Chapter-6

STABILITY OF THE POLYTROPIC SOLAR WIND

Abstract: *A linear stability analysis of a simple polytropic hydrodynamic model*† *for the solar wind dynamics is proposed. The proposed analysis is based on the data available from the Advanced Composition Explorer (ACE) spacecraft mission. We apply the usual variable-separation methodology in the dispersion analysis. It is seen that the growth rate of the fluctuations is an explicitly nonlinear function of the variable polytropic index (* α *) and the radial position (* r *) relative to the heliocenter. The growth attains a maximum near the solar corona (* $\alpha \sim 1$*); and so forth.*

6.1 INTRODUCTION

The Sun, its atmosphere and associated flow dynamics constitute an intriguing area of research because of their richness in various collective degrees of freedom exhibited through diverse waves, fluctuations and oscillations [1-6]. Due to many such gravito-electro-magnetic wave-related phenomena, the outer solar atmosphere is so hot that even its gravity cannot prevent it from continuously expanding [3-8]. The continuous stream of high energetic particles emanating from the Sun and flowing radially outward is known as the '*solar wind*'. The solar wind comprises mainly of two parts: the fast solar wind and slow solar wind, the intercoupling of which results in large-scale temporal dynamics. The origin of the solar wind, dynamical stability, the outflow dynamics of the solar wind acceleration mechanism are yet to be well understood [7-11]. Bold efforts have been made to see the solar wind flow dynamics and stability from different perspectives, like hydrodynamic model [1, 4, 8], magnetohydrodynamic (MHD) model [6, 7], kinetic model [11, 12], and so forth.

In spite of the anisotropic nature of the solar wind, the existing MHD models have been able to reproduce the global, average and physical sensible picture of the wind characteristics nicely [4-7]. The classic MHD models involve the energy and momentum

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equations with extra source or sink terms. However, there is another class of MHD models involving reduced adiabatic index known as '*polytropes*'. Polytropes are much simplified models based on the assumption that a power-law relationship exists between pressure and density in stars in a hydrodynamic equilibrium configuration. This equation of state is called the polytropic equation of state or polytropic energy equation or polytropic power law [13]. This excludes the need for solving the energy equation throughout the star [13-17]. Such models allow radial variation of the polytropic index, but within the validity limit of the total energy conservation of the radially symmetric expansion of the hot coronal plasma.

Moreover, a new and simple polytropic model for the solar wind, incorporating data from the Advanced Composition Explorer (ACE) spacecraft to set the model parameters, has recently been proposed [6, 7]. In this chapter, we present a simple linear fluctuation analysis of the polytropic solar wind by the well-known Fourier technique [14-17]. The main motivation behind the proposed analysis is to investigate the local nature of the solar plasma fluctuations with radial position, and also, with the variability of the associated polytropic index relative to the center of the Sun methodologically. A dispersion analysis of the associated fluctuations give the linear growth rate of the fluctuations as a nonlinear function of the variable polytropic index (α) and radial position (r) with respect to the center of the Sun. This is shown, both by analytical (approximate) and numerical (exact) techniques, that the growth of the fluctuations is stronger near the isothermal corona of the Sun, where $\alpha \sim 1$, relative to that observed elsewhere within the same solar plasma system.

6.2 PHYSICAL MODEL

We consider an idealized polytropic model of the solar wind for the stability analysis on the bounded Sun and its unbounded atmosphere. The model explicitly assumes a simple barotropic relationship in the usual form of $P \sim \rho^{\alpha}$ between the fluid pressure, P, and the fluid material density, ρ, with the cospatially variable polytropic index α. The pressure-density correlation is also termed as the polytropic energy equation, which gives the simplest analytical tool to allow spatial variations in the solar or stellar temperature [16, 17].

An example of such a polytropic model is the Lane-Emden equation of state describing the mass-pressure relationship inside a star, but with $\alpha = 1 + n^{-1}$, where n is called the polytropic index of the stellar structure [5]. The main advantage of adopting this model is that the equation of energy may be omitted from the set of structure equations to solve the solar wind stability problem [6]. Moreover, the polytropic index α is often considered to have a numerical value close to unity in order to represent the nearly isothermal corona of the Sun, and also, to obtain the acceleration of the solar wind. Under the polytropic assumption, the process is adiabatic only if the polytropic index satisfies the relation, $\alpha = \partial(\ln P)/\partial(\ln \rho)|_s$ at constant entropy *S*. This is because the quantity P/ρ^{α} is directly associated with the entropy *S* of the solar wind system. These model justifications are according to the analysis of Helios-1 data [5] that the solar wind behaves fairly polytropic between the radial distance range of 0.3 AU and 1 AU, with α = 1.46 as an average value for the polytropic index. This model, however, does not include the non-uniform temperature for the solar corona, any kind of anisotropy and density distribution over the solar surface.

6.3 MATHEMATICAL FORMULATION

In order to derive the dispersion relations for the solar wind fluctuations, both adiabatic and polytropic flow considerations are made [6]. The adiabatic and polytropic processes may be coupled together with the help of ideal gas approximation of the solar wind gas. Now, the equation of state for the ideal gas is $P \sim \rho^{\alpha}$, which implies $P = K\rho^{\alpha}$, where *K* is the polytropic constant and α is the adopted polytropic index [6]. For an adiabatic process, the conservative form of polytropic matter yields $D/Dt\left(P/\rho^{\gamma}\right) = 0$, where $D/Dt = (\partial/\partial t) + v_0(\partial/\partial x)$ represents commoving (material) derivative relative to the solar wind flow dynamics. Thus, using this, we get the following equation

$$\frac{DP}{Dt} - \frac{\gamma P}{\rho}\frac{D\rho}{Dt} = 0, \tag{6.1}$$

which can be also written in coordination space (*x*, *t*) for our inhomogeneous configuration as

$$\frac{\partial P}{\partial t} + v_0 \frac{\partial P}{\partial x} - \gamma \frac{P}{\rho}\left(\frac{\partial \rho}{\partial t} + v_0 \frac{\partial \rho}{\partial x}\right), \tag{6.2}$$

where, $\gamma = c_p/c_v$ is the ratio of specific heats of the solar wind (here, constant).

For linear stability analysis, we consider the plasma parameters to be linearly perturbed about the defined equilibrium as

$$\left.\begin{aligned} P &= P_0 + P_1 \\ \rho &= \rho_0 + \rho_1 \end{aligned}\right\}. \tag{6.3}$$

Since, we are interested in small perturbation analysis; therefore, the standard technique of the Fourier analysis is applied in order to see the spectral behavior of the linear fluctuations in the transformed Fourier space [14-17], characterized by wave frame (ω, k).

The perturbations can, according to the Fourier technique, be assumed to propagate periodically in the form of sinusoidal signals as ~ $\exp\{i(kx-\omega t)\}$, so that the linear differential operators get transformed as $\partial/\partial t = -i\omega$ and $\partial/\partial x = ik$ in the Fourier wave space (ω, k). Here, k is the angular wave number and ω is the angular frequency of the linear fluctuations. The equation of state, $P = K\rho^{\alpha}$, in the limit of small perturbation approximation $(P_0 >> P_1)$ now reduces to the following form

$$P_0 + P_1 = K\rho_0^{\alpha} + \alpha K\rho_0^{\alpha-1}\rho_1, \tag{6.4}$$

which, on separation of equilibrium and perturbation, yields

$$\left.\begin{aligned} P_0 &= K\rho_0^{\alpha} \\ P_1 &= \alpha K\rho_0^{\alpha-1}\rho_1 \end{aligned}\right\}. \tag{6.5}$$

Applying the perturbation (equation (6.3)) in equation (6.2), we have a linearized form as

$$\frac{\partial P_1}{\partial t} + v_0\frac{\partial P_1}{\partial x} - \frac{\gamma P_0}{\rho_0}\frac{\partial \rho_1}{\partial t} - \frac{\gamma P_1}{\rho_0}v_0\frac{\partial P_1}{\partial x} + \frac{\gamma P_0\rho_1}{\rho_0^2}\frac{\partial \rho_1}{\partial t} - \frac{\gamma P_0}{\rho_0}v_0\frac{\partial P_1}{\partial x} - \frac{\gamma P_1}{\rho_0}v_0\frac{\partial P_1}{\partial x} + \frac{\gamma P_0\rho_1}{\rho_0^2}v_0\frac{\partial P_1}{\partial x} = 0. \tag{6.6}$$

Using the standard Fourier transform technique [15] as mentioned earlier and with the help of equation (6.5), equation (6.6) can be written as

$$\begin{aligned} &\frac{\partial}{\partial t}(K\alpha)\left(\rho_0^{\alpha-1}\rho_1\right) + v_0\frac{\partial}{\partial x}(K\alpha)\left(\rho_0^{\alpha-1}\rho_1\right) - \frac{\gamma P_0}{\rho_0}(-i\omega)\rho_1 - \frac{\gamma P_1}{\rho_0}v_0(ik)\rho_1 \\ &+\frac{\gamma P_0\rho_1}{\rho_0^2}(-i\omega)\rho_1 - \frac{\gamma P_0}{\rho_0}v_0(ik)\rho_1 - \frac{\gamma P_1}{\rho_0}v_0(ik)\rho_1 + \frac{\gamma P_0\rho_1}{\rho_0^2}v_0(ik)\rho_1 = 0 \end{aligned}. \tag{6.7}$$

Since, our model is based on linear fluctuation analysis under small amplitude approximation, therefore $P_1\rho_1, P_1^2, \rho_1^2$, …. =0. Thus, equation (6.7) can be written as

$$\begin{aligned} &K\alpha\rho_0^{\alpha-1}\frac{\partial \rho_1}{\partial t} + K\alpha(\alpha-1)\rho_0^{\alpha-2}\rho_1 + v_0 K\alpha\rho_0^{\alpha-1}\frac{\partial \rho_1}{\partial x} \\ &+ v_0 K\alpha(\alpha-1)\rho_0^{\alpha-2}\rho_1 + i\omega\gamma\rho_1\frac{P_0}{\rho_0} - ikv_0\gamma\rho_1\frac{P_0}{\rho_0} = 0. \end{aligned} \tag{6.8}$$

In our modified analysis, *K* and α are considered to vary in the radial direction only because the acceleration is different in different types of solar wind [6]. Thus, the radial variation of the relevant polytropic parameters is taken into account. The explicit radial form of $\alpha = f(r)$ as a function of the radial coordinate *r* evolving between the initial point $r = r_1$ and the final point $r = r_2$ with all usual notations [6] is given by

$$\alpha(r) = \alpha_0 + (\alpha_1 - \alpha_0)\sin^2\left(\frac{\pi}{2}\frac{r - r_1}{r_2 - r_1}\right) \text{ for } r_1 \le r \le r_2 . \tag{6.9}$$

Similarly, the functional form of the factor *K(r)* can be written as

$$K(r) = K_0 n_0{}^{\alpha_0 - \alpha(r)} f(r), \tag{6.10}$$

where,

$$f(r) = 1 + (K_1 - 1)\sin^2\left(\frac{\pi}{2}\frac{r - r_1}{r_2 - r_1}\right) \text{ for } r_1 \le r \le r_2 , \tag{6.11}$$

$$\text{with } K_1 = \frac{K_{1AU}}{K_0 n_0^{\alpha_0 - \alpha_1}} . \tag{6.12}$$

Equation (6.8) can now be written as

$$\begin{aligned} &K\alpha\rho_0^{\alpha-1}(-i\omega)\rho_1 + K\alpha(\alpha-1)\rho_0^{\alpha-2}\rho_1 + v_0 K\alpha\rho_0^{\alpha-1}(ik)\rho_1 \\ &+ v_0 K\alpha(\alpha-1)\rho_0^{\alpha-2}\rho_1 + i\omega\gamma\rho_1\frac{P_0}{\rho_0} - ikv_0\gamma\rho_1\frac{P_0}{\rho_0} = 0, \end{aligned} \tag{6.13}$$

which on simplification yields

$$-i\omega\left(K\alpha\rho_0^{\alpha-1} - \gamma\frac{P_0}{\rho_0}\right) + ikv_0\left(K\alpha\rho_0^{\alpha-1} - \gamma\frac{P_0}{\rho_0}\right) + K\alpha(\alpha-1)\rho_0^{\alpha-2} + v_0 K\alpha(\alpha-1)\rho_0^{\alpha-2} = 0 . \tag{6.14}$$

Equation (6.14), after simplification, finally leads to the local dispersion relation as

$$\omega = kv_0 - \frac{iK\alpha(\alpha-1)\rho_0^{\alpha-2}(1+v_0)}{K\alpha\rho_0^{\alpha-1} - (\gamma P_0/\rho_0)} . \tag{6.15}$$

Now, substituting $\omega = \omega_r + i\omega_i$ in equation (6.15), where ω_r is the real part representing the actual frequency, and ω_i is the imaginary counterpart representing the instability behavior. This ω_i is of our interest, since, it measures physically the growth (or, decay) rate of the fluctuations within the solar plasma environment. Therefore, separating the real and imaginary parts in equation (6.15), we get the following expressions

$$\omega_r = k.v_o , \tag{6.16}$$

$$\omega_i = -\frac{K\alpha(\alpha-1)\rho_0^{\alpha-2}(1+v_0)}{K\alpha\rho_0^{\alpha-1} - (\gamma P_0/\rho_0)} \quad . \tag{6.17}$$

From equations (6.16) and (6.17), it is found that the phase velocity of the fluctuation is $v_p = \omega_r/k = v_0$, and their group velocity is $v_g = \partial\omega_r/\partial k = v_0$. Thus, the phase velocity and group velocity are equal, and the real speed of the fluctuation propagation comes out to be v_0. Equation (6.17) is further simplified to derive ω_i as a function of α along with *r* within this strategy. Hence, the final expression derived for ω_i is

$$\omega_i = -\left(K_0 n_0^{\alpha_0-\alpha(r)}\right)\left[1+(K_1-1)\sin^2\left(\frac{\pi}{2}\frac{r-r_1}{r_2-r_1}\right)\right]\times\left\{\frac{\alpha(\alpha-1)\rho_0^{\alpha-2}(1+v_0)}{(K\alpha\rho_0^{\alpha-1}-\gamma P_0/\rho_0)}\right\}. \tag{6.18}$$

Equation (6.18) shows that the polytropic solar wind fluctuations under investigation evolve with growth rates, which are nonlinear functions of the polytropic index (α), and radial distance (*r*) relative to the heliocentre. The quantitative details of the fluctuations can be seen with the help of graphical representation of equation (6.18) through numerical illustrative analysis within judiciously chosen multi-parametric variation as explained elsewhere [6].

6.4 RESULTS AND DISCUSSIONS

A simplified linear stability analysis of the polytropic solar wind model using the standard Fourier techniques is proposed. This is based on the application of a modified classic polytropic assumption with the inclusion of ideal gas-nature approximation of the wind. Thus, both the considered ideal adiabatic and polytropic flows are the underlying basic physical basis of the proposed model analysis. The associated growth rates are obtained by the Fourier techniques and then studied methodologically. It is seen that the growth rates of the solar wind fluctuations evolve dynamically as unique nonlinear functions of the spatially varying polytropic index (α) and the radial distance (*r*) relative to the heliocenter.

A detailed numerical analysis further elucidates the graphical behavior of the exact linear growth rates systematically by using some input data on relevant plasma parameters available from the ACE spacecraft [6]. The profile of the growth rate evolution as a function of the polytopic index is shown in figure 6.1. The input values taken for this numerical analysis are $K = (23.62\times10^{-22})^{3\alpha-2}$ N m, $\rho_0 = 1.43\times10^3$ kg m^{-3}, $P_0 = 11.682\times10^{-3}$ N m^{-2}, and $\gamma = 1$. Different lines correspond to the different cases with velocity v = 250 (blue line), 400 (red line), 550 (green line), and 700 km s^{-1} (black line); respectively. It is seen that greater the flow velocity of the wind, which is in the particular regime specified by $\alpha = 1-1.05$ (isentropic process), the greater the fluctuation growth-rate; and vice-versa. This is because the solar wind with greater flow velocity within the isentropic flow condition has greater kinetic energy, and vice-versa. It is seen that, if the wind flow is linearly perturbed, the equilibrium flow energy associated with each of the comoving wind elements (particles) undergo resonant coupling. This results in growths of the fluctuations in accordance with their phase and amplitude coordination as a coherent process. This happens only at the above α-regime, which corresponds to an isothermal process [6]. At different α-values, the isothermal condition is lost, and the wind fluctuations with different speeds get randomized (non-isentropic process). As a consequence, the wind fluctuations are averaged out to near-zero values, leading neither to growth, nor to any decay.

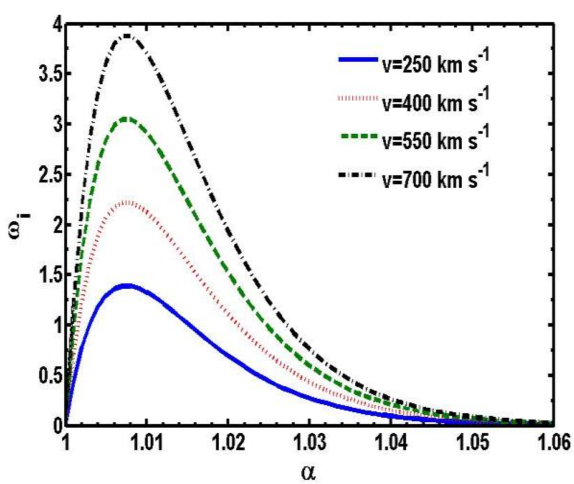
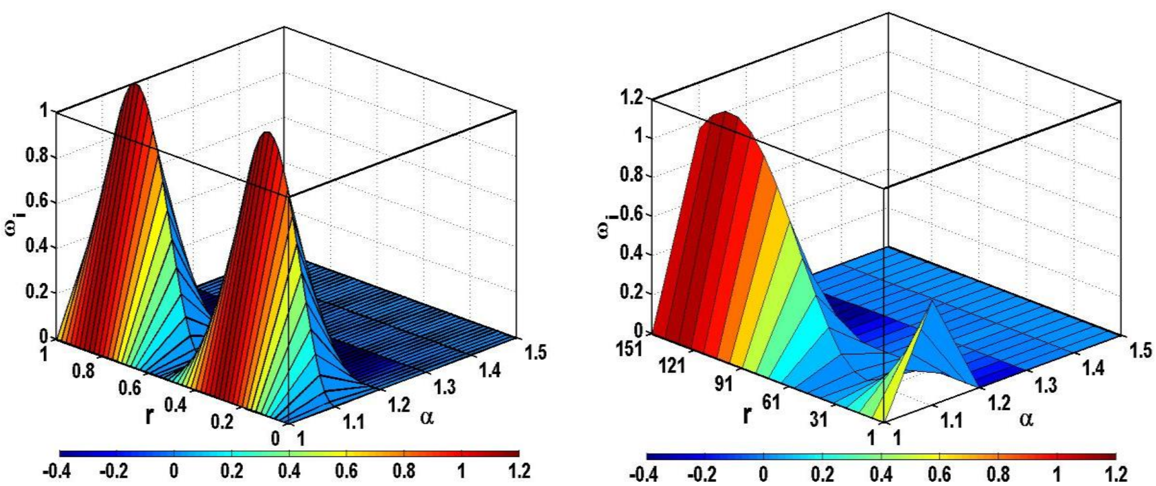


***Figure 6.1**: Profile of linear growth rate (ω_i) with variable polytropic index (α). Different lines correspond to different cases with velocity v = 250 (blue line), 400 (red line), 550 (green line), and 700 km s^{-1} (black line); respectively. The fine details are given in the text.*

The profile of the growth-rate as a function of polytropic index (α) and position (*r*) normalized by the solar radius (R_Θ) in the case of the solar interior is shown in figure 6.2(a).

The same is extended up to the solar exterior scale as displayed in figure 6.2(b). The different input values [6] taken for this computation are $K = (23.62 \times 10^{-22})^{3\alpha - 2}$ N m, $\rho_0 = 1.43 \times 10^3$ kg m^{-3}, $P_0 = 11.682 \times 10^{-3}$ N m^{-2}, $\gamma = 1$, $K_0 = 2.042 \times 10^{-17}$ J, $n_0 = 5.72 \times 10^{14}$ m^{-3}, α_0=1.26, α_1=1.424, and v = 400 km s^{-1}. It is realized that the growth rate starts increasing at the position specified by $r \sim 0$, and then. it reaches a maximum value at $r \sim 0.08$ $(\omega_{i,\max} = 0.6 \times 10^{-45}\, s^{-1})$. The growth rate for the weak fluctuations again comes down to a certain minimum value (~ 0) at $r \sim 0.5$, and then, it keeps on growing until it reaches a maximum value at $r \sim 0.7$ $(\omega_{i,\max} = 1.0 \times 10^{-45} s^{-1})$, as is clearly shown in figure 6.2(a). The growth achieves the maximum value $\left(\omega_i = 1.0 \times 10^{-45}\ s^{-1}\right)$ at $r \sim 150$ on the solar exterior scale [figure 6.2(b)], corresponding to $\alpha = 1.10$ (isentropic situation) near the solar corona.



(a) **(b)**

Figure 6.2*: Profile of linear growth rate (ω_i) with position (r normalised by R⊙) and polytropic index (α) for the solar interior (a) and for the solar exterior scale (b). The different input and initial values adopted here are described in the text.*

Thus, it is inferred that the solar fluctuations undergo resonant growths only in isentropic processes due to coherent (resonant) interaction among themselves at the cost of the free energy source coming from the defined dynamic equilibrium (uniformly flowing) itself within the expanding solar plasma volume. This may be considered as a new theoretical idea to give a qualitative explanation based on a simple approach about solar wind waves, oscillations and fluctuations, as compared with other different simple and complex theoretical models reported earlier [1, 6, 9]. However, our calculation scheme using a reduced and variable polytropic index excludes the sudden disturbances and fluctuations in the relevant solar parameters generated by the coronal mass ejections (CMEs) and thereby driven shocks in the solar plasma system. Moreover, nonlinear saturation mechanisms of the perturbations in the large-amplitude regime, due to balancing between nonlinear convection (caused by fluidity) and linear dispersion (caused by solar self-gravity amid large-scale dynamics) are also ignored for simplification of the proposed model.

6.5 CONCLUSIONS

In this chapter, a simplified theoretical model for the linear stability analysis of the polytropic solar wind in the framework of the well-known Fourier technique is methodically developed. The considered equilibrium is of the MHD-type (hydrodynamic) formed by a coupling of the adiabatic and polytropic processes in the limit of ideal gas approximation. The behavior of macro-instability in the solar wind is analyzed using the modified polytropic model, which forbids the conservation of energy, instead of using the basic MHD model. Based on the methodologically derived linear dispersion relation, the explicit form of the associated growth rate of the fluctuations is seen to evolve dynamically as a unique nonlinear function of the polytropic index and the anti-Sun-ward radial distance.

Applying a multi-parametric variation scheme, a constructive numerical analysis is carried out, which depicts the graphical nature of the local linear growth rates based on the ACE spacecraft data [5, 6]. The growths are found to be maximum only in the isentropic regions on the solar exterior scale; elsewhere, no growth saturation occurs. Such observations on growing fluctuations are mainly due to resonant interaction processes among the said fluctuations, plausibly due to their phase and amplitude coordination in isentropic conditions, which correspond to the isothermal corona of the Sun.

Moreover, the growth for higher flow velocity of the wind is interestingly found to be greater; and vice-versa. In addition, the fluctuations are randomized (with no phase and amplitude coordination), and hence, averaged out to zero in non-isentropic regions elsewhere in the solar wind system. This is in good agreement where normally source and sink terms need to be considered in the solar plasma flow dynamics. Our proposed theoretical model gives a simplified proposition for worthwhile understanding of the solar instability, involving various complex processes about helioseismic and asteroseismic wave activities within the framework of a new polytropic perspective. However, it is to be noted that such models fall short of explaining the coronal mass ejection driven shocks as energy conservation has not been used. Therefore, a full MHD simulation is more suitable to understand the shock propagation and shock-induced instabilities as compared to the polytropic models [14]. In addition, necessary modifications and refinements have to be considered to see the actual picture of the stability behaviors of the Sun and its atmosphere.

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