction}$ $(in\ \%)$		99.94	99.87	97.48	99.32

5.4.2.3 For top height growth from the Bowmont Norway spruce Thinning Experiment

The estimated parameters along with the observed and predicted values for top height growth data from Bowmont Norway spruce thinning experiment along with the statistical analysis are presented in **Table 5.5**. All parameters of the candidate models are logically consistent and biologically significant. The table values of χ^2 for 95% level of significance is found to be higher than the calculated χ^2 values for Chapman Richard models. The 95% confidence levels of all surviving results are also demonstrated in **Table 5.8**; where all surviving models are significantly different from zero. No results are rejected in the step IV as the adjusted determination coefficient values of the models are found to be 0.99. Finally, the Chapman Richard model with four parameters (method B) is found to be more appropriate for the top height growth data from Bowmont Norway spruce thinning experiment with $R^2_{prediction}$ and R^2 value 99.86 and 99.90 respectively.

Table 5.5: Fitting of the Chapman Richard growth models for the top height of Bowmont Norway spruce thinning experiment.

Age	Observed data	Estimated data			
		Chapman Richard with four parameter		Chapman Richard with three parameter	
		Method A	Method B	Method A	Method B
20	7.3	7.23	7.16	5.58	5.77
25	9	9.14	9.09	9.12	9.14
30	10.9	10.90	10.87	11.77	11.64
35	12.6	12.52	12.51	13.80	13.56
40	13.9	14.01	14.03	15.38	15.07
45	15.4	15.40	15.44	16.62	16.27
50	16.9	16.69	16.76	17.60	17.23
55	18.2	17.89	17.98	18.38	18.01
60	19	19	19.11	19.00	18.63
64	20	20.04	20.18	19.49	19.14
Parameters	A	35.1015	35.6324	21.4329	21.3405
	β	0.8821	0.8827	--	
	k	0.0643	0.0643	0.2231	0.2026
	d	0.9	0.9124	0.8359	0.7712
χ^2		.01274	0.01195	0.9735	0.7323
$RMSE$		0.1383	0.1307	0.981	0.8694
R^2 $(in\ \%)$		99.89	99.90	94.32	95.53
R^2_a		0.99	0.99	0.93	0.94
$R^2_{prediction}$ $(in\ \%)$		99.85	99.86	93.61	94.82

5.4.2.4 For mean diameter at breast height growth from the Bowmont Norway spruce Thinning Experiment

The estimation of parameters for the growth models along with the summary of statistical analysis to mean diameter at breast height are presented in **Table 5.6**. The eliminated results in each step are also highlighted accordingly. Here, all the estimated parameters are biologically realistic.

Table 5.6: Fitting of the Chapman Richard growth models for mean diameter at breast height of Bowmont Norway spruce thinning experiment.

Age	Observed data	Estimated data			
		Chapman Richard with four parameter		Chapman Richard with three parameter	
		Method A	Method B	Method A	Method B
20	8.40	6.74	7.09	5.69	6.09
25	10.40	9.65	10.08	9.61	10.54
30	12.35	12.35	12.77	12.87	14.05
35	14.74	14.88	15.23	15.68	16.86
40	17.13	17.26	17.51	18.14	19.13
45	19.50	19.50	19.63	20.32	20.97
50	21.49	21.62	21.63	22.26	22.47
55	23.82	23.64	23.48	23.99	23.68
60	25.55	25.55	25.24	25.55	24.67
64	26.50	27.37	26.89	26.95	25.47
Parameters	A	66.6035	60.3632	40.6147	29.0682
	β	0.9618	0.9676	--	--
	k	0.0427	0.0435	0.0922	0.2011
	d	0.9000	0.8207	0.8094	0.9179
χ^2		0.502	0.307	1.563	1.782
$RMSE$		0.646	0.528	1.080	1.475
R^2 (in %)		98.87	99.24	96.85	94.14
R_a^2		0.98	0.99	0.96	0.92
$R_{prediction}^2$ (in %)		98.69	99.17	96.94	93.26

For mean diameter at breast height data, it is also observed that Chapman Richard growth model with three parameters (method B) are eliminated as its calculated χ^2 value less then tabulated value at 95% level of significance. Chapman Richard with four parameters (method A and method B) is promoted to the next step with the least value of RMSE in step III. In step IV, Chapman Richard with four parameters (method A) is eliminated as some of their parameters are not significantly different from zero (**Table 5.8**). And finally the best result is chosen and it is found as Chapman Richard with four parameters (method A) with $R^2_{prediction}$ and R^2 values 99.17 and 99.24 respectively.

5.4.2.5 For cumulative basal area production from the Bowmont Norway spruce Thinning Experiment

The estimation of parameters for the growth models and the summary of statistical analysis to cumulative basal area production are presented in **Table 5.7**. The best result is selected and found as Chapman Richard with four parameters (method B) with $R^2_{prediction}$ and R^2 values 99.76 and 99.83 respectively. The eliminated results in each step are highlighted accordingly in the **Table 5.7**.

Table 5.7: Fitting of the Chapman Richard growth models for cumulative basal area production of Bowmont Norway spruce thinning experiment.

Age	Observed data	Estimated data			
		Chapman Richard with four parameter		Chapman Richard with three parameter	
		Method A	Method B	Method A	Method B
20	37.99	37.50	36.66	32.63	29.15
25	49.00	49.35	48.52	48.53	49.53
30	60.41	60.41	59.60	61.21	65.32
35	68.91	70.79	70.02	72.17	77.79
40	78.73	80.58	79.85	82.00	87.72
45	89.83	89.83	89.14	91.02	95.68
50	98.60	98.59	97.95	99.42	102.08
55	107.00	106.91	106.31	107.32	107.23
60	114.80	114.80	114.26	114.80	111.39
64	119.54	122.31	121.82	121.94	114.75
Parameters	A	283.3189	284.1495	8951.47	129.0303
	β	0.9336	0.9350	--	--
	k	0.0431	0.0432	0.0001	0.2095
	d	0.9000	0.9067	0.5725	0.8929
χ^2		0.164	0.156	1.241	5.775
$RMSE$		1.223	1.093	2.425	5.859

$R^2\ (in\ \%)$	99.79	99.83	99.17	95.18
R_a^2	0.99	0.99	0.99	0.94
$R_{prediction}^2\ (in\ \%)$	99.69	99.76	99.06	94.37

Chapman Richard with three parameters (method A) is eliminated in step I due to the non-logical estimation of the parameters. Chapman Richard with three parameters (method B) is also eliminated in step II as its calculated χ^2 value less then tabulated value at 95% level of significance.

Table 5.8: 95% Confidence intervals of the parameters of Chapman Richards growth models.

Data	Models	Method	A		β		k		d	
			Lower limit	Upper limit	Lower limit	Lower limit	Upper limit	Upper limit	Lower limit	Upper limit
Top height growth of babul trees.	Chapman Richards Four Parameters	*A*	20.262	21.927	0.802	1.067	0.277	0.440	0.582	1.218
		B	20.496	21.466	0.309	1.405	0.309	0.405	0.709	1.078
Maximum diameter growth of babul trees	Chapman Richards Four Parameters	*A*	41.403	44.831	0.979	1.074	0.255	0.361	0.736	1.064
		B	40.166	46.507	0.955	1.117	0.212	0.405	0.610	1.199
Top height growth data from Bowmont	Chapman Richards Four Parameters	*A*	28.501	41.702	0.787	0.977	0.033	0.096	0.667	1.133
		B	28.944	42.321	0.790	0.976	0.033	0.095	0.680	1.145
mean diameter at breast height	Chapman Richards Four Parameters	*A*	0.533	132.674	0.843	1.080	-0.03	0.125	0.443	1.357
		B	37.932	82.794	0.925	1.010	0.009	0.077	0.656	0.985
Cumulative basal area	Chapman Richards Four Parameters	*A*	147.012	419.625	0.838	1.029	0.001	0.086	0.625	1.175
		B	155.244	413.055	0.847	1.023	0.003	0.083	0.649	1.164

Chapman Richard with four parameters (method A and method B) is promoted to the next step with the least value of RMSE in step III. In step IV, no result is eliminated as all of the parameters are significantly different from zero (**Table 5.8**).From the results, it is observed that for top height growth data of babul tree, top height age data, the mean diameter at breast height data and for cumulative basal area production from the Bowmont Norway spruce thinning; Chapman Richards growth model with four parameters (method B) is found to be more suitable than the remaining growth models. Whereas for maximum diameter growth data of babul tree, the Chapman Richards growth model with four parameters (method A) provides a good fit. Chapman Richards growth model with three parameters (method A and method B) unable us to provide fit for all sets of data either due to failed in chi-square test or because of the unrealistic estimate of the parameters.

5.5 Conclusion

The basic focus of this Chapter is to account for the Chapman Richards growth model and introduced few methods of estimation to fit the model which are easy to use and free from any cost. The strong literature review and discussion on the evolution of the Chapman Richards model may help to understand the model intimately. This Chapter also discuss about various parameterizations and limiting case of the model. Finally, the methods of estimation develop in this study are very well-fitted and can compete with the existing methods.

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