Chapter 4

Generalized two-parameter Poisson-Lindley distribution

4.1 Introduction

In this chapter, generalized two-parameter Poisson-Lindley distribution has been studied. The negative-binomial distribution, Poisson-Lindley distribution and geometric distribution have been obtained as a particular case of generalized two-parameter Poisson-Lindley (GTPL) distribution. Various properties of the distribution such as recurrence relations, moments, estimation of parameters etc. have been studied. In section 4.2, the derivation of the generalized two parameter Poisson-Lindley (GTPL) distribution has been discussed. Graphical representation of the pmf of GTPL distribution has been discussed in section 4.3. Some properties of GTPL distribution has been derived in section 4.4, 4.5, 4.6 and 4.7. In section 4.8, size-biased GTPL distribution has been obtained. Estimation method of GTPL distribution has been discussed in section 4.9. Goodness of fit and discussions have been included in section 4.10 and 4.11.

Poisson-Lindley distribution introduced by Sankaran [85] has only one parameter so it does not provide enough flexibility for analyzing different types of life time data. Thus, Shanker and Mishra [93] in order to increase the flexibility obtained a two parameter Poisson-Lindley distribution.

4.2 Derivation of the generalized two-parameter Poisson-Lindley (GTPL) distribution

Definition: Let $X|\lambda$ be a random variable following a Poisson distribution with parameter λ having the probability mass function

$$g(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!};\ x = 0, 1, 2, \dots;\ \lambda > 0$$

then X is said to be generalized two-parameter Poisson Lindley (GTPL) distribution if it follows the representation,

$$X|\lambda \sim \text{Poisson}\ (\lambda)\ .$$

$\lambda|\theta, \alpha \sim$ Two-parameter Lindley (TPL) distribution with parameter (θ, α) given by the density function,

$$f(\lambda;\ \theta, \alpha) = \frac{\theta^2}{(\alpha\theta+1)}(\alpha + \lambda)e^{-\theta\lambda};\ \lambda > 0, \theta > 0, \alpha > 0\ .$$

Preposition: Let X be a random variable following the GTPL $(\theta,\ \alpha)$distribution. Then, the pmf of X is,

$$P(X = x) = \frac{\theta^2}{(\theta+1)^{x+1}(\alpha\theta+1)}\left(\alpha + \frac{x+1}{\theta+1}\right); x = 0,1,2 \dots, \theta > 0, \alpha > 0\ .$$

Proof: Since $X|\lambda \sim$ Poisson (λ) distribution and $\lambda|\theta, \alpha \sim$ TPL $(\theta,\ \alpha)$ distribution

Then, the marginal pmf of X is given as,

$$P(X = x) = \int_0^\infty P(x|\lambda)f(\lambda;\ \theta, \alpha)d\lambda\ ,$$

$$= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\alpha\theta+1}(\alpha + \lambda)e^{-\theta\lambda}d\lambda\ ,$$

$$= \frac{\theta^2}{(\alpha\theta+1)x!}\left(\int_0^\infty e^{-\lambda(1+\theta)}\lambda^x d\lambda + \alpha\int_0^\infty e^{-\lambda(1+\theta)}\lambda^{x+1}d\lambda\right),$$

$$= \frac{\theta^2}{(\theta+1)^{x+1}(\alpha\theta+1)}\left(\alpha + \frac{x+1}{\theta+1}\right);\ x = 0,1,2, \dots, \theta > 0, \alpha > 0\ . \quad (4.2.1)$$

The resultant distribution obtained in equation (4.2.1) is the pmf GTPL distribution obtained by Shanker and Mishra [93].

Particular cases

(i) For $\alpha \to 0$, equation (4.2.1) reduces to negative binomial distribution with parameter $r = 2$ and $p = \frac{\theta}{\theta+1}$.

(ii) For $\alpha = 1$, it reduces to one parameter Poisson- Lindley distribution.

(iii) For $\alpha \to \infty$, it reduces to geometric distribution with parameter $p = \frac{\theta}{\theta+1}$.

Let us denote the density function of GTPL distribution by $f(x;\ \theta, \alpha)$. The first and second derivatives of GTPL distribution may be obtained as,

$$f'(x) = \frac{-\theta^2}{(\alpha\theta+1)}\{log(\theta+1)(\alpha\theta+\alpha+x+1)-1\}\ ,$$

and $$f''(x) = \frac{\theta^2}{(\alpha\theta+1)(\theta+1)^{x+2}}\{log(\theta+1)(\alpha\theta+\alpha+x+1)-2\}\ .$$

To determine the mode of GTPPL distribution we solve the equation,

$$f'(x) = 0\ .$$

i.e. $\frac{-\theta^2}{(\alpha\theta+1)}\{log(\theta+1)(\alpha\theta+\alpha+x+1)-1\} = 0.$

which implies $\hat{x} = \frac{1}{log(\theta+1)} - (\alpha+1+\alpha\theta)$,

and, $f''(\hat{x}) = \frac{-\theta^2(\theta+1)}{(\alpha\theta+1)}\left(\alpha+\alpha\theta-1-\frac{1}{log(\theta+1)}\right) < 0.$

This implies that $\hat{x} = \frac{1}{log(\theta+1)} - (\alpha+\alpha\theta+1)$is the unique critical point for $\hat{x} > 0, \theta, \alpha > 0$ at which GTPPL distribution is maximum.

Thus the mode of GTPPL distribution may be written as

$$\text{Mode} = \frac{1}{log(\theta+1)} - (\alpha+\alpha\theta+1);\ \theta > 0,\ \ \alpha > 0.$$

4.3 Graphical representation of GTPL distribution

For different values of α and θ, the nature and behavior of GTPL distribution are explained graphically in Figure 4.1 and Figure 4.2. In Figure 4.1 graphical representation of GTPPL distribution for $\alpha =$ 0.01, 1.1 and $\theta =$0.3, 0.7, 1.1 has been shown. Figure 4.2 represents graphical representation for $\alpha =$1.1 and $\theta =$0.3, 0.7, 1.1

and graphical representation for α =5.0, 9.0 and θ =0.3, 0.7, 1.1 has been represented in Figure 4.3

It is clear from the graphs of the pmf below that GTPL distribution is monotonically decreasing for increasing values of the parameters α and θ. For increasing α and constant θ, the graph of pmf of generalized two- parameter Poisson-Lindley (GTPL) distribution move upward and then decreases slowly as the value of x increases. For constant α and increasing θ, the graph of pmf decreases fast from higher values with increasing values of x. Thus, we may conclude that θ is the dominating parameter as the parameter θ makes a difference in the shape of the pmf of generalized two-parameter Poisson-Lindley distribution while changing its values. The change in parameter α does not influence much on the shape of the pmf of generalized two-parameter Poisson-Lindley distribution.

Figure 4.1 pmf plot of GTPL distribution for $\alpha = 0.01$ and $\theta = 0.3,0.7, 1.1$

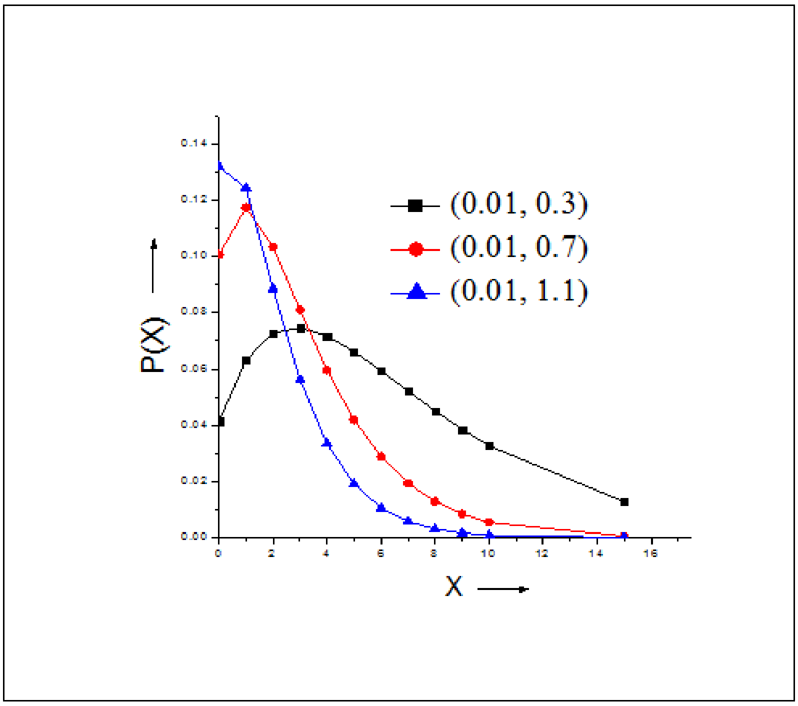


Figure 4.1: pmf plot of GTPL distribution for $\alpha = 1.0$ and $\theta = 0.3, 0.7, 1.1$

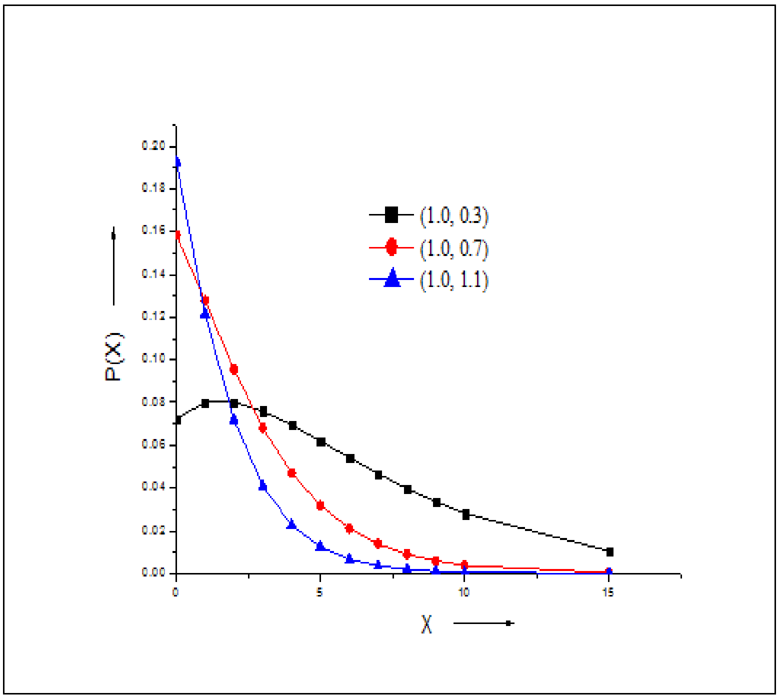
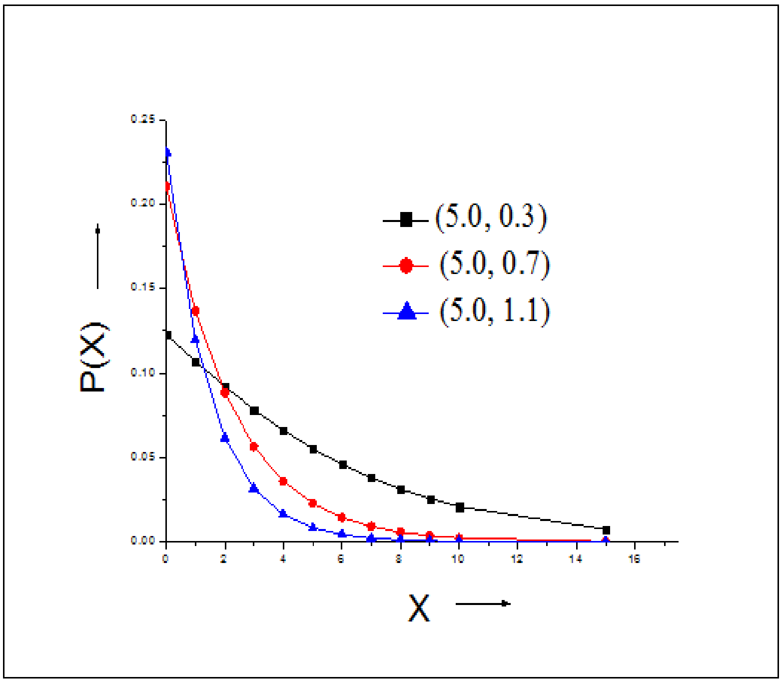


Figure 4.2: pmf plot of GTPL for $\alpha = 5.0$ and $\theta = 0.3, 0.7, 1.1$



4.4 Shape of the Probability Function

Since,

$$P(0) = \frac{\theta^2(\alpha\theta+\alpha+1)}{(1+\theta)^2(1+\alpha\theta)},$$

$$\frac{P(x+1)}{P(x)} = \frac{1}{(1+\theta)}\left\{1 + \frac{1}{\alpha(\theta+1)+x+1}\right\}, \ x = 1,2, \ldots \quad .$$

which is a decreasing function in x, GTPPL distribution is log-concave. Therefore, it has an increasing hazard rate and is unimodal, [Johnson et al. [56]].

4.5 Recursive expressions of GTPL distribution

4.5.1 Probability generating function

The probability generating function (pgf) of GTPL distribution has been obtained as

$$g(t) = E(t^x),$$

$$= \sum_{x=0}^{\infty} t^x \frac{\theta^2}{(\theta+1)^{x+1}(\alpha\theta+1)}\left[\alpha + \frac{x+1}{\theta+1}\right],$$

$$= \frac{\theta^2}{(\alpha\theta+1)(\theta+1-t)^2}\{\alpha(\theta+1-t)+1\}. \tag{4.5.1}$$

Differentiating equation (4.5.1) w.r.t $'t'$ and expanding it we have obtained as,

$$g'(t) = \frac{2g(t)}{(\theta+1-t)} - \frac{\alpha\theta^2}{(\alpha\theta+1)(\theta+1-t)^2}. \tag{4.5.2}$$

From equation (4.5.2) equating the coefficient of t^r, we have obtained the recurrence relation for probabilities as,

$$p_{r+1} = \frac{[2(\theta+2)p_r - p_{(r-1)}]}{(\theta+1)^2}, r > 1 \tag{4.5.3}$$

Where $p_0 = \frac{\theta^2}{(\theta+1)^2(\alpha\theta+1)}(\alpha+\alpha\theta+1)$,

and $p_1 = \frac{\theta^2}{(\theta+1)^3(\alpha\theta+1)}(\alpha+\alpha\theta+2)$,

From equation (4.5.3) higher order probabilities may be obtained for $r = 2,3, \ldots$

4.5.2 Factorial moment generating function

The factorial moment generating function (fmgf) of GTPL distribution may be obtained as,

$$G(t) = g(1+t)\ ,$$

$$G(t) = \frac{\theta^2}{(\alpha\theta+1)(\theta-t)^2}\{\alpha(\theta-t)+1\}. \tag{4.5.4}$$

Differentiating equation (4.5.4) and equating the co-efficient of $'\frac{t^r}{r!}'$ we obtain the recursive expression for factorial moments as,

$$\mu'_{(r+1)} = \frac{\theta(1+2r)\mu'_{(r)} - r^2\mu'_{(r-1)}}{\theta^2}, r > 1\ , \tag{4.5.5}$$

where, $\mu'_{(1)} = \frac{\alpha\theta+2}{\theta(\alpha\theta+1)}$,

Higher factorial moments can be obtained from the recursive expression of equation (4.5.5) for $r = 2, 3$ ... etc.

4.5.3 Moment generating function

The moment generating function (mgf) of GTPL distribution may be obtained as,

$$m(t) = \frac{\theta^2}{(\alpha\theta+1)(\theta+1-e^t)^2}\{\alpha(\theta+1-e^t)+1\}, |t| < 1\ . \tag{4.5.6}$$

The recurrence relation for raw moments from equation (4.5.6) has been obtained as,

$$\mu'_{r+1} = \frac{\left[\alpha B\{(\theta+1)-2^r\}+2B+\sum_{j=1}^{r}\mu'_j\binom{r}{r+1-j}\{3(\theta+1)^2-3(\theta+1)2^{r+1-j}+3^{r+1-j}\}\right]}{\theta^3}, r \geq 1 \tag{4.5.7}$$

where $B = \frac{\theta^2}{(\alpha\theta+1)}$,

and $\mu'_1 = \frac{\alpha\theta+2}{\theta(\alpha\theta+1)}$.

4.5.4 Cumulant generating function

The cumulant generating function (cgf) of GTPL distribution may be obtained from moment generating function of equation (4.5.6) as,

$$K(t) = \log m(t) \,,$$

$$K(t) = \log\left\{\frac{\theta^2\{\alpha(\theta+1-e^t)+1\}}{(\alpha\theta+1)(\theta+1-e^t)^2}\right\} . \qquad (4.5.8)$$

Expanding the equation of (4.5.8) we have,

$$K(t) = \log\{\theta^2(\alpha(\theta+1-e^t)+1)\} - \log(\alpha\theta+1) - 2\log(\theta+1-e^t) . \qquad (4.5.9)$$

Differentiating equation (4.5.9) w.r.t $'t'$ and equating the coefficient of $'\frac{t^r}{r!}'$ the recurrence relation for cumulants has been obtained as

$$k_{r+1} = \frac{\{\alpha(1+\theta)+2-\alpha 2^r\}+\sum_{j=1}^{r} K_j\binom{r}{r-j+1}\{2\alpha\theta-\alpha 2^j+1\}}{\alpha\theta(\theta+1)}, r > 1$$

where, $k_1 = \frac{\alpha\theta+2}{\theta(\alpha\theta+1)}$ (Mean),

$$k_2 = \frac{\alpha^2\theta^3+3\alpha\theta^2+\alpha^2\theta^2+2\theta+4\alpha\theta+2}{\theta^2(\alpha\theta+1)^2} \text{ (Variance).}$$

4.6 Moments of GTPL distribution

The factorial moment may be obtained from equation (4.5.4) as

$$\mu'_{(1)} = \frac{\alpha\theta+2}{\theta(\alpha\theta+1)} ,$$

$$\mu'_{(2)} = \frac{2(\alpha\theta+3)}{\theta^2(\alpha\theta+1)},$$

$$\mu'_{(3)} = \frac{6(\alpha\theta+3)}{\theta^2(\alpha\theta+1)} + \frac{6(\alpha\theta+4)}{\theta^3(\alpha\theta+1)} ,$$

$$\mu'_{(4)} = \frac{14(\alpha\theta+3)}{\theta^2(\alpha\theta+1)} + \frac{36(\alpha\theta+4)}{\theta^3(\alpha\theta+1)} + \frac{24(\alpha\theta+5)}{\theta^4(\alpha\theta+1)} .$$

From the factorial moments the raw moments may be obtained using the relationship between the raw and factorial moment.

The central moments μ_2, μ_3 and μ_4 have been obtained as

$$\mu_2 = \mu_2' - {\mu_1'}^2 \ ,$$

$$\mu_2 = \frac{\alpha^2\theta^3+3\alpha\theta^2+\alpha^2\theta^2+2\theta+4\alpha\theta+2}{\theta^2(\alpha\theta+1)^2},$$

$$\mu_3 = \frac{\alpha^3\theta^5+\theta^4(4\alpha^2+3\alpha^3)+\theta^3(5\alpha+15\alpha^2+2\alpha^3)+\theta^2(2+18\alpha+2\alpha^2)+\theta(6+12\alpha)+4}{\theta^3(\alpha\theta+1)^3},$$

$$\mu_4 = \frac{\begin{array}{c}\alpha^4\theta^7+\theta^6\alpha^3(5+10\alpha)+\theta^5\alpha^2(20\alpha^2+60\alpha+9)+\theta^4\alpha(7+116\alpha+120\alpha^2+9\alpha^4)\\ +\theta^3(2+92\alpha+240\alpha^2+72\alpha^3)+\theta^2(26+180\alpha+132\alpha^2)+\theta(48+168\alpha)+24\end{array}}{\theta^4(\alpha\theta+1)^4}.$$

4.7 Skewness and Kurtosis

Skewness gives an idea about symmetricity of the distribution. It tells whether a distribution is positively skewed or negatively skewed. It may be obtained as,

$$\gamma = \frac{\mu_3}{{\mu_2}^{3/2}} = \frac{\alpha^3\theta^5+\theta^4(4\alpha^2+3\alpha^3)+\theta^3(5\alpha+15\alpha^2+2\alpha^3)+\theta^2(2+18\alpha+2\alpha^2)+\theta(6+12\alpha)+4}{(\alpha^2\theta^3+3\alpha\theta^2+\alpha^2\theta^2+2\theta+4\alpha\theta+2)^{3/2}}.$$

Kurtosis enables us to determine peakedness of a distribution. It may be defined as,

$$\beta_2 = \frac{\mu_4}{{\mu_2}^2} = \frac{\begin{array}{c}\alpha^4\theta^7+\theta^6\alpha^3(5+10\alpha)+\theta^5\alpha^2(20\alpha^2+60\alpha+9)+\theta^4\alpha(7+116\alpha+120\alpha^2+9\alpha^4)\\ +\theta^3(2+92\alpha+240\alpha^2+72\alpha^3)+\theta^2(26+180\alpha+132\alpha^2)+\theta(48+168\alpha)+24\end{array}}{\left((\alpha\theta+2)(\theta+1)(\alpha\theta+1)+\alpha\theta\right)^2}.$$

4.8 Size biased generalized two-parameter Poisson-Lindley (SBGTPL) distribution

As already mentioned in the previous chapter that size biased distribution arises when observations in a sample are recorded with unequal probability proportional to some measure of unit size. This distribution arises from the weighted distribution by replacing weight$w(x) = x$. The pmf of size biased distributions have the functional form

$$P^w(x) = \frac{xP(x)}{\mu},$$

where, $P(x)$: pmf of SBGTPL distribution,

μ: mean of SBGTPL distribution.

Then, the pmf of SBGTPL distribution may be obtained as

$$P^w(x) = \frac{x\theta^3(\alpha(\theta+1)+x+1)}{(\theta+1)^{x+2}(\alpha\theta+2)}, x = 1,2,\ldots;\ \theta > 0, \alpha > 0 \tag{4.8.1}$$

4.8.1 Probability generating function

The probability generating function (pgf) of SBGTPL distribution has been obtained as

$$g(t) = E(t^x)\ ,$$

$$= \sum_{x=0}^{\infty} t^x P^w(x)\ ,$$

where, $P^w(x)$ is the pmf of SBGTPL distribution given in equation (4.8.1).

$$= \sum_{x=0}^{\infty} t^x \left\{\frac{x\theta^3(\alpha(\theta+1)+x+1)}{(\theta+1)^{x+2}(\alpha\theta+2)}\right\},$$

$$= \frac{\theta^3 t(\alpha\theta+\alpha+2-\alpha t)}{(1+\theta-t)^3(\alpha\theta+2)}, |t| < 0. \tag{4.8.2}$$

Note: It has been observed that the probability generating function of SBGTPL distribution is same as size biased Poisson-Lindley (SBPL) distribution when $\alpha = 1$.

Expanding equation (4.8.2) and equating the co-efficient of t^r we have the recurrence relation for probabilities has been obtained as

$$p_{r+1} = \frac{\{(1+\theta)^2(3r+1)p_r-(1+\theta)(3r-1)p_{r-1}+(r-1)p_{r-2}\}}{(1+\theta)^3(r+1)}, r \geq 2 \tag{4.8.3}$$

where, $p_1 = \frac{\theta^3(\alpha(\theta+1)+2)}{(\theta+1)^3(\alpha\theta+2)}$,

$$p_2 = \frac{2\theta^3(\alpha(\theta+1)+3)}{(\theta+1)^4(\alpha\theta+2)},$$

The higher order probabilities may be obtained from equation (4.8.3) for $r = 3, 4,$ ...etc.

4.8.2 Factorial moment generating function

The factorial moment generating function may be obtained as,

$$G(t) = g(1+t)\ ,$$

$$G(t) = \frac{\theta^3(1+t)(\alpha\theta+2-\alpha t)}{(\theta-t)^3(\alpha\theta+2)}. \tag{4.8.4}$$

The recurrence relation for factorial moments may be obtained as

$$\mu'_{(r)} = \frac{\left[3\theta^2 r\mu'_{(r-1)} - 3\theta r(r-1)\mu'_{(r-2)} + 3r(r-1)(r-2)\mu'_{(r-3)}\right]}{\theta^3}, r > 2 \tag{4.8.5}$$

where, $\mu'_{(1)} = \frac{\alpha\theta(\theta+2)+2(\theta+3)}{\theta(\theta\alpha+2)}$,

$$\mu'_{(2)} = \frac{2[\alpha\theta(2\theta+3)+6(\theta+2)]}{\theta^2(\theta\alpha+2)}.$$

From equation (4.8.5) higher order factorial moments may be obtained for $= 3, 4, \ldots$.

4.9 Estimation of parameters of GTPL distribution

The following methods are used to estimate the parameters of GTPL distribution.

4.9.1 Method of Moments

Suppose, considering a sample of size n, say $x_1, x_2, \ldots \ldots \ldots, x_n$ from equation (4.2.1), the moment estimates $\hat{\theta}$ and $\hat{\alpha}$ of θ and α can be obtained by solving the equations

$$m_1 = \frac{\alpha\theta+2}{\theta(\alpha\theta+1)},$$

and $$m_2 = \frac{\theta(\alpha\theta+2)+2(\alpha\theta+3)}{\theta^2(\alpha\theta+1)},$$

where m_1 and m_2 denote the first and second raw moments. Solving the above two equations we have obtained,

$$\hat{\alpha} = \frac{2-m_1\theta}{(m_1\theta-1)\theta}.$$

Substituting the value of α in m_2 we have obtained a quadratic equation of θ in terms of m_1 and m_2.

Thus we have obtained θ as $\hat{\theta} = \frac{2m_1 \pm \sqrt{4m_1{}^2+2m_1-2m_2}}{(m_2-m_1)}$.

We have chosen $\hat{\theta} = \frac{2m_1 + \sqrt{4m_1{}^2+2m_1-2m_2}}{(m_2-m_1)}$ as otherwise θ would be negative.

Theorem 4.9.1 The estimator $\hat{\theta}$ of θ is positively biased, for fixed α, i.e. $E(\hat{\theta}) > \theta$.

Proof: Let $\hat{\theta} = g(\bar{X})$ and $g(t) = \frac{(\alpha-t)+\sqrt{t^2+6\alpha t+\alpha^2}}{2\alpha t}$ for $t > 0$.

Then, $g''(t) = \frac{1}{t^3}\left[1 + \frac{15\alpha t^2+9\alpha^2 t+3t^3+\alpha^3}{(t^2+6\alpha t+\alpha^2)^{3/2}}\right] > 0.$

Therefore, $g(t)$ is strictly convex. Thus, by Jensen's inequality we have,

$$E\{g(X)\} > g\{E(X)\}.$$

Hence, we get $E(\hat{\theta}) > \theta$ since, $g\{E(\bar{X})\} = g\left(\frac{\alpha\theta+2}{\theta(\alpha\theta+1)}\right) = \theta.$

Theorem 4.9.2: The moment estimate $\hat{\theta}$ of θ is consistent and asymptotically normal, for fixed values α, and is distributed as,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \nu^2(\theta)),$$

where, $\nu^2(\theta) = \frac{\theta^2(\alpha\theta+1)^2\{(\theta+1)(\alpha^2\theta^2+3\alpha\theta+2)+\alpha\theta\}}{(4\alpha\theta+\alpha^2\theta^2+2)^2}.$

4.9.2 Maximum Likelihood Estimation

Suppose $x_1, x_2, x_3, \dots\dots, x_n$ are random sample of size n from GTPL distribution and let f_x be the observed frequency in the sample corresponding to $X = x (x = 1, 2, \dots, k)$ such that $\sum_{x=1}^{k} f_x = n$ where k is the largest observed value having non-zero frequency.

Then the likelihood function may be written as,

$$L = \left(\frac{\theta^2}{(\alpha\theta+1)}\right)^n \prod_{x=1}^{k}\left[\frac{1}{(\theta+1)^{x+1}}\left(\alpha + \frac{x+1}{\theta+1}\right)\right]^{f_x}$$

The log likelihood function is

$$\log L = 2n\log\theta - \sum_{x=1}^{k}(x+1)f_x\log(1+\theta) - n\log(\alpha\theta+1) + \sum_{x=1}^{k} f_x \log\left(1 + \frac{x+1}{\theta+1}\right).$$

Differentiating $\log L$ w.r.t θ and α we get,

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{\sum_{x=1}^{k} x f_x}{1+\theta} - \frac{n\alpha}{\alpha\theta+1} - \sum_{x=1}^{k}\frac{(x+1)f_x}{\{1+\theta+(x+1)\}(1+\theta)}$$

$$\frac{\partial \log L}{\partial \alpha} = -\frac{n\theta}{\alpha\theta+1}$$

The second derivatives are,

$$\frac{\partial^2 \log L}{\partial^2 \theta} = \frac{n\alpha^2}{(\alpha\theta+1)^2} - \frac{2n}{\theta^2} + \frac{\sum_{x=1}^{k} x f_x}{(1+\theta)^2} + \sum_{x=1}^{k} \frac{(x+1)(3+\theta+x) f_x}{(1+\theta)^2 (2+\theta+x)^2}$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = \frac{-n(2\alpha+1)}{(\alpha\theta+1)^2}$$

$$\frac{\partial^2 \log L}{\partial^2 \alpha} = \frac{n\theta^2}{(\alpha\theta+1)^2}$$

The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial^2 \theta} & \frac{\partial^2 \log L}{\partial \theta \partial \alpha} \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial^2 \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \theta} \\ \frac{\partial \log L}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$

where θ_0 and α_0 are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently closed values of $\hat{\theta}$ and $\hat{\alpha}$ can be obtained.

4.10 Applications

The estimation of parameters plays an important role in fitting of probability distribution. The method of moment is the simplest procedure for estimating the parameters out of all the procedure. The method of maximum likelihood despite of being considered to be more accurate for fitting a probability distribution has been used very rarely due to its complications in calculation. Therefore, the parameters of GTPL distribution has been estimated by the method of moment. The first two sample moments are used to estimate the parameters.

The GTPL distribution has been fitted to two data sets and found that it provides much closer fit than Poisson-Lindley distribution by Sankaran [85]. The fitted frequencies of Poisson-Lindley distribution and GTPL distribution are shown in Table 4.1 and Table 4.2.

Table 4.1: Observed and expected frequency of PL and GTPL distribution which is regarding mistakes in copying groups of random digits. [data from Kemp and Kemp [62]]

No. of errors per group	Observed frequencies	Expected frequencies	
		PL	GTPL
0	35	33.1	32.4
1	11	15.3	15.9
2	8	6.8	7.0
3	4	2.9	3.0
4	2	1.2	1.2
Total	60	60	60
Parameter estimates		$\hat{\theta}$ =1.743	$\hat{\theta}$ =5.2308 $\hat{\alpha}$ =2.6154
χ^2		2.20	2.5118
d.f		1	1
p-value		0.14	0.11

In Table 4.1, we have considered data set due to Kemp and Kemp [62] which is regarding the distribution of mistakes in copying groups of random digits. The expected frequencies are computed from the recursive expression for probabilities and the χ^2 goodness of fit has been calculated. The GTPL distribution has been compared with PL distribution based on the value of χ^2.

Table 4.2: Observed and expected frequency of PL and GTPL distribution which is regarding the distribution of Pyrausta nublilalis. [data from Beall [7]]

No. of insects	Observed frequencies	Expected frequencies	
		PL	GTPL
0	33	31.5	31.90
1	12	14.2	13.80
2	6	6.1	6.00
3	3	2.5	2.53
4	1	1.0	1.07
5	1	0.7	0.45
Total	56	56	56
Parameter estimate		$\hat{\theta}$ =1.8081	$\hat{\theta}$ = 0.3920 $\hat{\alpha} = 0.2569$
χ^2		0.53	0.3527
d.f		1	1
p-value		0.42	0.6123

The data set in Table 4.2 is due to Beall [7] regarding the distribution of Pyrausta nublilalis in 1937. The parameters are estimated by the method of moment and the χ^2 goodness of fit has been obtained from the observed and expected frequencies.

4.11 Conclusion:

The comparison of observed and expected frequencies of the fitted distribution are given in Table 4.1 and Table 4.2 for comparison of observed and expected frequencies and the χ^2 values is used to test the goodness of fit. It has been observed from the values of χ^2 of GTPL distribution obtained in Table 4.1 and Table 4.2 that GTPL distribution gives a better fit as compared with Poisson-Lindley (PL) distribution. So, we may conclude that the derived distribution fits the data well.