Chapter 2

Linear and Nonlinear QCD Evolution Equations

2.1 Linear evolution equations

QCD induces higher order corrections to the naive parton model which eventually lead to a breaking of scaling violations. Thus QCD enables the explicit estimation of the dependence of the structure function on Q^2, however, it does not reveal the specific value of F_2 for a given Q^2, but preferably portrays in what manner F_2 varies with Q^2 from a given input. The Q^2 dependence of the PDFs can be computed perturbatively as long as Q^2 is adequately large so that α_s continues to be small. The standard and the basic theoretical frameworks employed to study the scale dependence of the PDFs and eventually the DIS structure functions are the linear Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) [1-4] evolution equations. One can calculate the PDFs for any value of Q^2 making use of the DGLAP equations considering that an initial condition for the PDFs is indeed available at a given initial scale Q_0^2 and then evolving to higher Q^2. The DGLAP approach sums up higher-order α_s contributions enhanced by the logarithm of photon virtuality, i.e. $\alpha_s^n \ln(Q^2)^n$ in the perturbative expansion. Nevertheless, at small-x contributions enhanced by the logarithm of a small momentum fraction, x, carried by gluons, turns out to be essential. Accordingly a different approach is needed to explain the situation of high-energy or in other words small-x scattering. The leading logarithm (LL) contributions of $(\alpha_s \ln(1/x))^n$ are summed up by the Balitsky-Fadin-Kuraev-Lipatov (BFKL) evolution equation

[5-7]. Another evolution equation to study the linear evolution of PDFs in the small-x regime is the so called Catani-Ciafaloni-Fiorani-Marchesini (CCFM) equation [8-10]. The CCFM approach retains the components of both the DGLAP and BFKL realms in the LL approximation. All the aforementioned evolution equations are linear in parton density which have to be modified in a suitable way to add the higher twist approximations at very small-x. A brief account of the linear evolution equations is given below.

2.1.1 DGLAP equation

The evolution of the structure functions or more precisely the quark and gluon distribution functions with Q^2 can be described by the DGLAP evolution equations [1-4]. These equations sum all leading Feynman diagrams that give rise to the logarithmically enhanced $\ln(Q^2)$ contributions to the cross section in order to neglect any kind of higher twist corrections. The associated perturbative resummation is organized in powers $\alpha_s^n \ln(Q^2)^n$. They are the conventional and the fundamental theoretical frameworks for all of the phenomenological perspectives used to interpret hadron interactions at short distances. The DGLAP equations for quark and gluon density can be written as

$$\frac{\partial}{\partial \ln Q^2}\begin{pmatrix} q_i(x,Q^2) \\ g(x,Q^2) \end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi}\sum_{i=1}^{2N_f}\int_x^1 \frac{dy}{y}\begin{pmatrix} P_{qq}(\frac{x}{y}) & P_{qg}(\frac{x}{y}) \\ P_{gq}(\frac{x}{y}) & P_{gg}(\frac{x}{y}) \end{pmatrix}\begin{pmatrix} q_i(y,Q^2) \\ g(y,Q^2) \end{pmatrix}, \tag{2.1}$$

where the sum runs over all flavors of quarks and anti-quarks. Here $q_i(x,Q^2)$ stands for quark density whereas $g(x,Q^2)$ represents gluon density. P_{qq}, P_{qg}, P_{gq} and P_{gg} are the splitting functions whose interpretations are graphically displayed in Fig.2.1. The splitting functions are elucidated as the probability for finding a parton (quark or gluon) of type i having momentum fraction x arising from a parton (quark or gluon) j with larger momentum fraction $y > x$. They are independent of the quark flavors and are identical for quarks and antiquarks. The leading order splitting functions are given by [11]

$$P_{qq}^{(0)}(z) = C_F\Big(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\Big),$$

$$P_{qg}^{(0)}(z) = \frac{1}{2}\big(z^2 + (1-z)^2\big),\;\; P_{gq}^{(0)}(z) = C_F\Big(\frac{1+(1-z)^2}{z}\Big) \text{ and}$$

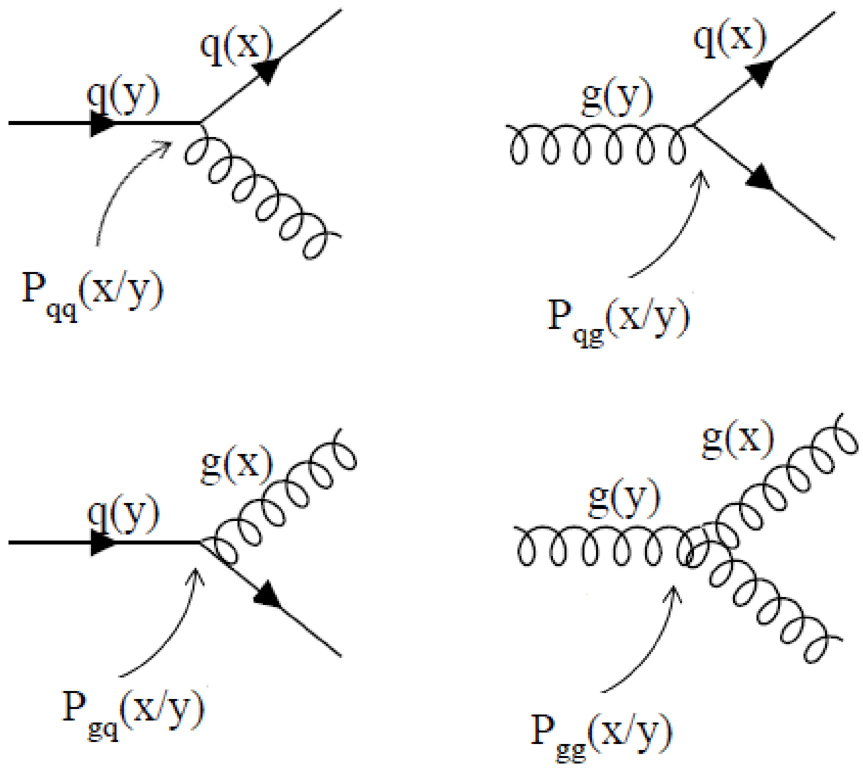

Figure 2.1: Splitting functions

$$P_{gg}^{(0)}(z) = 2N_c\Big(\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\Big) + \Big(\frac{11}{2} - \frac{N_f}{3}\Big)\delta(1-z), \tag{2.2}$$

with $C_F = \frac{(N_c^2-1)}{2N_c}$. The "+" distribution is defined by the property [11]

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z)-f(1)}{(1-z)}, \tag{2.3}$$

where $(1-z)_+ = 1-z$ for $z<1$ but is infinite for $z=1$. The discrepancy at $z=1$ complements the radiation of soft gluons and is balanced out by the virtual gluon loop contributions.

In perturbation theory, the splitting functions can be expressed as a power series of $\alpha_s(Q^2)$ [11, 12]

$$P_{ab}\big(z,Q^2\big) = P_{ab}^{(0)}(z) + \Big(\frac{\alpha_s(Q^2)}{2\pi}\Big)P_{ab}^{(1)}(z) + \Big(\frac{\alpha_s(Q^2)}{2\pi}\Big)^2 P_{ab}^{(2)}(z) + \cdots, \tag{2.4}$$

with $z = \frac{x}{y}$. These functions are at present working up to next-to-next-to-leading order (NNLO) accuracy. The leading order (LO) expressions $P^{(0)}$ are the well-known Altarelli-Parisi splitting functions [4, 11]. On the other hand, the next-to-leading order (NLO) functions $P^{(1)}$ have been estimated during the time 1977-1980 [13-16], whereas the NNLO terms $P^{(2)}$ are calculated in the period 2004 [17, 18]. The

LO DGLAP evolution sums up the leading log contributions $(\alpha_s \ln(Q^2))^n$, the NLO evolution incorporates the sum of the $\alpha_s(\alpha_s \ln(Q^2))^{n-1}$ terms and so on.

The derivation of the DGLAP equation is founded on the QCD collinear factorization in gluon emission to legitimize the resummation of logarithms in the transverse scale. In consonance with the traditional collinear factorization approach the hadronic observables can be expressed as the convolution of the PDFs with partonic hard-scattering coefficients. The partonic coefficients are computed with the assumption that the hard scattering is originated by a parton collinear to its parent hadron. Customarily the large logarithms are obtained from the region in phase space where

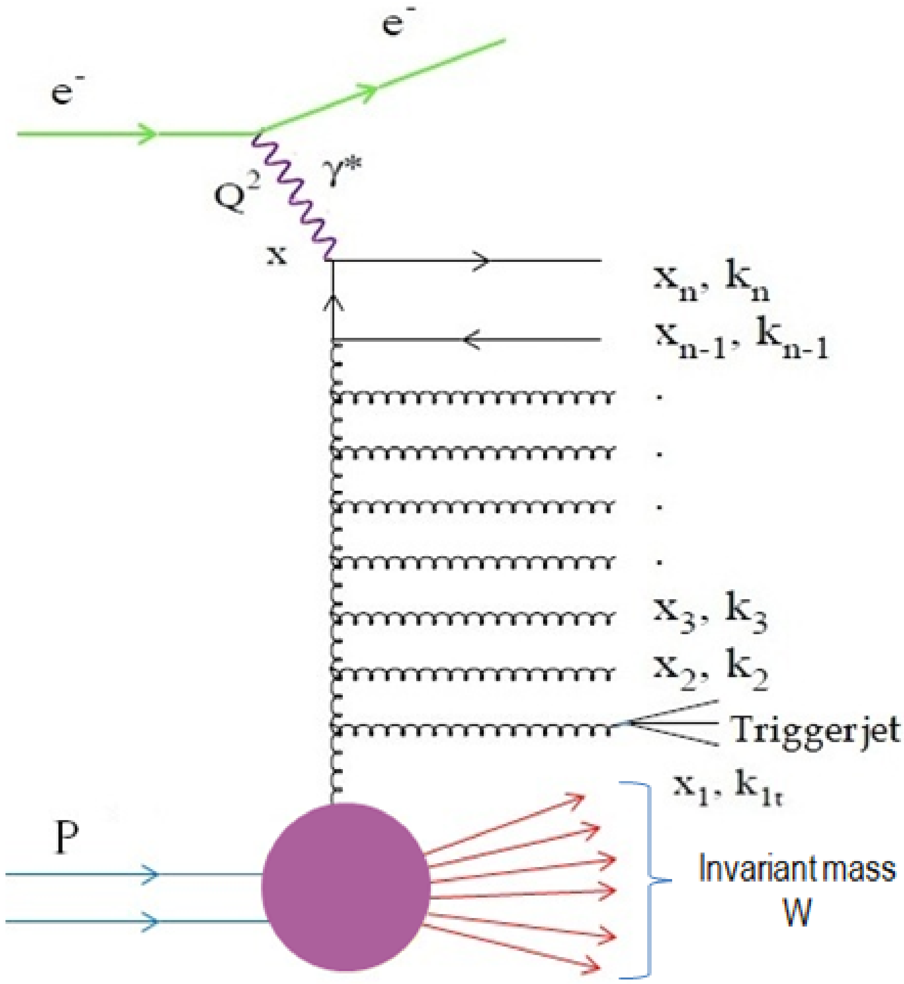

Figure 2.2: Ladder-diagram in LLQ^2 application of DIS

the multiple emissions are strongly set in order in transverse momenta with succeeding emissions having larger momenta, i.e. $Q^2 >> k_n^2 >> \cdots >> k_2^2 >> k_1^2$. Fig.2.2 exhibits a schematic ladder diagram of the quark and gluon emissions in LL(Q^2) application of DIS.

The non-singlet and singlet combinations of the quark flavor group can be defined as [11]

$$q_{NS} \equiv q_i - q_j, \tag{2.5}$$

$$q_S \equiv \sum_i q_i. \tag{2.6}$$

The DGLAP equations for non-singlet and singlet quark distributions are

$$\frac{\partial q_{NS}(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dy}{y} P_{qq}(x/y) q_{NS}(y,Q^2), \tag{2.7}$$

$$\frac{\partial q_S(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dy}{y}\Big(\sum_{i=1}^{2N_f} P_{qq}(x/y) q_i(y,Q^2) + P_{qg}(x/y) g(y,Q^2)\Big). \tag{2.8}$$

As an illustration, the first term of Eq.(2.8) mathematically articulates the fact that a quark with momentum fraction x characterized by $q(x, Q^2)$ (on the left hand side) could have originated form a parent quark with a momentum fraction $y > x$ depicted by $q(y, Q^2)$ (on the right-hand side) which has radiated a gluon. The probability of occurrence of this process is proportional to $\alpha_s P_{qq}(x/y)$. The second term deals with the prospect that a quark with momentum fraction x is the consequence of $q\bar{q}$ pair creation by a parent gluon with momentum fraction $y > x$ and the probability that it happens is proportional to $\alpha_s P_{qg}(x/y)$. The integration appears because of the consideration that the secondary quark with momentum x can come from a parent quark with any momentum fraction $y > x$ [11].

On the other hand, since the gluon distribution does not carry any flavor quantum numbers, it is a flavor singlet and the DGLAP equation for gluon distribution is given by

$$\frac{\partial g(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dy}{y}\Big(\sum_{i=1}^{2N_f} P_{gq}(x/y) q_i(y,Q^2) + P_{gg}(x/y) g(y,Q^2)\Big). \tag{2.9}$$

As soon as the x dependence of quark or gluon distributions are known at some initial scale Q_0^2 then they can be determined for any higher value of Q^2 by using the DGLAP equations. The initial distributions are at present have to be computed from experiment presuming an input form in x which complies with the QCD sum rules. This strategy is adopted in the global analyses of PDFs [19, 20]. As an alternative, one may produce the parton distributions dynamically originating from an input distribution for the valence quarks and a valence-like input for the sea quarks and gluons [21].

The DGLAP equations neglect higher order contributions of the form $\alpha_s \ln(1/x)$. However, at finite order, the large logarithms in $1/x$ become important in the perturbative expansion at small vales of x, where the evolution is dominated by the gluon cascade and accordingly these leading $\ln(1/x)$ terms have to be resummed. For large-Q^2 this is achieved by the double leading logarithmic approximation (DLLA), which resums the terms that include the leading $\ln(1/x)$ and the leading $\ln(Q^2)$ simultaneously. As a result at small-x one may consider the DLLA of the DGLAP evolution to choose the major contribution to the gluon density growth, analogous to the contribution of the $(\alpha_s \ln(Q^2)\ln(1/x))^n$ terms. The DLLA is valid when $\alpha_s \ln(1/x)\ln(Q^2) \sim 1$ but $\alpha_s \ln(1/x)$ and $\alpha_s \ln(Q^2)$ individually are small. But if Q^2 is not extremely large, then as we move towards smaller values of x the DGLAP equation no longer has its legitimacy. In that case alternative evolution equations, described below, which are appropriate in different regions may be taken into account

2.1.2 BFKL equation

The BFKL equation [5-7] was initially suggested by Balitsky, Fadin, Kuraev, and Lipatov to delineate the high-energy behaviour of processes involving hadrons. Recalling that $x \sim Q^2/s$, where Q^2 is the hard scale of the process and s is c.m.s. energy squared, at small-x, it is essential to sum the terms of the perturbation series enhanced by powers of $\ln(1/x)$. This equation sums up all the leading logarithm contributions of the type $(\alpha_s \ln(1/x))^n$ on the basis of gluon Reggeization. The BFKL approach is usually associated with the evolution equation for the unintegrated gluon distribution, $f(x, k_t)$, which depends on two independent variables, the proton momentum fraction x carried by a gluon and its transverse momentum k_t. An important characteristic of this evolution is distribution of the gluon density in $ln(k_t)$ space. The general form of BFKL evolution equation in LO is

$$\begin{aligned} f(x,k_t^2) &= f^0(x,k_t^2) + \frac{3\alpha_s(k_t^2)}{\pi} k_t^2 \int_x^1 \frac{dz}{z} \int_0^\infty \frac{dk_t'^2}{k_t'^2} \Big[\frac{f(x/z,k_t'^2) - f(x/z,k_t^2)}{\mid k_t'^2 - k_t^2 \mid} \\ &\quad + \frac{f(x/z,k_t^2)}{\sqrt{4k_t'^4 + k_t^4}} \Big], \end{aligned} \tag{2.10}$$

where the function $f^0(x, k_t^2)$ is a suitably defined inhomogeneous term and $k_t'^2$, k_t^2 are the transverse momenta squared of the gluon in the initial and final states respectively. In comparison to the DGLAP equation, this is a more intricate problem on the grounds that the BFKL equation literally involves contributions from operators of higher twists. The BFKL equation, in its ordinary form, not only represents the high-energy behaviour of cross-sections but also describes the amplitudes at non-zero momentum transfer.

2.1.3 CCFM equation

The CCFM equation [8-10] is a theoretical framework proposed by Catani, Ciafaloni, Fiorani and Marchesini (CCFM) which effectively interpolates between the the BFKL evolution and the more familiar DGLAP evolution equations. The primary objective of the CCFM approach is to provide accurate description of both the large-x region, where the summation of $\ln(Q^2)$ dominates, as well as the small-x region, where the large logarithms $\ln(1/x)$ are important. It depends on the comprehensible emission of gluons, that gives rise to an angular arrangement of the gluons along a series of multiple emissions. Similar to the BFKL equation, the CCFM equation is also defined in respect of a unintegrated gluon density f, which determines the possibility of finding a gluon with longitudinal momentum fraction x and transverse momentum k_t. Nonetheless, this distribution has a further dependance on some external scale Q. The CCFM equation is

$$\begin{aligned} f(x, k_t^2, Q^2) &= f^0(x, k_t^2, Q^2) \\ &+ \int_x^1 dz \int \frac{d^2q}{\pi q^2} \Theta(Q - zq) \Delta_S(Q, z, q) \tilde{P}(z, k_t, q) f(x/z, k_t'^2, q^2). \end{aligned} \tag{2.11}$$

The inhomogeneous contribution $f^0(x, k_t^2, Q^2)$ is of non-perturbative origin and is assumed to contribute only for $k_t^2 < q_0^2$. The remaining terms contribute in the region $k_t^2 > q_0^2$. The function $\tilde{P}$ is the gluon-gluon splitting function

$$\tilde{P} = \frac{3\alpha_s}{\pi} \Big(\frac{1}{1-z} + \Delta_R \frac{1}{z} - 2 + z(1-z) \Big), \tag{2.12}$$

where the factors Δ_S and Δ_R are the Sudakov and Regge form factors respectively. The multiplicative factors Δ_S and Δ_R counteract the singularities which are apparent as $z \to 1$ and $z \to 0$ respectively. Unlike Δ_S, the Regee form factor Δ_R not only depends on the branching variables, but also on the history of the cascade. At large-x one can get the usual DGLAP equation for gluon evolution by fixing $\Delta\Delta = 1$ and evolving Δ_S. On the other hand, at small-x keeping only the $1/z$ piece of P_{gg} and by setting $\Delta_S = 1$ and evolving $\Delta\Delta$ one can obtain the BFKL equation.

2.2 Nonlinear evolution equations

It is very fascinating to observe that the linear QCD evolution equations for parton densities , both the DGLAP and BFKL equations, prognosticate a steep rise of quark and gluon densities in the small-x region which is perceived in the DIS experiments at HERA as well. This sharp growth generates cross sections which in the high-energy limit fail to comply with the unitarity bound or in particular the Froissart bound [22, 23] on and so it will have to eventually slow down in order to restore unitarity. It is a known fact that the hadronic cross sections should obey the Froissart bound which derives from the general assumptions of the analyticity and unitarity of the scattering amplitude. Accordingly, the increasing number of gluon densities, so as to approach small-x, demands a formulation of the QCD at high density, where unitarity corrections are suitably taken into account.

Following DGLAP, the growing number of small-x gluons graphically conforms to higher density of individuals in the same approved region and thus differs from a diluted system at moderate values of x. As a result, at very small values of x the likelihood of interaction between two gluons can no longer be overlooked and it sooner or later engenders a situation in which individual partons inevitably overlap or shadow each other. We recall that, at very high energies, one can get into the region of smaller and smaller values of x and, under these situations, the gluon recombination being more effectual balances gluons splitting at some point. As a result, the abrupt growth of gluon distribution is eventually subdued due to the correlative interactions between gluons. This process is normally referred to as saturation of gluon density and it occurs when the possibility of gluon recombination, i.e. the process $gg \to g$, is as significant as that for a gluon to split into two gluons i.e. $g \to gg$. In deriving the linear DGLAP equations, the correlations among the initial gluons in the physical process of interaction and recombination of gluons are not taken into account. It is indispensable to point out that the linear DGLAP dynamics consider only the splitting processes in the partonic evolution, i.e. the processes $q \rightarrow qg$, $g \rightarrow q\bar{q}$ and $g \rightarrow gg$. However at small-x, the modifications due to the correlations among initial gluons to the evolutionary amplitude should be treated accordingly. The multiple gluon interactions induce nonlinear or shadowing corrections in the linear evolution equation and so the standard linear DGLAP evolution equation will have to be modified in order to include the contributions of recombination mechanism in the small-x regime.

The DGLAP evolution equations can delineate the available experimental data in a decent manner covering a large domain of x and Q^2 with appropriate parameterizations. But despite the remarkable achievement of the DGLAP approach, some issues come into sight when trying to generate the best possible global fits to the H1 data [24] concurrently in the region of large-Q^2 ($Q^2 > 4$ GeV^2) as well as in the region of small-Q^2 ($1.5 < Q^2 < 4$ GeV^2) [25]. In the NLO analysis of MRST2001 [26] an overall good fit is obtained including both the regions but resulting a negative gluon distribution at $Q_0 = 1$ GeV , thus creating an ambiguity in the interpretation of the PDFs as probability or number density distributions. On the other hand, in the CTEQ collaboration [27], where a slightly higher input scale of $Q_0 = 1.3$ GeV is considered, a very good compatibility with the data are observed in the large-Q^2 region whereas, the consistency with data in the small-Q^2 region becomes poor. The matter of negative gluon distributions also arises in the NLO set of CTEQ6M when evolving backwards to 1 GeV. Nevertheless, the negative gluon distributions are not empowered in LO. These emerging enigmas are really very appealing as they can provide a signal of gluon recombination towards smaller values of x and Q^2. In Ref.[25] the effects of including nonlinear GLRMQ corrections to the LO DGLAP evolution equations are studied by using the HERA data for the structure function $F_2(x, Q^2)$ of the free proton and the PDF sets CTEQ5L and CTEQ6L as a baseline. With the inclusion of the nonlinear corrections, the agreement with the $F_2(x, Q^2)$ data is exhibited to be improved in the region of $x \leq 3 \times 10^{-5}$ and $Q^2 \leq 1.5$ GeV^2, but managing the good fit to the data obtained in the global analyses at large-x and Q^2. Moreover, in Ref.[28] an analysis of HERA $F_2(x, Q^2)$ data is presented adding the effect of absorptive corrections due to parton recombination on the parton distributions. The small-x gluon distribution is found to be enhanced at small scales due to the absorptive effects, which may possibly avoid the need of a negative gluon distribution at 1 GeV. The gluon recombination effects lead to the nonlinear corrections to the linear DGLAP evolution equations due to multiple gluon interactions and as a result, in the very small-x region the conventional linear evolution equations are likely to breakdown. The nonlinear terms tame the abrupt growth of the gluon distribution in the kinematic region where α_s continues to be small but the density of gluons becomes very high so that a perturbative treatment is possible. Accordingly, the corrections of the higher order QCD effects, which suppress or shadow the growth of the parton densities, turns out to be the center of rigorous studies in the last few years.

The first perturbative QCD calculations reporting the recombination of two gluon ladders into one were carried out by Gribov, Levin and Ryskin (GLR), and Mueller and Qiu (MQ). They suggested that the nonlinear or shadowing corrections due to gluon recombination could be depicted in a new evolution equation with an additional nonlinear term quadratic in the gluon density. This equation, widely known as the GLR-MQ equation [29, 30], can be regarded as the updated version of the usual DGLAP equations with the corrections for gluon recombination. There are several other nonlinear evolution equations reporting the corrections of gluon recombination to the DGLAP and BFKL evolutions. They are the Modified-DGLAP (MD-DGLAP) [31, 32], Balitsky-Kovchegov (BK) [33, 34], Modified-BFKL (MD-BFKL) [35], Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK) [36-38] equations. The BK equation is the most widely studied among these. The nonlinear equations viz. Modified-BFKL (MD-BFKL), BK and JIMWLK are based on BFKL evolution, whereas the MD-DGLAP equation is based on DGLAP evolution. A concise description of all the above mentioned nonlinear evolution equations is given below.

2.2.1 GLR-MQ equation

The shadowing corrections of gluon recombination to the parton distributions were first investigated by Gribov, Levin and Ryskin and then by Mueller and Qiu at the twist-4 level in their pioneering papers [29, 30]. They provided the idea that the nonlinear corrections due to gluon recombination could be portrayed in a new evolution equation with an additional nonlinear term quadratic in gluon density. This equation, widely known as the GLR-MQ equation, can be considered as the improved version of the usual DGLAP equations with the corrections for gluon recombination. The pictorial representation of the corrections arising from gluon recombination processes is shown in Fig.2.3. Gribov et al. first suggested qualitative modification of the DGLAP

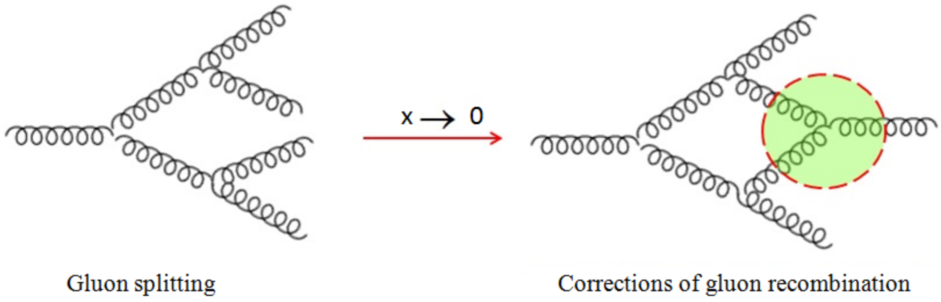

Figure 2.3: Corrections of gluon recombination

gluon evolution equation in order to include the gluon recombination effects based on the Abramovsky-Gribov-Kancheli (AGK) cutting rules [39]. Afterwards Mueller and Qiu completed the equation numerically using a perturbative calculation of the recombination probabilities in the DLLA, which is a significant achievement as it enables the GLR-MQ equation to be applied phenomenologically. This equation was generalized to incorporate the contributions from more higher order corrections in the Glauber-Mueller formula [40].

The GLR-MQ equation is based on two processes in the parton cascade:

(i) The splitting of gluons generated by the QCD vertex : $g \to g + g$;

(ii)The recombination of gluons promoted by the same vertex : $g + g \to g$.

For splitting process $1 \to 2$, the probability is proportional to $\alpha_s \rho$, whereas the probability for recombination process $2 \to 1$ is in proportion to $\alpha_s r^2 \rho^2$. Here, $\rho = \frac{xg(x,Q^2)}{\pi R^2}$ is the gluon density in the transverse plane, πR^2 is the target area, and R is the correlation radius between two interacting gluons [40]. It is worthwhile to mention that R is non-perturbative in nature and therefore all phenomena that occur at distance scales larger than R is non-perturbative [41]. Here, r is the size of the gluon induced in the recombination process and for DIS $r \sim \frac{1}{Q}$. For, $x \sim 1$ only the emission of gluons is influential since $\rho \ll 1$. At $x \to 0$, on the other hand, the density of gluons

ρ happens to be so high that the recombination of gluons should also be taken into account. Considering a cell of volume $\Delta \ln Q^2 \Delta \ln(1/x)$ in the phase space, number of gluons increases through splitting and decreases through recombination and this picture allows one to write the modification of the gluonic density as [41, 42]

$$\frac{\partial^2 xg(x,Q^2)}{\partial \ln(1/x) \partial \ln Q^2} = \frac{\alpha_s N_c}{\pi} xg(x,Q^2) - \frac{\alpha_s^2 \gamma}{\pi Q^2 R^2} [xg(x,Q^2)]^2, \tag{2.13}$$

which is known as the GLR-MQ equation. The factor γ is found to be $\gamma = 81/16$ for $N_c = 3$, as evaluated by Mueller and Qiu [30]. Here the gluon distribution is represented by $G(x,Q^2) = xg(x,Q^2)$, where $g(x,Q^2)$ is the gluon density. The quark-gluon emission diagrams are ignored because of their negligible influence in the gluon-dominated small-x domain. The first term in the right hand side of Eq.(2.15) represents the usual DGLAP term in the DLLA and hence linear in gluon field. The second term, having a negative sign controls the growth of the gluon distribution generated by the linear term at small-x and consequently delineates shadowing corrections emerging from recombination of two gluons into one. Likewise, the GLR-MQ equation for sea quark distribution can be written as

$$\frac{\partial xq(x,Q^2)}{\partial \ln(1/x) \partial \ln Q^2} = \frac{\partial G(x,Q^2)}{\partial \ln Q^2}\Big|_{DGLAP} - \frac{27}{160} \frac{\alpha_s^2(Q^2)}{R^2 Q^2} \left[G\left(\frac{x}{\omega}, Q^2\right) \right]^2 + HT, \tag{2.14}$$

where HT denotes a higher-dimensional gluon distribution term suggested by Mueller and Qiu [30].

In the linear QCD evolution of DIS structure functions like the DGLAP or BFKL only the splitting of quarks and gluons is considered. This leads to a constant increase of the parton densities at small-x eventually violating the unitarity bound and are therefore expected to be tamed by the inverse recombination processes. Therefore, in order to account for gluon recombination processes, apart from the production diagrams, the GLR-MQ equations also include the dominant non-ladder contributions denoted as the fan diagrams. The fan diagrams take into consideration some of the gluon recombination processes that turn significant at small-x and therefore plays the key role in the restoration of unitarity. These diagrams are depicted in Fig.2.4.

The gluon recombination term in the GLR-MQ equation contains a factor $1/Q^2$, whose dimension is balanced by the parameter R representing the size of the region containing the recombining gluons. The size of the nonlinear term varies as $1/R^2$. The value of R depends on how the gluons are distributed within the proton or how

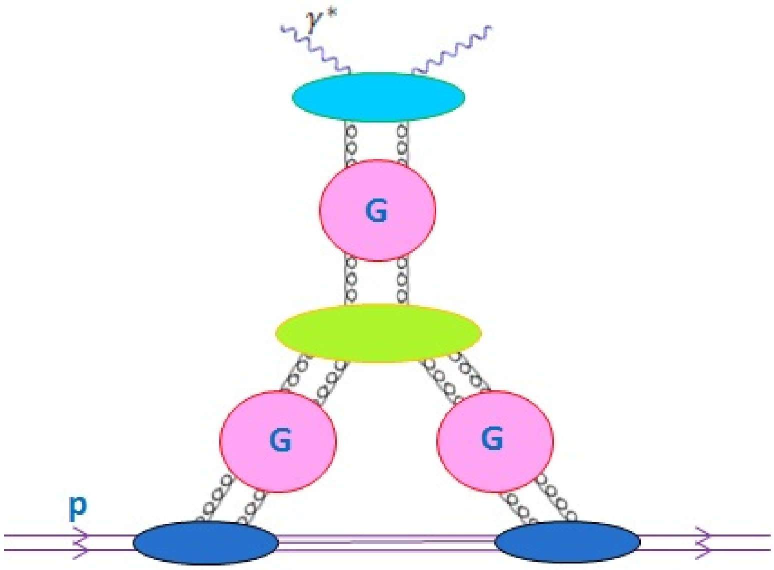

Figure 2.4: Fan-diagrams contributing to the GLR-MQ equation

the gluon ladders couple to each other. The gluon ladders may emerge from different constituents of the proton or from the same constituent. The gluons are supposed to be distributed uniformly across the whole of the proton if the gluon ladders emerge off distinct valence quarks. In that case R is of the order of proton radius R_h, that is to say $R \sim 5$ GeV^{-1} and recombination or shadowing correction is negligibly small [29]. On the other hand, if the gluon ladders couple to the same parton then it leads to a higher gluon density in the parton's vicinity. Such smaller regions within the proton where the gluon density is higher than the average are known as the so-called 'hot spots' [43, 44]. The hot spots could specify the fast onset of gluon-gluon interactions in the environs of the emitting parton and so boost the recombination effect. The value of R for such hot spot, is considered to be of the order of the transverse size of a valence quark i.e $R \sim 2$ GeV^{-1}.

A remarkable feature of the GLR-MQ equation is that it predicts the saturation momentum in the asymptotic region $x \to 0$. Moreover, it predicts a critical line separating the perturbative regime from saturation regime and it is valid only in the vicinity of this critical line [42]. The general benchmark of this equation is that the nonlinear corrections should be small as compared to the linear term, otherwise further corrections must be taken into account and non-perturbative effects could be of significance. As the GLR-MQ equation only includes the first nonlinear term, so this equation is not legitimate in very high density region where the contributions from the higher order terms become crucial.

2.2.2 MD-DGLAP equation

The MD-DGLAP equation [31, 32] sums up all possible twist-4 cut diagrams in the LL(Q^2) approximation and describes the corrections of parton recombination to the QCD evolution equation. These equations are advocated by Zhu and Ruan. This equation is obtained by aggregating the Feynman diagrams in the framework of the time-ordered perturbation theory (TOPT) [45] instead of the AGK cutting rule [39]. The MD-DGLAP equation for gluon distribution is [31, 32]

$$\begin{aligned}\frac{dxg(x,Q^2)}{dln(Q^2)} &= P_{gg}\otimes G(x,Q^2)+P_{gq}\otimes S(x,Q^2)\\ &+\frac{\alpha_s^2 k}{Q^2}\int_{x/2}^{x} dx_1 x x_1 G^2(x_1,Q^2)\sum_i P_i^{gg\to g}(x_1,x)\\ &-\frac{\alpha_s^2 k}{Q^2}\int_{x}^{1/2} dx_1 x x_1 G^2(x_1,Q^2)\sum_i P_i^{gg\to g}(x_1,x)\end{aligned} \tag{2.15}$$

and for sea quark distribution is [31, 32]

$$\begin{aligned}\frac{dxq(x,Q^2)}{dln(Q^2)} &= P_{qg}\otimes G(x,Q^2)+P_{qq}\otimes S(x,Q^2)\\ &+\frac{\alpha_s^2 k}{Q^2}\int_{x/2}^{x} dx_1 x x_1 G^2(x_1,Q^2)\sum_i P_i^{gg\to q}(x_1,x)\\ &-\frac{\alpha_s^2 k}{Q^2}\int_{x}^{1/2} dx_1 x x_1 G^2(x_1,Q^2)\sum_i P_i^{gg\to q}(x_1,x),\end{aligned} \tag{2.16}$$

where P are the evolution kernels of the linear DGLAP equation. The recombination functions are

$$\sum_i P_i^{gg\to g}(x_1,x)=\frac{27}{64}\frac{(2x_1-x)(-136xx_1^3-64x_1x^3+132x_1^2x^2+99x_1^4+16x^4)}{xx_1^5}, \tag{2.17}$$

$$\sum_i P_i^{gg\to q}(x_1,x)=\frac{1}{48}\frac{(2x_1-x)(36x_1^3+49x_1x^2-14x^3-60x^2x)}{x_1^5}. \tag{2.18}$$

The nonlinear coefficient k depends on the definition of the double parton distribution and the geometric distributions of partons inside the target. The positive third terms on the right-hand side of both Eq.(2.17) and Eq.(2.18) represent the anti-shadowing effect, whereas the negative fourth term is the result of the shadowing correction. The concurrence of shadowing and anti-shadowing in the QCD evolution of the parton densities is a usual demand for the local momentum conservation. The shadowing and anti-shadowing terms are defined on distinct kinematic regions $[x, 1/2]$ and $[x/2, x]$ respectively. Hence, the overall recombination effects in Eq.(2.17) are not only associated to the value of gluon density, but also depend on the slope of the gluon distribution in the space $[x/2, x]$. This implies that a steeper gluon distribution has an intense antishadowing effect as compared to a lower gluon distribution.

2.2.3 BK equation

The BK evolution equation [33, 34] is based on the BFKL equation and was derived by Balitsky and Kovchegov in the large-N_c limit, with N_c being the number of colors. The BK equation is an upgraded form of the GLR-MQ equation and it determines the saturation of parton densities at very small-x. This equation is written for the scattering amplitude N. It provides an explanation of the more specific triple-pomeron vertex [46, 47] and can be utilized for the non-forward amplitude. The BK equation is obtained in the leading $\ln(1/x)$ approximation of perturbative QCD, i.e. it sums all contributions of the order $(\alpha_s \ln(1/x))^n$. The contributions of the orders $\alpha_s(\alpha_s \ln(1/x))^n$ and $\alpha_s \ln Q^2(\alpha_s \ln(1/x))^n$ are not included in this equation. The phenomenological analysis of this equation is performed in the dipole model [48, 49] approximation, where the nonlinear terms are supposed to be formed by the dipole splitting and the screening or shadowing effects are emerged from the double scattering of the probe on the final states. The BK equation reads

$$\begin{aligned}\frac{\partial N(r,Y;b)}{\partial Y} &= \frac{\bar{\alpha_s}}{2\pi}\int \frac{d^2r'r^2}{(r-r')^2r'^2} \\ &\times\Big[2N\big(r',Y;b+\frac{1}{2}(r-r')\big)-N(r,Y;b) \\ &-N\big(r',Y;b-\frac{1}{2}(r-r')\big)N\big(r-r',Y;b-\frac{1}{2}r'\big)\Big], \qquad (2.19)\end{aligned}$$

where $\bar{\alpha_s} = (\alpha_s N_c)/\pi$, $N(r, Y; b)$ is the scattering amplitude of interaction for the dipole with the size r and rapidity $Y = \ln(1/x)$, at impact parameter b. In the large N_c limit $C_F = N_c/2$, where N_c is the number of colors.

Eq.(2.21) is an integro-differential equation and it presents the scattering amplitude $N(r, Y; b)$ at all rapidities $Y > 0$ provided the initial condition at $Y = 0$ is known. The physical significance of Eq.(2.21) is that the dipole of size r decays in two dipoles of sizes r' and $r - r'$ which interact with the target. The linear part of Eq.(2.21) is the usual LO BFKL equation [5-7], which accounts for the evolution of the multiplicity of the color dipoles of fixed size in respect of the rapidity Y. The nonlinear term considers a coexisting interaction of two produced dipoles with the target and it sums the high twist contributions. An outstanding feature of the BK equation is that its solution predicts a limiting form of the scattering amplitude resulting in parton saturation.

2.2.4 MD-BFKL equation

The nonlinear MD-BFKL [35] equation was suggested by Zhu, Shen and Ruan to describes the corrections of the gluon recombination to the BFKL equation, but it differs from the BK equation. The MD-BFKL equation forecasts an intense shadowing effect, which subdues the gluon density. Surprisingly, it generates the extinction of gluons below the saturation region. This unforeseen effect of gluon extinction below the saturation region is induced by an apparent chaotic solution of the equation as suggested in [35]. The MD-BFKL is defined as

$$\begin{aligned} -x\frac{\partial f(x,k_{b0})}{\partial x} &= \frac{\alpha_s N_c}{2\pi^2}\int d^2k_{bc}\frac{k_{b0}^2}{k_{bc}^2 k_{c0}^2}2f(x,k_{bc}) - \frac{\alpha_s N_c}{2\pi^2}f(x,k_{b0})\int d^2k_{bc}\frac{k_{b0}^2}{k_{bc}^2 k_{c0}^2} \\ &\quad -\frac{18\alpha_s^2}{\pi^2 R^2}\frac{N_c^2}{N_c^2-1}\int d^2k_{bc}\frac{1}{k_{bc}^2}\frac{k_{b0}^2}{k_{bc}^2 k_{c0}^2}f^2(x,k_{bc}) \\ &\quad +\frac{9\alpha_s^2}{\pi^2 R^2}\frac{N_c^2}{N_c^2-1}f^2(x,k_{b0})\int d^2k_{bc}\frac{1}{k_{b0}^2}\frac{k_{b0}^2}{k_{bc}^2 k_{c0}^2}. \end{aligned} \tag{2.20}$$

The nonlinear part of the MD-BFKL equation has an infra red (IR) divergence very much alike the BFKL kernel and as a matter of course, it requires the regularization scheme alike the BFKL equation. The evolution kernels in the linear and nonlinear parts of the MD-BFKL equation are fixed by using the same procedure of summations of the real and virtual processes. This equation is derived on the basis of the TOPT cutting rules just as the MD-DGLAP equation to include the contributions from the virtual processes in the linear and nonlinear parts of the MD-BFKL equation. The MD-BFKL and BK equations differ from each other in their assumptions of regularization schemes. In MD-BFKL equation the singularities in the nonlinear real part are aborted by the contributions from the complementary virtual processes, whereas such singularities are assimilated into the double amplitude NN in BK equation.

2.2.5 JIMWLK equation

The JIMWLK evolution equation [36-38], advocated by Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner, is the renormalization group equation (RGE) for the Color Glass Condenstate and describes the small-x hadronic physics in the regime of very high gluon density. This is a functional Fokker-Planck equation regarding a classical random color source, which defines the color charge density of the partons with large-x [38]. This equation controls the evolution with rapidity of the statistical weight function for the color glass field.

The JIMWLK equation in the compact form is [50]

$$\partial\tau \hat{Z}_\tau[U] = -\frac{1}{2} i\nabla^a_x \chi^{ab}_{xy} i\nabla^b_y Z_\tau[U], \tag{2.21}$$

where $\hat{Z}_\tau[U]$ is the weight functional and it governs the correlators $O[U]$ of U fields conforming to $\langle O[U]\rangle_\tau = \int \hat{D}[U]O[U]Z_\tau[U]$, with $\hat{D}[U]$ being a functional Haar measure [50]. ∇^a_x are functional form of the left-invariant vector fields affecting U_y in accordance with

$$i\nabla^a_x U_y = U_x t^a \delta^{(2)}_{xy}, \tag{2.22}$$

where U_x are the Wilson line variables representing the kinematically enhanced degrees of freedom. Again

$$\chi^{ab}_{xy} = \frac{\alpha_s}{\pi^2} \int d^2 z K_{xyz}[(1 - \tilde{U}^\dagger_x \tilde{U}_z)(1 - \tilde{U}^\dagger_z \tilde{U}_x)]^{ab},$$

and

$$K_{xyz} = \frac{(x-z).(z-y)}{(x-z)^2(z-y^2)}. \tag{2.23}$$

The deduction of the JIMWLK equation demands an analytic estimation to all orders in the environment of a classical gluon field for a random light cone source. The solution of the JIMWLK equation is normally anticipated to enable the saturation momentum to raise constantly as $y \to \infty$. Moreover its solution is supposed to be universal. In the restrain of weak field the JIMWLK equation scales down to the BFKL equation, whereas in the large N_c limit, it grows to be equivalent to the BK equation.

2.3 Solutions of evolution equations

The QCD evolution equations are the underlying theoretical tools to compute the quark and gluon distributions and eventually the DIS structure functions. On that account the solutions of the evolution equations are drawing attention substantially. The solutions of the DGLAP equation for the QCD evolution of PDFs have been discussed considerably over the past years. There exist two main classes of approaches: those that solve the equation directly in x-space, and those that solve it for Mellin transforms of the parton densities and subsequently invert the transforms back to x-space. Some available programs that deal with DGLAP evolution are CANDIA [51] based on the logarithmic expansions in x-space, QCD PEGASUS [52], which is based on the use of Mellin moments, HOPPET [53] and QCDNUM [54]. HOPPET is a Fortran package for carrying out QCD DGLAP evolution and other common manipulations of PDFs. The Fortran package QCD PEGASUS provides fast, flexible and accurate solutions of the evolution equations for unpolarized and polarized parton distributions of hadrons in perturbative QCD. Similarly QCDNUM is a Fortran program that numerically evolves parton densities or fragmentation functions up to NNLO in perturbative QCD. Most of the methods used for the solution of DGLAP equation are numerical. Laguerre polynomials [55, 56], Brute-Force method [57], Matrix method [58], Mellin transformation [59, 60] etc. are different methods used to solve DGLAP evolution equations. The shortcomings common to all are the computer time required and decreasing accuracy for $x \to 0$. More precise approach to the solution of the DGLAP evolution equations is the matrix approach, but it is also a numerical solution. A numerical solution does not provide the full control on the employed phenomenological parameters, and the transparency and simplicity of physical interpretation are lost if one relies only on the numerical solutions.

As an alternative to the numerical solution, one can study the behavior of quarks and gluons via analytic solutions of the evolution equations. Even though exact analytic solutions of the DGLAP equations cannot be obtained in the whole range of x and Q^2, such solutions are possible under definite conditions and are fairly successful as far as the HERA small-x data are concerned. In recent years, such a scheme in the analytic study of the DGLAP equations has been reported with considerable phenomenological success [61-67]. The Taylor series expansion method, the method of characteristic and the Regge theory methods are some of the very simple and frugal analytical methods that have been utilized widely to obtain the solutions of DGLAP equations. Part I of this thesis also reports the analytical solutions of the DGLAP equations for DIS structure functions upto NNLO with significant phenomenological triumph.

In contrast to the DGLAP equation, it is very difficult to the solve the BFKL and CCFM equations. Although the solution of the LO BFKL evolution is known, but regardless of a number of attempts, it seems that an exact analytical solution of the NLO BFKL equation, or a general all-order BFKL equation in QCD is still unavailable. Nevertheless, in a conformal field theory without the running of the coupling, i.e. in the N = 4 super Yang-Mills theory, the form of the solution of the BFKL equation to all-order has been identified [68, 69]. The numerical solution of CCFM equation can be obtained by monte-carlo approach CASCADE to study the small-x regime. Although in the single loop approximation the CCFM equatiion can be solved analytically [70], but due the non Sudakov form-factor the solution beyond the single-loop approximation is less apparent.

The solutions of the nonlinear evolution equations, on the other hand, are particularly important for understanding the nonlinear effects of gluon-gluon interaction due to the high gluon density at small-x. The solution of nonlinear evolution equations also provide the determination of the saturation momentum, which incorporates physics in addition to that of the linear evolution equations commonly used to fit the DIS data. It is very difficult to solve the nonlinear equations analytically, unlike the linear DGLAP equations. However the studies on the solutions and viable generalizations of the GLR-MQ type equations in different approaches have been revealed in the last few years [25, 28, 29, 71-80]. In Refs. [29, 71-74] the solutions of GLR-MQ type nonlinear equations are reported in semi classical approach using characteristics method which leads to existence of a critical line separating the perturbative regime from the nonperturbative one. Here it is shown that all characteristics in the region of small-x cannot cross this line but can approach it. Again a new equation is proposed in Ref.[75] which generalizes the GLR equation and allows to probing into smaller distance in the dense parton system considering the shadowing effects more exclusively by including multigluon correlations. The general solution to the new equation is obtained in in an eikonal approach and fixed α_s. A new approach for searching a solution of the nonlinear GLR-MQ evolution equation in the nonperturbative part of the small-x region is discussed in Ref [76]. Here it is justified that the suggested solution satisfies all physics restrictions and there is only one solution that complements the perturbative DGLAP evolution. A color dipole approach to the solution to the nonlinear GLR-MQ like equation for high parton density is suggested in the full kinematic region including $x \to 0$ in Ref.[77]. The solution replicates the saturation of the gluon density. However due to moderate dependence on the impact parameter, the saturation gives rises to the dipole-target total cross section proportional to $\ln(1/x)$ in the region of very small-x. A numerical analysis of the GLR-MQ equation is presented in Ref. [78] where the signatures of gluon recombination are discussed. They also provide a simple and qualitative idea to explore the H1 [79] experimental data for evidence of gluon recombination. Similarly, a numerical solution of GLR-MQ equation is suggested in Ref.[25], where the effects of the first nonlinear corrections to the DGLAP evolution equations are studied by using the recent HERA data for the structure function $F_2(x, Q^2)$. It is argued in this paper that the nonlinear corrections become important at $x \leq 10^{-3}$ and $Q^2 \leq 10$ GeV^2, but become negligible at large-x and large-Q^2. In Ref.[28] The effect of absorptive corrections due to parton recombination on the parton distributions of the proton is discussed at small-x in a more precise version of the GLRMQ equations using a truncated version of the MRST2001 NLO analysis [26]. Moreover the approximate analytical solutions of the nonlinear GLR-MQ evolution equation have also been reported in recent years [80, 81]. In part II of this thesis we present a semi-analytical approach to solve the GLR-MQ equation in the vicinity of saturation and make a deliberate attempt to explore the effect of nonlinear or shadowing corrections in the kinematic region of small-x and moderate Q^2.

Unlike GLR-MQ the other nonlinear equations are comparatively complicated to solve. The numerical solutions to BK or JIMWLK nonlinear equations in the presence of the impact parameter is very challenging. The JIMWLK equation is difficult to solve, even numerically as it consist of an infinite hierarchy of coupled evolution equations. The BK hierarchy is a special case of the JIMWLK equation where the primary projectile is set and captured by a quark-antiquark pair. For practical calculations one may use the average field approximation and thereby diminish a full infinite hierarchy to a single closed equation. Even if the full analytical solution of the BK equation is not known, a number of its general properties, such as the existence and shape of limiting solutions, have been determined in both analytical [82-84] and numerical [85-89] approaches in recent years. On the other hand, in Ref.[90] numerical solutions to the MD-DGLAP equation are reported in the small-x region using a with GRV like input distributions. Here the the small-x behavior parton distributions in the nucleus and free proton are predicted numerically and it is seen that gluon recombination at the twist-4 level suppresses the rapid increase of parton densities towards small-x. It is further claimed that saturation and partial saturation occur sooner than the saturation scale Q_s^2 is reached.

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