Chapter 7

Comparative Analysis of Various Nonlinear Evolution Equations

7.1 Introduction

The growth of total hadronic cross sections at very high energies is one of the most challenging problems of QCD and accordingly the study of the high density QCD turns out to be the center of intensive studies in the last few years. The attempts to understand the aspects of the higher twist phenomena led to many different kinds of model in the past times. The corrections of the higher order QCD effects, which suppress or shadow the growth of the parton densities, leading to a possible restoration of the Froissart bound on physical cross-section in the very small-x region are at the onset accounted for by Gribov, Levin and Ryskin, and Mueller and Qiu in the GLR-MQ [1-3] equations. Several other nonlinear evolution equations are proposed in later times reporting the corrections of the gluon recombination to the linear DGLAP [4-6] and BFKL [7-9] evolutions, viz. the Modified-DGLAP (MD-DGLAP) [10, 11], Balitsky-Kovchegov (BK) [12, 13], Modified-BFKL (MD-BFKL) [14] and Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK) [15-17] equations. The nonlinear equations viz. Modified-BFKL (MD-BFKL), BK and JIMWLK are based on BFKL evolution, whereas, the MD-DGLAP equation is based on DGLAP evolution. The BK and the MD-DGLAP equations are the most widely studied among these . The GLR-MQ equation takes the double leading logarithmic approximation (DLLA) for both Q^2 and $1/x$, keeping only the $\ln(Q^2/\Lambda^2)\ln(1/x)$ factor in the solutions of the evolution equation, whereas, the MD-DGLAP equation is derived under the leading logarithmic LL(Q^2) approximation. Unlike the GLR-MQ equation, the MD-DGLAP equation sums the Feynman diagrams in the framework of the time-ordered perturbation theory (TOPT) [18] instead of using the AGK cutting rule [19]. Moreover, apart from the shadowing corrections, the MD-DGLAP equation also takes into account the antishadowing effects which balance the momentum lost in the shadowing process. The antishadowing corrections may change the predictions of the GLR-MQ equations. On the other hand, the BK equation is an upgraded version of the GLR-MQ equation and it determines the saturation of parton densities at very small-x. The BK equation considers the more precise triple-pomeron vertex [20, 21] and can be used for the non-forward amplitude. The BK equation is obtained in the leading $\ln(1/x)$ approximation of perturbative QCD, i.e. it sums all contributions of the order $(\alpha_s \ln(1/x))^n$.

In this chapter we present a comparative analysis of the GLR-MQ equation with the MD-DGLAP and BK equations. Here the gluon distribution function obtained from the semi analytical solution of the GLR-MQ equation discussed in chapter 5 are compared with the results of MD-DGLAP and BK equations in the region of small-x. To compare our predictions in the GLR-MQ approach with those of MD-DGLAP and BK equations we have used the results of Ref.[22] and Ref.[23] respectively where the numerical analysis of these equations are presented.

7.2 Formalism

The GLR-MQ equation for the gluon distribution function can be expressed as [1-3, 24]

$$\frac{\partial G(x,Q^2)}{\partial \ln Q^2} = \frac{\partial G(x,Q^2)}{\partial \ln Q^2}\Big|_{DGLAP} - \frac{81}{16}\frac{\alpha_s^2(Q^2)}{R^2 Q^2}\int_x^1 \frac{d\omega}{\omega}\Big[G\Big(\frac{x}{\omega}, Q^2\Big)\Big]^2, \tag{7.1}$$

In chapter 5 we have solved this equation semi analytically and investigated the effect of shadowing corrections on the behaviour of small-x and Q^2-dependence of gluon distribution function using a simple form of Regge like ansatz. Here we have used these results to perform a comparative analysis of the small-x dependence of gluon distribution function obtained in the GLR-MQ approach with the results of MD-DGLAP and BK equations respectively. For convenience, we rewrite here some of the important results of chapter 5.

By incorporating the Regge like behaviour of the gluon distribution function, i.e., $G(x, Q^2) = H(Q^2)x^{-\lambda_G}$ with the Regge intercept λ_G, the solution of Eq.(7.1) is obtained as

$$G(x,t) = \frac{t^{\gamma_1(x)}}{C + \gamma_2(x)\int t^{\gamma_1(x)-2}e^{-t}dt}. \tag{7.2}$$

Here $t = \ln(Q^2/\Lambda^2)$ and the constant C is determined from initial boundary conditions. So we use the physically plausible boundary condition at some high $x = x_0$ in Eq.(7.2) and obtain the x dependence of the gluon distribution function as

$$G(x,t) = \frac{t^{\gamma_1(x)}G(x_0,t)}{t^{\gamma_1(x_0)} + \left[\gamma_2(x)\int t^{\gamma_1(x)-2}e^{-t}dt - \gamma_2(x_0)\int t^{\gamma_1(x_0)-2}e^{-t}dt\right]G(x_0,t)}. \tag{7.3}$$

This equation helps us to predict the effect of shadowing corrections to small-x behaviour of nonlinear gluon distribution function by picking out suitable input distribution at an initial value of $x = x_0$. The Regge type solution of the GLR-MQ equation is found to be valid in the kinematic region $1 \leq Q^2 \leq 30$ GeV^2 as well as $10^{-5} \leq x \leq 10^{-2}$ as discussed in chapter 5.

The MD-DGLAP equation [10, 11] derived by Zhu and Ruan sums up all possible twist-4 cut diagrams in the LL(Q^2) approximation and describes the corrections of parton recombination to the QCD evolution equation. For gluon distribution the MD-DGLAP equation is given by [22]

$$\begin{aligned}\frac{dxG(x,Q^2)}{d\ln(Q^2)} &= P_{gg}\otimes G(x,Q^2) + P_{gq}\otimes S(x,Q^2)\\ &\quad+\frac{\alpha_s^2 k}{Q^2}\int_{x/2}^{x} dx_1 x x_1 G^2(x_1,Q^2)\sum_i P_i^{gg\to g}(x_1,x)\\ &\quad-\frac{\alpha_s^2 k}{Q^2}\int_{x}^{1/2} dx_1 x x_1 G^2(x_1,Q^2)\sum_i P_i^{gg\to g}(x_1,x)\end{aligned} \tag{7.4}$$

where P_{gg} and P_{gq} are the evolution kernels of the linear DGLAP equation. The explicit form of the recombination function is

$$\sum_i P_i^{gg\to g}(x_1,x) = \frac{27}{64}\frac{(2x_1 - x)(-136xx_1^3 - 64x_1x^3 + 132x_1^2x^2 + 99x_1^4 + 16x^4)}{xx_1^5}. \tag{7.5}$$

The nonlinear coefficient k is based on the definition of the double parton distribution and the geometric distributions of partons inside the target. The positive third term on the right-hand side represents the anti-shadowing effect, whereas the negative fourth term is the result of the shadowing correction.

In Ref.[22] an analysis of MD-DGLAP equation is presented by W. Zhu et al., where the parton distributions in the small-x region in the nucleus and free proton are numerically predicted considering the GRV-like input distributions with and without anti-shadowing corrections. Here the Q^2 and x behaviour of the parton distributions at high gluon density are studied in LL(Q^2) approximation using the MD-DGLAP equation. The initial gluon density in the GRV98LO set is used as the input distribution at $Q_0^2 = 0.34$ GeV^2, i.e.,

$$xg(x, Q_0^2) = 17.47x^{1.6}(1-x)^{3.8}, \tag{7.6}$$

with the representation $G(x, Q^2) = xg(x, Q^2)$. The results obtained in Ref.[22] show that the growth of the predicted gluon distribution in the proton toward small-x is slower than $\ln(1/x)$ for $x < 10^{-6}$ which implies that the gluon recombination at twist 4 level suppresses the rapid growth of gluon densities with decrease in x. We consider the results of Ref.[22] for a comparative analysis of our predictions of gluon distribution obtained from the solution of GLR-MQ equation with the MD-DGLAP results.

The BK equation [12, 13] is derived by Balitsky and Kovchegov in the LL($1/x$) approximation of perturbative QCD, i.e. it sums all contributions of the order $(\alpha_s \ln(1/x))^n$. This equation is written in coordinate space in terms of the dipole scattering amplitude N. This equation provides the basic indication of the fact that the correct degrees of freedom at high energies in QCD are colour dipoles. It provides an explanation of the more specific triple-pomeron vertex [20, 21] and can be utilized for the non-forward amplitude. The BK equation reads

$$\begin{aligned}\frac{\partial N(r,Y;b)}{\partial Y} &= \frac{\bar{\alpha_s}}{2\pi}\int \frac{d^2r' r^2}{(r-r')^2 r'^2} \\ &\quad \times\Big[2N\big(r',Y;b+\frac{1}{2}(r-r')\big) - N(r,Y;b) \\ &\quad -N\big(r',Y;b-\frac{1}{2}(r-r')\big)N\big(r-r',Y;b-\frac{1}{2}r'\big)\Big],\end{aligned} \tag{7.7}$$

where $\bar{\alpha_s} = (\alpha_s N_c)/\pi$, $N(r, Y; b)$ is the scattering amplitude of interaction for the dipole with the size r and rapidity $Y = \ln(1/x)$, at impact parameter b. In the large N_c limit $C_F = N_c/2$, where N_c is the number of colors. Eq.(7.7) implies that the dipole of size r decays in two dipoles of sizes r' and $r - r'$ which interact with the target. The linear part of Eq.(7.7) represents the conventional LO BFKL equation [7-9]. The non-linear term accounts for the simultaneous interaction of two produced dipoles with the target and the high twist contributions. A fascinating characteristics of the BK equation is that its solution predicts a limiting form of the scattering amplitude resulting in parton saturation. For small dipole densities N the quadratic term in the brackets is negligible and Eq.(7.7) reduces to the conventional BFKL equation, whereas, sauration is reached when $N = 1$.

In Ref.[23] the solution of the LO BK equation is reported where the authors include the impact parameter dependence of the amplitude at initial values of rapidity $Y = \ln(1/x)$ and find the amplitude in each point of impact parameter space. The gluon density is related to the dipole amplitude as

$$G(x, Q^2) = \frac{4}{\pi^3} \int_x^1 \frac{dx'}{x'} \int_{4/Q^2}^{\infty} \frac{dr^2}{r^2} \int d^2b 2N(r, x'; b), \tag{7.8}$$

where the representation $G(x, Q^2) = xg(x, Q^2)$ is used. The calculated results of the gluon density function in Ref.[23] are found to be in good agreement with the GRV parametrization. Here we use the results of Ref.[23] to perform a comparative analysis of our results of gluon distribution obtained from the solution of GLR-MQ equation with those of the BK equation.

7.3 Result and discussion

The x dependence of gluon distribution function with shadowing corrections calculated in the framework of GLR-MQ equation is compared with the results of MD-DGLAP and BK equations taken from the Refs.[22] and [23] respectively. We perform these comparisons in the kinematic region $1 \leq Q^2 \leq 30$ GeV^2 and $10^{-5} \leq x \leq 10^{-2}$ as our predicted solution of GLR-MQ equation is found to be valid only in this domain. In Figure 7.1 the gluon distribution function calculated from Eq.(7.3) at the hot spots $R = 2$ GeV^{-1} are plotted as a function of x for fixed values of Q^2=2.2, 3, 5, 10 and 20 GeV^2 respectively. Our results manifest that the gluon density increases with the decreasing x but this behavior is tamed as x grows smaller due to nonlinear or shadowing corrections. For each Q^2 our predictions obtained in the framework of GLR-MQ equation are in very good agreement with the results of the BK equation.

Moreover, concerning the shape of the curves we observe that the shapes of the curves found in the GLR-MQ approach are very similar to the shape of the BK curves. On the other hand, we note that our predictions do not match with the results of MD-DGLAP equation, as the MD-DGLAP curves have opposite concavities in the region of $x > 10^{-3}$. However in the region $x \leq 10^{-3}$ the shape of our results is almost similar to that of the MD-DGLAP equation with a completely different slope. The MD-DGLAP equation predicts a steeper gluon distribution towards small-x which implies the presence of strong antishadowing effect in the results of MD-DGLAP equation, whereas our predictions show significant effect of shadowing corrections as a consequence of gluon recombination processes towards small-x which results in a flatter gluon distribution.

7.4 Summary

To summarize, the gluon distribution function obtained in the framework of nonlinear GLR-MQ equation in leading twist approximation is compared with the MD-DGLAP and BK equations. We make the comparison in the kinematic domain $1 \leq Q^2 \leq 30$ GeV^2 and $10^{-5} \leq x \leq 10^{-2}$ as the predicted solution of GLR-MQ equation is found to be valid only in this region. It is a very captivating finding that the predictions of nonlinear gluon density obtained from the GLR-MQ equation are very compatible with the results of the BK equation. Our results of nonlinear gluon density are also found to almost comparable with those of the MD-DGLAP equation but with a completely different slope. The MD-DGLAP equation predicts a steeper gluon distribution due to a relatively stronger antishadowing effect, whereas a flatter gluon distribution is observed in our predictions due to significant shadowing corrections at small-x. In this work we have not considered other nonlinear equations such as the JIMWLK equation for comparative analysis with the GLR-MQ equation, owing to the fact that the JIMWLK equation deals with the process dependent unintegrated parton distributions and the cross sections whereas the GLR-MQ equation considers the shadowing in the process independent parton distributions.

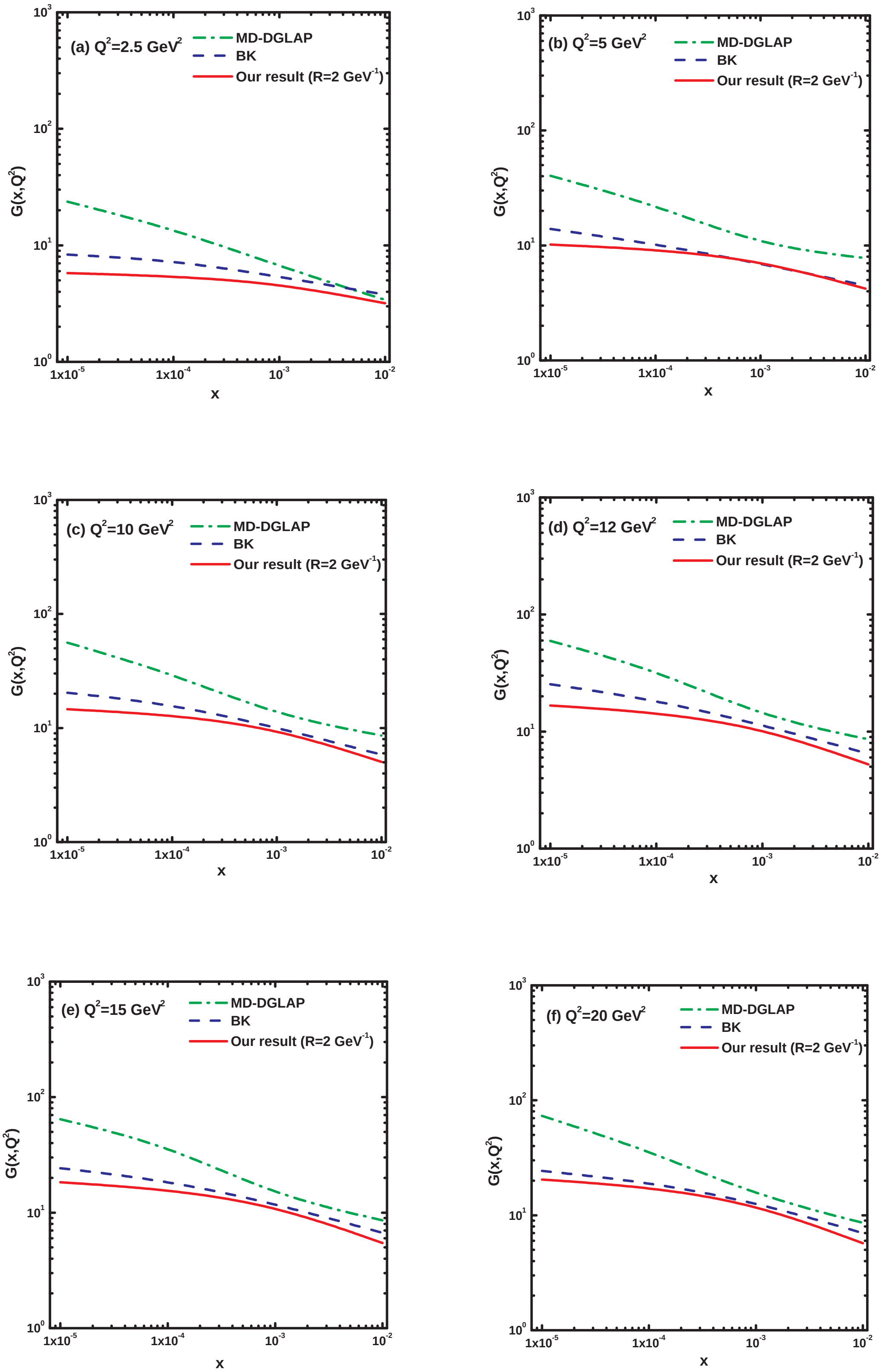

Figure 7.1: Comparison of the gluon distribution function obtained from Eq.(7.3) in the GLR-MQ approach with the MD-DGLAP results [22] as well as the BK results [23].

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