Chapter 2

New quasi Poisson-Lindley distribution: Properties and Applications

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2.1 Introduction

A new quasi Poisson-Lindley (NQPL) distribution has been studied in this chapter of which the Poisson-Lindley (PL) distribution introduced by Sankaran [82] to model count data has been obtained as a particular case. As Poisson-Lindley distribution has one parameter so it does not provide enough flexibility for analysing different types of lifetime data. It will be better if we consider further alternatives to this distribution so as to increase the flexibility for modelling purposes.

In this chapter, a new quasi Poisson-Lindley (NQP) distribution has been proposed by mixing Poisson distribution and new quasi-Lindley distribution [87]. Section 2.2 deals with the derivation of NQPL distribution. In section 2.3, the graphical representation of NQPL distribution to study the behavior of PL distribution for different values of the parameters has been discussed. Certain properties of the distribution have been studied in section 2.4. Zero modified QPL distribution along with its generating function has been obtained in section 2.5. In section 2.6 and 2.7, the method of estimation of parameters and goodness of fit has been discussed.

2.2 New Quasi Poisson–Lindley (NQPL) distribution

Suppose, the parameter λ of the Poisson distribution having the probability mass function (pmf)

$$g(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1, \ldots;\ \lambda > 0$$

follows the new quasi-Lindley distribution obtained by Shanker and Ghebretsadik [87] with probability density function (pdf) given as,

$$f(x) = \frac{\theta^2(\theta+\alpha x)}{\theta^2+\alpha} e^{-\theta x};\quad x > 0,\ \theta > 0,\ \alpha > 0.$$

Then, the probability mass function (pmf) of NQPL distribution may be obtained as,

$$P(X = x) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} \frac{\theta^2}{\theta^2+\alpha} (\theta + \alpha\lambda) e^{-\theta\lambda}\, d\lambda\ ,$$

$$= \frac{\theta^2}{(\theta^2+\alpha)x!} \int_0^\infty e^{-\lambda(1+\theta)} \lambda^x (\theta + \alpha\lambda) d\lambda\ ,$$

$$= \frac{\theta^2}{(\theta^2+\alpha)x!} \left(\theta \int_0^\infty \lambda^x\, e^{-\lambda(1+\theta)} d\lambda + \alpha \int_0^\infty \lambda^{x+1} e^{-\lambda(1+\theta)} d\lambda\right),$$

$$= \frac{\theta^2}{(\theta^2+\alpha)x!} \left(\theta \frac{\Gamma(x+1)}{(1+\theta)^{x+1}} + \alpha \frac{\Gamma(x+2)}{(1+\theta)^{x+2}}\right),$$

$$= \frac{\theta^2}{(1+\theta)^{x+2}} \left(1 + \frac{\theta+\alpha x}{\theta^2+\alpha}\right), x = 0,1, \ldots, \theta > 0 \text{ and } \alpha > 0\ . \quad (2.2.1)$$

The compounded distribution that is obtained in equation (2.2.1) is the new quasi Poisson-Lindley (NQPL) distribution.

Particular Cases

(i) For $\alpha = \theta$, (2.2.1) reduces to one-parameter Poisson-Lindley distribution.

(ii) For $\alpha \to 0$, (2.2.1) reduces to geometric distribution with probability $\frac{\theta}{1+\theta}$.

2.3. Graphical representation of NQPL distribution

The pmf has been shown graphically to study the behavior of NQPL distribution using equation (2.2.1) for different values of parameter (α, θ) and possible values of x. From the figure 2.1, figure 2.2 and figure 2.3 it has been clearly observed that the behavior of NQPL distribution is monotonically decreasing for fixed α and different values of θ.

Figure 2.1 pmf plot of NQPL distribution for $\alpha = 0.5$ and $\theta = 0.5, 1.5$ and 2.0

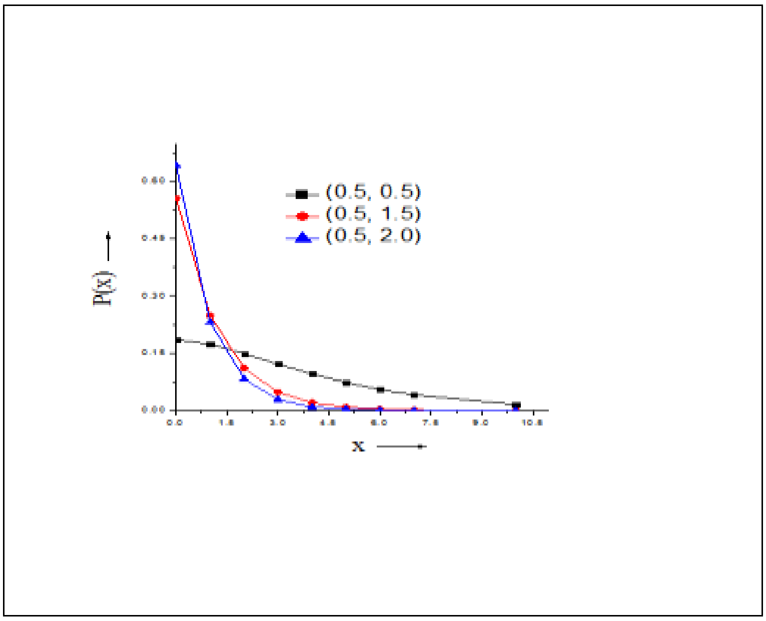


Figure 2.2 pmf plot of NQPL distribution for $\alpha = 1.0$ and $\theta = 0.3, 2.0$ and 3.5

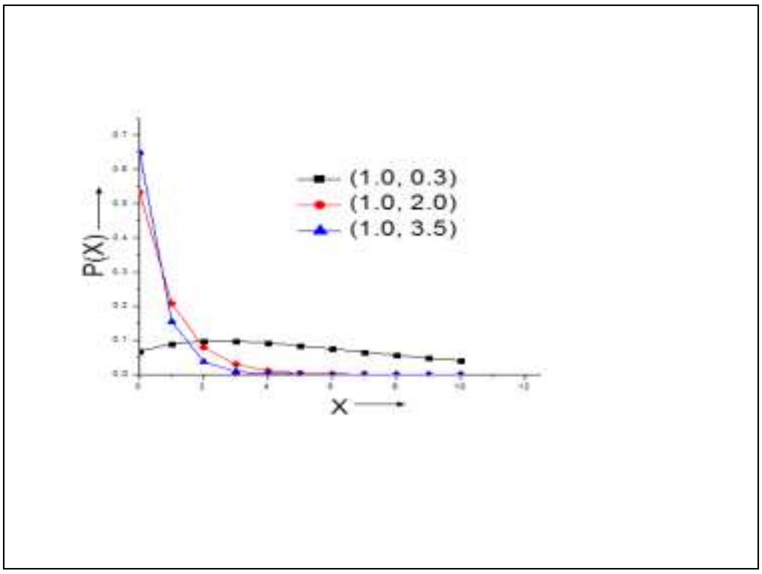
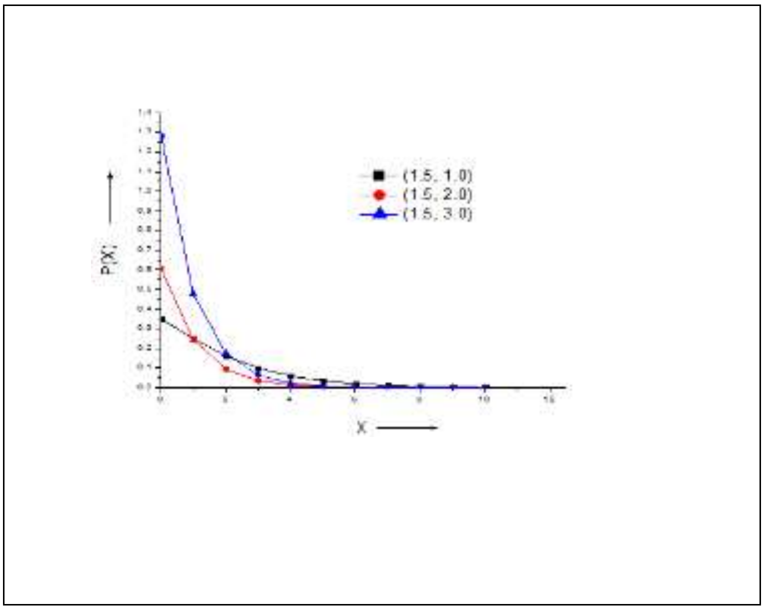


Figure 2.3 pmf plot of NQPL distribution for $\alpha = 1.5$ and $\theta = 1.0, 2.0$ and 3.0



2.4 Distributional properties

In this section certain statistical properties of NQPL distribution has been studied specifically the shape of the probability function, moments, recurrence relation for probability.

2.4.1 Shape of the probability function

To obtain the shape of the NQPL distribution, the ratio of the probability mass function at the point "$x + 1$" and "x" has been obtained.It can be seen that

$$\frac{P(x+1;\theta,\alpha)}{P(x;\theta,\alpha)} = \frac{1}{1+\theta}\left\{1 + \frac{\alpha}{\theta^2+\theta+\alpha+\alpha x}\right\},$$

which is clearly a decreasing function in x. Therefore, $P(x;\theta,\alpha)$ is log-concave. Therefore, the NQPL distribution (4.2.1) distribution is unimodal. [Johnson et al. [56]].

2.4.2 Moments of NQPL distribution

The r^{th} factorial moment may be obtained as

$$\mu'_{(r)} = E\left[E\left(X^{(r)}|\lambda\right)\right],$$

where, $X^{(r)} = X(X-1)\ldots\ldots(X-r+1)$ is the descending factorial.

Then,

$$\mu'_{(r)} = \int_0^\infty \left\{\sum_{x=0}^\infty x^{(r)} \frac{e^{-\lambda}\lambda^x}{x!}\right\} \frac{\theta^2}{\theta^2+\alpha}(\theta+\alpha\lambda)e^{-\theta\lambda}d\lambda\ ,$$

$$\mu'_{(r)} = \int_0^\infty \lambda^r \left\{\sum_{x=r}^\infty \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right\} \frac{\theta^2}{\theta^2+\alpha}(\theta+\alpha\lambda)e^{-\theta\lambda}d\lambda. \qquad (2.4.1)$$

Taking $(x+r) = x$ in (2.4.1), we have

$$\mu'_{(r)} = \int_0^\infty \lambda^r \left\{\sum_{x=0}^\infty \frac{e^{-\lambda}\lambda^{x}}{x!}\right\} \frac{\theta^2}{\theta^2+\alpha}(\theta+\alpha\lambda)e^{-\theta\lambda}d\lambda. \qquad (2.4.2)$$

The expression within the second bracket in equation (2.4.2) is equal to unity as it is the sum of pmf of Poisson distribution.

Thus, from equation (2.4.2) we have obtained,

$$\mu'_{(r)} = \frac{\theta^2}{\theta^2+\alpha}\int_0^\infty \lambda^r\,(\theta+\alpha\lambda)e^{-\theta\lambda}d\lambda\ ,$$

which is a gamma integral and on simplification we get the r^{th} factorial moment as

$$\mu'_{(r)} = \frac{\Gamma(r+1)\{\theta^2+\alpha(r+1)\}}{\theta^r(\theta^2+\alpha)}, r = 1,2,\ldots.$$

The r^{th} moment about the origin may be obtained as,

$$\mu'_r = E[E(X^r|\lambda)]\ ,$$

$$\mu'_r = \int_0^\infty \left\{\sum_{x=0}^\infty x^r \frac{e^{-\lambda}\lambda^x}{x!}\right\} \frac{\theta^2}{\theta^2+\alpha}(\theta+\alpha\lambda)e^{-\theta\lambda}d\lambda, \qquad (2.4.3)$$

The expression within the bracket in equation (2.4.3) is the r^{th} moment about the origin of the Poisson distribution. Taking $r = 1$, in equation (2.4.3) and using the mean i.e. the first raw moment of Poisson distribution we can get the first raw moment (mean) of new quasi Poisson Lindley (NQPL) distribution as

$$\mu'_1 = \int_0^\infty \lambda \frac{\theta^2}{\theta^2+\alpha}(\theta+\alpha\lambda)e^{-\theta\lambda}d\lambda\ ,$$

$$\mu_1' = \frac{\theta^2+2\alpha}{\theta(\theta^2+\alpha)}.$$

Taking $r = 2$, in equation (2.4.3) and using the second raw moment of the Poisson distribution we can get the second moment about origin of NQPL distribution as,

$$\mu_2' = \int_0^\infty (\lambda^2 + \lambda) \frac{\theta^2}{\theta^2+\alpha} (\theta + \alpha\lambda) e^{-\theta\lambda} d\lambda,$$

$$\mu_2' = \frac{\theta^2+2\alpha}{\theta(\theta^2+\alpha)} + \frac{2(\theta^2+3\alpha)}{\theta^2(\theta^2+\alpha)}.$$

Taking $r = 3, 4$ in equation (2.4.3) and using the third and fourth raw moments of Poisson distribution we can get the third and fourth moment of NQPL distribution as

$$\mu_3' = \frac{\theta^2+2\alpha}{\theta(\theta^2+\alpha)} + \frac{6(\theta^2+3\alpha)}{\theta^2(\theta^2+\alpha)} + \frac{6(\theta^2+4\alpha)}{\theta^3(\theta^2+\alpha)},$$

$$\mu_4' = \frac{\theta^2+2\alpha}{\theta(\theta^2+\alpha)} + \frac{14(\theta^2+3\alpha)}{\theta^2(\theta^2+\alpha)} + \frac{36(\theta^2+4\alpha)}{\theta^3(\theta^2+\alpha)} + \frac{24(\theta^2+5\alpha)}{\theta^4(\theta^2+\alpha)}.$$

From the raw moments using the identity,

$$\mu_r = E(X-\mu)^r = \sum_{k=0}^{r} \binom{r}{k} \mu'_k \left(-\mu'_1\right)^{r-k},$$

the central moment $\mu_2, \mu_3$and μ_4 has been obtained as follows:

$$\mu_2 = \sigma^2 = \frac{\theta^5+\theta^4+3\theta^3\alpha+4\theta^2\alpha+2\theta\alpha^2+2\alpha^2}{\theta^2(\theta^2+\alpha)^2},$$

$$\mu_3 = \frac{\left[2\alpha^3(\theta+1)(\theta+2)+\alpha^2\left(5\theta^4+18\theta^3+12\theta^2\right)+\alpha\left(4\theta^6+15\theta^5+12\theta^4\right)+\theta^6(\theta+1)(\theta+2)\right]}{\theta^3(\theta^2+\alpha)^3},$$

$$\mu_4 = \frac{\left[\begin{array}{c}\alpha^4\left(2\theta^3+4\theta^2+48\theta+216\right)+\alpha^3\left(7\theta^5+40\theta^4+180\theta^3+624\theta^2\right)+\\ \alpha^2\left(9\theta^7+4\theta^6+240\theta^5+476\theta^4+336\theta^2\right)+\alpha\left(5\theta^9+126\theta^7+264\theta^6+72\theta^4+96\theta^2\right)\\ +\left(\theta^{11}+18\theta^9+33\theta^8+24\theta^4\right)\end{array}\right]}{\theta^4(\theta^2+\alpha)^4}.$$

Some other indices of the shape of the NQPL distribution are skewness, kurtosis, index of dispersion and coefficient of variation respectively.

2.4.3 Co-efficient of Skewness and Kurtosis

The co-efficient of skewness denoted by $\sqrt{\beta_1}$ may be written as, Skewness $\left(\sqrt{\beta_1}\right) = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$

Hence, $(\sqrt{\beta_1}) = \frac{(\theta+1)(\theta+2)\{2\alpha^3+\theta^6\}+\alpha^2(5\theta^4+18\theta^3+12\theta^2)+\alpha(4\theta^6+15\theta^5+12\theta^4)}{(\theta^5+\theta^4+3\theta^3\alpha+4\theta^2\alpha+2\alpha^2(\theta+1))^{3/2}}$.

The co-efficient of kurtosis may be written as,

Kurtosis $(\beta_2) = \frac{\mu_4}{\sigma^4} = \frac{A\alpha^4+B\alpha^3+C\alpha^2+D\alpha+E}{(\theta^5+\theta^4+3\theta^3\alpha+4\theta^2\alpha+2\theta\alpha^2+2\alpha^2)^2}$,

where, $A = 2\theta^3 + 26\theta^2 + 48\theta + 24$,

$B = 7\theta^5 + 92\theta^4 + 180\theta^3 + 96\theta^2$,

$C = 9\theta^7 + 116\theta^6 + 240\theta^5 + 132\theta^4$,

$D = 5\theta^9 + 60\theta^8 + 126\theta^7 + 72\theta^6$,

$E = \theta^{11} + 10\theta^{10} + 18\theta^9 + 9\theta^8$.

2.4.4 Index of Dispersion (ID) and Coefficient of Variation (CV)

The index of dispersion (ID) (Johnson et al. [56]) is defined as the ratio of variance and mean is useful in determining whether a distribution is over dispersed or under dispersed for ID > (<) 1.

The index of dispersion is given as,

$$\text{I. D} = \frac{\sigma^2}{\mu} = 1 + \frac{\theta^4+4\theta^2\alpha+2\alpha^2}{\theta(\theta^2+\alpha)(\theta^2+2\alpha)} > 1,$$

for $\theta > 0$ and fixed α. Therefore we may conclude that the NQPL(α, θ) distribution is over-dispersed.

Coefficient of variation may be defined as the ratio of standard deviation to mean and is used to compare variability. With higher level of co-efficient of variation, the level of dispersion around the mean increases. The coefficient of variation may be written as,

$$C.V = \frac{\sigma}{\mu} = \frac{\sqrt{2\alpha^2(\theta+1)+\alpha(3\theta^3+4\theta^2)+\theta^5+\theta^4}}{\theta^2+2\alpha} .$$

2.4.5 Probability recurrence relation

The probability generating function (pgf) of NQPL distribution derived from equation (2.2.1) may be written as

$$g(t) = E(t^x), |t| < 1$$

Thus, $g(t) = \sum_{t=0}^{\infty} t^x P(x)$, where $P(x)$ is the pmf of NQPL distribution.

$$g(t) = \frac{\theta^2}{(\theta^2+\alpha)(\theta+1-t)^2}\{\theta(\theta+1-t)+\alpha\}. \tag{2.4.4}$$

Differentiating equation (2.4.4) w.r.t "t" we get,

$$g'(t) = \frac{\theta^2(\theta^2+\theta-\theta t+2\alpha)}{(\theta^2+\alpha)(\theta+1-t)^3}.$$

Therefore, $g'(t)(\theta+1-t)^3 = G(t)(\theta+1-t)^2 + \frac{\alpha\theta^2}{\theta^2+\alpha}$, (2.4.5)

Equating the co-efficient of "t^r"on both sides of equation (2.4.5), the recurrence relation may be obtained as,

$$p_{r+1} = \frac{(\theta+1)^2(3r+1)p_r-(\theta+1)(3r-1)p_{r-1}+(r-1)p_{r-2}}{(\theta+1)^3(r+1)}, r \geq 2 \tag{2.4.6}$$

where, $p_0 = \left(\frac{\theta}{\theta+1}\right)^2\left\{\frac{\theta(\theta+1)+\alpha}{\theta^2+\alpha}\right\}$,

$$p_1 = \frac{\theta^2}{(\theta+1)^3}\left\{\frac{\theta(\theta+1)+2\alpha}{\theta^2+\alpha}\right\},$$

$$p_2 = \frac{\theta^2}{(\theta+1)^4}\left\{\frac{\theta(\theta+1)+3\alpha}{\theta^2+\alpha}\right\}.$$

Putting $r = 2, 3, ...$ in equation (2.4.6) higher order probabilities may be obtained as

$$p_3 = \frac{(\theta+1)^2(3r+1)p_2-(\theta+1)(3r-1)p_1+(r-1)p_0}{(\theta+1)^3(r+1)},$$

$$p_4 = \frac{(\theta+1)^2(3r+1)p_3-(\theta+1)(3r-1)p_3+(r-1)p_1}{(\theta+1)^3(r+1)} \text{ etc.}$$

The r^{th} probability of NQPL distribution may be written as,

$$P_r = \frac{\theta^2}{(1+\theta)^{r+2}}\left\{\frac{\theta(\theta+1)+r\alpha}{\theta^2+\alpha}\right\}.$$

2.4.6 Expression for factorial moment

The factorial moment generating function (fmgf) may be written as,

$$G(t) = g(1+t).$$

Therefore, $G(t) = \frac{\theta^2}{(\theta^2+\alpha)(\theta-t)^2}\{\theta(\theta-t)+\alpha\}$. (2.4.7)

Expanding equation (2.4.7) and equating the co-efficient of " $\frac{t^r}{r!}$ " we have obtained

$$\mu'_{(r)} = \frac{r}{\theta^2}\left\{2\theta\mu'_{(r-1)} - (r-1)\mu'_{(r-2)}\right\}, r > 1$$

where $\mu'_{(1)} = \frac{\{\theta^2+2\alpha\}}{\theta(\theta^2+\alpha)}$,

$$\mu'_{(2)} = \frac{2\{\theta^2+3\alpha\}}{\theta^3(\theta^2+\alpha)}.$$

2.5 Zero-modified new quasi Poisson-Lindley (ZMNQPL) distribution

It has been observed that in many real-life situations that the number of zeros shown up in the count data with a greater tendency than is expected. Zero modified distribution has been used as many distributions obtained in the course of experimental investigation often have an excess frequency of observed event at Zero Point.

Zero-modified distribution has been obtained by combining the pmf of the original distribution together with the probabilities of degenerate distribution concentrated at zero. The pmf of the zero-modified distribution has been given as

$$P[X=0] = w + (1-w)P_0\ , \tag{2.5.1}$$

and $P[X=x] = (1-w)P_x; x \geq 1$. (2.5.2)

In equations (2.5.1) and (2.5.2), w is the parameter assuming an arbitrary value in the interval $0 < w < 1$. It may also be possible that $w < 0$ provided $w + (1-w)P_0 \geq 0$ [Johnson et al. [56]]. P_0 is the probability at zero of the original distribution and P_x is the pmf of the original distribution.

Then, the pmf of zero-modified NQPL distribution may be obtained as

$$P[X=0] = w + (1-w)\frac{\theta^2}{(1+\theta)^2}\left(\frac{\theta^2+\alpha+\theta}{\theta^2+\alpha}\right),$$

and $$P[X=x] = (1-w)\frac{\theta^2}{(1+\theta)^{x+2}}\left(\frac{\theta^2+\alpha+\alpha x+\theta}{\theta^2+\alpha}\right); x \geq 1.$$

where, P_0 is the probability at zero of NQPL distribution and P_x is the pmf of NQPL distribution.

2.5.1 Recurrence relation for probabilities

The probability generating function (pgf) of ZMNQPL distribution may be obtained as,

$$g(t) = w + (1-w)g_1(t),$$

where, $g_1(t)$ is the pgf of NQPL distribution given in equation (2.2.1).

Thus, $$g(t) = w + (1-w)\frac{\theta^2}{(\theta^2+\alpha)(\theta+1-t)^2}\{\theta(\theta+1-t)+\alpha\}. \qquad (2.5.1)$$

Now, differentiating equation (2.5.1) w.r.t $'t'$ and equating the coefficient of $'t^r'$ we get,

$$p_{r+1} = \frac{[3(1+\theta)^2 r p_r - 3(1+\theta)(r-1)p_{r-1} + 3(r-2)p_{r-2}]}{(1+\theta)^3(r+1)}, r > 1$$

2.5.2 Recurrence relation for factorial moment generating function (fmgf)

The factorial moment generating function (fmgf) of ZMNQPL distribution may be obtained as,

$$G(t) = g(1+t),$$

$$G(t) = w + \frac{\theta^2\{\theta(\theta-t)+\alpha\}(1-w)}{(\theta^2+\alpha)(\theta-t)^2}. \quad (2.5.2)$$

Now from equation (2.5.2) on differentiating $G(t)$ and equating the co-efficient of $\frac{t^r}{r!}$ we have obtained the recursive expression for fmgf as,

$$\mu'_{(r+1)} = \frac{\left[3\theta^2\mu'_{(r)} - 3\mu'_{(r-1)} + \mu'_{(r-2)}\right]}{\theta^3}.$$

2.6 Estimation of Parameters of NQPL distribution

The following methods have been used for estimating the parameters of NQPL distribution.

2.6.1 Method of Moments

The first two moment's i.e. μ'_1 and μ'_2 are required to estimate the parameters of NQPL distribution by the method of moments.

$$\frac{\mu'_2 - \mu'_1}{\mu'^2_1} = \frac{2(\theta^2+3\alpha)(\theta^2+\alpha)}{(\theta^2+2\alpha)^2} = k\ (say) \quad . \quad (2.6.1)$$

Substituting $\alpha = b\theta^2$, in (2.6.1) a quadratic equation in 'b' is obtained as

$$(6-4k)b^2 + 4(2-k)b + (2-k) = 0. \quad (2.6.2)$$

Replacing the first two population moment by the respective sample moments in (2.6.1) the estimate of k may be obtained. Substituting the estimate of k in (2.6.2.) the estimate $\hat{b}$ can be obtained from the quadratic equation (2.6.2).

We have,

$$\bar{x} = \frac{\theta^2+2\alpha}{\theta(\theta^2+\alpha)}. \quad (2.6.3)$$

Substituting $\alpha = b\theta^2$ in the expression (2.6.3) we have,

$$\bar{x} = \frac{1+2\hat{b}}{\theta(1+b)}.$$

Hence, $\hat{\theta} = \left(\frac{1+2\hat{b}}{1+\hat{b}}\right)\frac{1}{\bar{x}},$

and $\hat{\alpha} = b\hat{\theta}^2 = \frac{\hat{b}\left(1+2\hat{b}\right)^2}{\left(1+\hat{b}\right)^2(\bar{x})^2}$, where $\hat{b}$ can be obtained from equation (2.6.2).

2.6.2 Maximum Likelihood Estimates

Let $x_1, x_2, \dots, x_n$ be a random sample of size n from the NQPL distribution (2.2.1) and let f_x be the observed frequency in the sample corresponding to $X = x\ (x = 1, 2, \dots, k)$ such that $\sum_{x=1}^{k} f_x = n$.

The likelihood function L of NQPL distribution may be written as,

$$L = \left(\frac{\theta^2}{\theta^2+\alpha}\right)^n \frac{1}{(1+\theta)^{\sum_{x=1}^{k}(x+2)f_x}} \prod_{x=1}^{k} [\theta(\theta+1) + \alpha(x+1)]^{f_x}.$$

The log likelihood function becomes

$$\log L = n\log\left(\frac{\theta^2}{\theta^2+\alpha}\right) - \sum_{x=1}^{k}(x+2)f_x \log(1+\theta) + \sum_{x=1}^{k} f_x \log[\theta(\theta+1) + \alpha(x+1)] .$$

Then the derivatives of log likelihood equations are obtained as,

$$\frac{\partial \log L}{\partial \alpha} = -\frac{n}{\theta^2+\alpha} + \sum_{x=1}^{k} \frac{(x+1)f_x}{[\theta(\theta+1)+\alpha(x+1)]} = 0 ,$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{2n}{\theta^2+\alpha} - \sum_{x=1}^{k} \frac{(x+2)f_x}{1+\theta} + \sum_{x=1}^{k} \frac{(2\theta+1)f_x}{[\theta(\theta+1)+\alpha(x+1)]} = 0 .$$

The above two equations could be solved using Fisher's scoring method. We have,

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{n}{(\theta^2+\alpha)^2} - \sum_{x=1}^{k} \frac{(x+1)^2 f_x}{[\theta(\theta+1)+\alpha(x+1)]^2} ,$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \theta} = \frac{2n\theta}{(\theta^2+\alpha)^2} - \sum_{x=1}^{k} \frac{(x+1)(2\theta+1)f_x}{[\theta(\theta+1)+\alpha(x+1)]^2} ,$$

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{2n}{\theta^2} + \frac{4n\theta}{(\theta^2+\alpha)^2} + \sum_{x=1}^{k} \frac{(x+2)f_x}{(1+\theta)^2} + \sum_{x=1}^{k} \frac{\left(2\theta^2+2\theta+1-2\alpha(x+1)\right)f_x}{[\theta(\theta+1)+\alpha(x+1)]^2} .$$

The following system of equations can be solved using any numerical method for $\hat{\theta}$ and $\hat{\alpha}$ iteratively until close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained, where θ_0 and α_0 are the initial values of θ and α respectively.

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \theta} & \frac{\partial^2 \log L}{\partial \theta^2} \end{bmatrix} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \alpha} \\ \frac{\partial \log L}{\partial \theta} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} .$$

2.7 Goodness of fit

The NQPL distribution has been fitted to some reported data and it is believed that these distribution gives a reasonably close fit to some numerical data for which various distributions were fitted earlier viz. Poisson-Lindley (PL), two-parameter Poisson-Lindley (TPPL), discrete gamma and negative binomial (NB) distribution. The parameter θ and α of NQPL distribution have been estimated by the method of moment because of the complexity of maximum likelihood method of estimation. Three sets of data have been considered to see the suitability and applicability of NQPL distribution and have obtained the expected frequency to calculate the χ^2 goodness of fit. This distribution is then compared with other distributions as measured by χ^2 criterion.

Table 2.1: Observed and expected frequency of Poisson-Lindley (PL), two-parameter Poisson-Lindley (TPPL) and new quasi Poisson-Lindley (NQPL) distribution on Pyrausta nublilalis in 1937. [data from Beall [7]].

No. of Insects	Observed frequencies	Expected frequencies		
		PL	TPPL	NQPL
0	33	31.49	31.9	31.90
1	12	14.16	13.8	13.80
2	6	6.09	5.9	5.92
3	3	2.54	2.5	2.73
4	1	1.04	1.1	1.09
≥5	1	0.42	0.8	0.56
Total	56	56	56	56
Parameters		$\hat{\theta} = 1.8082$	$\hat{\alpha} = 0.2573$ $\hat{\theta} = 0.39249$	$\hat{\alpha} = 0.3920$ $\hat{\theta} = 1.5255$
χ^2		4.82	0.36	0.3203
d.f		2	1	1
p-value		0.0898	0.5485	0.5714

In Table 2.1, we have considered the data by Beall [7] on the Pyrausta nublilalis, to which PL and TPPL distribution were fitted earlier by Shanker and Mishra (2014). It has been observed that sample mean $\bar{x} = 0.75$ and sample variance $\mu_2' = 1.8571$. Using the method of moment we have estimated the parameters from the data set for NQPL distribution as $\hat{\alpha} = 0.3920$ and $\hat{\theta} = 1.5255$.

Table 2.2: Observed and expected frequency of discrete gamma ($d\gamma$), negative-binomial (NB) distribution and new quasi Poisson-Lindley (NQPL) distribution of number of European red mites on apple leaves. [data from Bliss et al. [11]]

European red Mites	Observed frequencies	Expected frequencies		
		$d\gamma(k,\theta)$	NBD(r,p)	NQPL (θ,α)
0	70	69.67	69.49	67.75
1	38	37.49	37.6	38.51
2	17	20.02	20.1	20.99
3	10	10.67	10.7	11.11
4	9	5.69	5.69	5.77
5	3	3.03	3.02	3.14
6	2	1.61	1.6	1.68
7	1	.86	0.85	.84
8	0	.96	0.95	.30
Total	150	150	150	150
Parameters		$\hat{k} = 1.0078$ $\hat{\theta} = 1.5830$	$\hat{r} = 1.0245$ $\hat{p} = 0.5281$	$\hat{\alpha} = 0.9121$ $\hat{\theta} = 1.2076$
χ^2		2.89	2.91	2.7592
d.f		4	4	4
p-value		0.5901	0.5730	0.5981

In Table 2.2 we have considered the distribution of number of European red mites on apple leaves (Bliss et al. [11]) for which discrete gamma ($d\gamma$) distribution and negative binomial (NB) distribution have been fitted earlier by Chakraborty and Chakravarty (2012). From the data set the sample mean and sample variance have been obtained as $\bar{x} = 1.1467$ and $\mu_2' = 3.5733$. From the mean and sample variance the parameters of NQPL distribution for the data on number of European red mite's apple leaves have been obtained as $\hat{\alpha} = 0.9121$ and $\hat{\theta} = 1.2076$.

Table: 2.3: Observed and expected frequencies of Poisson-Lindley, two-parameter Poisson-Lindley and new quasi Poisson-Lindley distributions for the mistakes in copying groups of random digits. [data from Kemp and Kemp [62]].

No. of errors per group	Observed frequencies	Expected frequencies		
		PL	TPPL	NQPL
0	35	33.1	32.4	33
1	11	15.3	15.8	15.8
2	8	6.8	7.0	7.1
3	4	2.9	2.9	3.0
4	2	1.2	1.9	1.2
Total	60	60	60	60
Parameters		$\hat{\theta}$ =1.7434	$\hat{\theta}$ =1.9997 $\hat{\alpha}$ =0.3829	$\hat{\theta}$ =2.1008 $\hat{\alpha}$ =0.3829
χ^2		2.20	2.11	2.08
d.f		1	1	1
p-value		0.14	0.15	0.15

In Table 2.3, in order to examine the flexibility of NQPL distribution we have considered data by Kemp and Kemp [62] regarding mistakes in copying groups of random digits for which PL and TPPL distributions have been fitted earlier by Sankaran [85]. Based on the data of mistakes in copying groups of random digits the parameters of NQPL distribution estimated by the method of moments have been obtained as $\hat{\theta} = 2.1008$ and $\hat{\alpha} = 0.3829$.

2.8 Conclusion:

In Table 2.1, Table 2.2 and Table 2.3 the values of χ^2 have been estimated from the observed and expected frequencies of NQPL distribution. Then, comparing the values of χ^2 and p-value it has been observed that for all the three data sets the NQPL distribution gives a closer fit than PL, TPPL, $d\gamma$ and NB distributions. Thus we may conclude that the NQPL distribution fits the data well.