Chapter 5

A comparative study on some zero-truncated distributions

5.1 Introduction

In probability theory, zero-truncated distributions may be defined as discrete distributions which supports the set of positive integers. Zero truncated distributions arise when the data to be modeled originate from a mechanism which generates data that excludes zero counts.

Suppose, $P(x;\,\theta,\alpha)=\frac{P_0(x;\,\theta,\alpha)}{1-P_0(0;\,\theta,\alpha)}$, where $P_0(x;\,\theta,\alpha)$ is the pmf of the original distribution.

In this chapter, we have studied on zero-truncated Poisson-Lindley (ZTPL) distribution, zero-truncated new generalized Poisson-Lindley (ZTNGPL) distribution and zero-truncated generalized two-parameter Poisson-Lindley (ZTGTPL) distribution. Various properties of the distributions have been studied. Lastly, methods of estimation of parameters have been discussed.

5.2 Zero-truncated Poisson-Lindley (ZTPL) distribution

The Poisson-Lindley (PL) distribution having the probability mass function (pmf) given as,

$$P(x;\ \theta,\alpha\ ) = \frac{\theta^2(x+\theta+2)}{(1+\theta)^{x+3}},\ \ x = 0,1,2,\dots,\theta > 0. \tag{5.2.1}$$

has been obtained by Sankaran[85] by compounding Poisson distribution with the Lindley distribution.

The pmf of ZTPL distribution has been obtained by considering its zero-truncated form as,

$$P(X = x) = p(x) = \frac{P_0(x;\theta,\alpha)}{1-P_0(0;\theta,\alpha)}\,,$$

where, $P_0(x;\ \theta,\alpha)$ is the p.m.f of PL distribution in equation (5.2.1)

and $P_0(0;\ \theta,\alpha) = \frac{\theta^2(\theta+2)}{(1+\theta)^3}$ is the pmf of PL distribution at point $x = 0$.

Thus, the pmf of ZTPL distribution has been obtained as

$$P(X = x) = p(x) = \frac{\theta^2(x+\theta+2)}{(1+\theta)^x(\theta^2+3\theta+1)};\ \ \ x = 1,2,3,\dots,\theta > 0\ . \tag{5.2.2}$$

The resultant distribution that is obtained in equation (5.2.2) is the zero truncated version of PL distribution obtained by Ghitany et al. [41].

5.2.1 Recursive expression for probabilities

If X follows ZTPL distribution then the probability generating function of X is

$$g(t) = E(t^x)\ ,$$

$$= \sum_{x=1}^{\infty} t^x\, p(x),$$

where $p(x)$ is the pmf of zero truncated Poisson Lindley distribution as given in equation (5.2.2).

Thus, $g(t) = \sum_{x=1}^{\infty} t^x \frac{\theta^2(x+\theta+2)}{(1+\theta)^x(\theta^2+3\theta+1)}$,

$$= \frac{\theta^2 t}{(\theta^2+3\theta+1)} \left\{ \frac{(\theta+3)(\theta+1) - t(\theta+2)}{(1+\theta-t)^2} \right\}. \qquad (5.2.3)$$

Expanding equation (5.2.3) and equating the co-efficient of t^r we have obtained the recursive expression for probability as

$$p_r = \frac{\{2(1+\theta)p_{r-1} - p_{r-2}\}}{(\theta+1)^2}, \qquad (5.2.4)$$

where, $p_1 = \frac{\theta^2(\theta+3)}{(1+\theta)(\theta^2+3\theta+1)}$,

$$p_2 = \frac{\theta^2(\theta+4)}{(1+\theta)^2(\theta^2+3\theta+1)}.$$

The higher probabilities may be obtained from equation (5.2.4) putting $r = 3,4, \ldots$.

5.2.2 Recursive expression for moments

The moment generating function of ZTPL distribution may be obtained as

$$m(t) = \frac{\theta^2 e^t}{(\theta^2+3\theta+1)} \left\{ \frac{(\theta+3)(\theta+1) - e^t(\theta+2)}{(1+\theta-e^t)^2} \right\},$$

The recurrence relation for moment generating function of ZTPL distribution has been obtained as,

$$\mu'_{r+1} = A\{2^r(3\theta+7) + (\theta+3)(\theta+1)\} + 3\{(1+\theta)^2 - 2^{r-j+1} + 3^{r-j}\} \sum_{j=1}^{r} \binom{r}{r-j+1} \mu'_j, r > 1,$$

where, $A = \frac{(\theta+1)}{\theta(\theta^2+3\theta+1)}$,

$$\mu'_1 = \frac{(\theta+1)^2(\theta+2)}{\theta(\theta^2+3\theta+1)},$$

$$\mu'_2 = \frac{(\theta+1)^2(\theta^2+4\theta+6)}{\theta^2(\theta^2+3\theta+1)}.$$

5.2.3 Recursive expression for factorial moment generating function (fmgf)

The fmgf may be been obtained as

$G(t) = g(1+t)$,

$$= \frac{\theta^2(1+t)}{(\theta^2+3\theta+1)}\left\{\frac{(\theta^2+3\theta+1)-t(\theta+2)}{(\theta-t)^2}\right\}. \qquad (5.2.5)$$

The recursive expression for factorial moment generating function may be written as,

$$\mu'_{(r)} = \frac{1}{\theta^2}\left\{2\theta r\mu'_{(r-1)} - r(r-1)\mu'_{(r-2)}\right\},\ r > 2 \qquad (5.2.6)$$

where, $\mu'_{(1)} = \frac{(\theta+1)^2(\theta+2)}{\theta(\theta^2+3\theta+1)}$,

$$\mu'_{(2)} = \frac{(\theta+1)^2(\theta+3)}{\theta^2(\theta^2+3\theta+1)}.$$

From equation (5.2.6) the higher factorial moments may be obtained.

From the factorial moments using the relationship between the factorial moments and raw moments, the raw moment has been obtained as,

$$\mu'_1 = \frac{(\theta+1)^2(\theta+2)}{\theta(\theta^2+3\theta+1)},$$

$$\mu'_2 = \frac{(\theta+1)^2\{\theta(\theta+2)+2(\theta+3)\}}{\theta^2(\theta^2+3\theta+1)},$$

$$\mu'_3 = \frac{(\theta+1)^2\{\theta(\theta^2+6\theta+6)+2(\theta^2+9\theta+12)\}}{\theta^3(\theta^2+3\theta+1)},$$

$$\mu'_4 = \frac{(\theta+1)^2\{\theta(\theta^3+14\theta^2+36\theta+24)+2(\theta^3+21\theta^2+72\theta+6)\}}{\theta^4(\theta^2+3\theta+1)}.$$

The central moment μ_2 has been obtained as

$$\mu_2 = \frac{(\theta+1)^2(\theta^3+6\theta^2+10\theta+2)}{\theta^2(\theta^2+3\theta+1)^2}.$$

5.2.4 Index of dispersion and co-efficient of variation

The index of dispersion of ZTPL distribution has been denoted by γ and may be defined as,

$$\text{I.D} = \frac{\sigma^2}{\mu} = \frac{\theta^3+6\theta^2+10\theta+2}{\theta(\theta+2)(\theta^2+3\theta+1)}.$$ [Ghitany et al. [40]]

The co-efficient of variation may be defined as,

$$\text{C.V} = \frac{\sigma}{\mu} = \frac{\sqrt{(\theta^3+6\theta^2+10\theta+2)}}{(\theta+1)(\theta+2)}$$

5.3 Zero-truncated new generalized Poisson-Lindley (ZTNGPL) distribution

Shanker and Mishra [93] obtained discrete two-parameter Poisson-Lindley distribution having the pmf as,

$$p_0(x;\ \theta,\alpha) = \frac{\theta^2\{1+\theta+\alpha(x+1)\}}{(\theta+\alpha)(1+\theta)^{x+2}};\ x = 0,1,2\ldots;\ \theta > 0, \alpha > 0\ . \qquad (5.3.1)$$

The distribution has been further studied by Bhati et al. [10] and renamed it as new generalized Poisson-Lindley (NGPL) distribution and investigated certain additional properties.

The pmf of ZTNGPL may be obtained as,

$$P^*(X = x) = p^*(x) = \frac{P_0(x;\ \theta,\alpha)}{1-P_0(0;\ \theta,\alpha)},$$

where, $P_0(x;\ \theta,\alpha)$ is the pmf as given in equation (5.3.1).

$P_0(0;\ \theta,\alpha)$ is the pmf of NGPL distribution at $x = 0$.

Thus the pmf of ZTNGPL distribution has been obtained as

$$P^*(X = x) = p^*(x) = \frac{\theta^2\left(1+\theta+\alpha(x+1)\right)}{(\theta^2+2\theta\alpha+\alpha+\theta)(\theta+1)^x}, x = 1,2,\ldots;\ \theta > 0, \alpha > 0\ .$$

5.3.1 Probability generating function

If X~ZTNGPL distribution then the probability generating function (pgf) of X may be written as,

$$g(t) = E(t^x), |t| < 1$$

$$= \sum_{x=1}^{\infty} t^x p^*(x)\ ,$$

$$= \frac{\theta^2 t\{(\alpha+\theta+1)(1+\theta-t)+\alpha t(1+\theta)\}}{(\theta^2+\alpha+\theta+2\alpha\theta)(1+\theta-t)^2}, \theta > 0, \alpha > 0\ .$$

The recurrence relation for probabilities may be obtained as

$$p_{r+1} = \frac{\{3r(1+\theta)^2 p_r - 3(1+\theta)(r-1)p_{r-1} + (r-2)p_{r-2}\}}{(r+1)(1+\theta)^3}, r > 1$$

where, $p_1 = \frac{\theta^2(1+\theta+\alpha)}{(\theta^2+\alpha+\theta+2\alpha\theta)(1+\theta)}$,

$$p_2 = \frac{\theta^2(1+\theta+2\alpha)}{(\theta^2+\alpha+\theta+2\alpha\theta)(1+\theta)^2}.$$

5.3.2 Recursive expression for moment generating function

The moment generating function (mgf) may be obtained as,

$$m(t) = E(e^{tx}),$$

$$= \frac{\theta^2 e^t\{(\alpha+\theta+1)(1+\theta-e^t)+\alpha e^t(1+\theta)\}}{(\theta^2+\alpha+\theta+2\alpha\theta)(1+\theta-e^t)^2}. \quad (5.3.2)$$

Differentiating equation (5.3.2) w.r.t t and equating the co-efficient of $\frac{t^r}{r!}$ we have

$$\mu'_{r+1} = \frac{A(1+\theta)\left[2^r B+C(1+\theta)+\sum_{j=1}^{\infty}\mu'_j\{3(1+\theta)^2-3(1+\theta)2^{r+1-j}+3^{r+1-j}\}\right]}{\theta^3},$$

where, $A = \frac{\theta^2}{(\theta^2+\alpha+\theta+2\alpha\theta)}$,

$$B = (2\theta\alpha+\alpha+\theta+1),$$

$$C = \alpha+\theta+1.$$

5.3.3 Recursive expression for factorial moment generating function

The factorial moment generating function (fmgf) may be written as,

$$G(t) = g(1+t),$$

$$= \frac{\theta^2(1+t)\{(1+\theta+\alpha)(\theta-t)+\alpha(1+t)(1+\theta)\}}{(\theta^2+\alpha+\theta+\alpha\theta)(\theta-t)^2}. \quad (5.3.3)$$

Expanding equation (5.3.3) and equating the co-efficient of $\frac{t^r}{r!}$ we have obtained the recurrence relation for factorial moment generating function as,

$$\mu'_{(r)} = \frac{r\left[2\theta\mu'_{(r-1)}-(r-1)\mu'_{(r-2)}\right]}{\theta^2}, r > 2 \quad (5.3.4)$$

where, $\mu'_{(1)} = \frac{(\theta+1)^2(\theta+2\alpha)}{\theta(\theta^2+\alpha+\theta+2\alpha\theta)}$,

$$\mu'_{(2)} = \frac{2(\theta+1)^2(\theta+3\alpha)}{\theta^2(\theta^2+\alpha+\theta+2\alpha\theta)} .$$

The higher order probabilities may be obtained from equation (5.4.4) for $r = 3, 4, ...$

From the factorial moments the raw moments have been obtained as

$$\mu'_1 = \frac{(\theta+1)^2(\theta+2\alpha)}{\theta(\theta^2+\alpha+\theta+2\alpha\theta)} ,$$

$$\mu'_2 = \frac{(\theta+1)^2(\theta^2+2\alpha\theta+2\theta+6\alpha)}{\theta^2(\theta^2+\alpha+\theta+\alpha\theta)} ,$$

$$\mu'_3 = \frac{(\theta+1)^2(\theta^3+2\alpha\theta^2+6\theta^2+18\alpha\theta+6\theta24\alpha)}{\theta^3(\theta^2+\alpha+\theta+\alpha\theta)} ,$$

$$\mu'_4 = \frac{(\theta+1)^2(\theta^4+2\alpha\theta^3+14\theta^3+42\alpha\theta^2+36\theta^2+144\alpha\theta+24\theta+120\alpha)}{\theta^3(\theta^2+\alpha+\theta+\alpha\theta)} .$$

5.4 Zero-truncated generalized two-parameter Poisson-Lindley (ZTGTPL) distribution

Two-parameter Poisson-Lindley (TPPL) distribution has been obtained by Shanker et al. [93]. Again in the previous chapter we have revisited (TPPL) distribution and named it as generalized two-parameter Poisson-Lindley (GTPL) distribution having the pmf as,

$$P_0(x;\, \theta, \alpha) = \frac{\theta^2}{(\theta+1)^{x+1}(\alpha\theta+1)}\left(\alpha + \frac{x+1}{\theta+1}\right), x = 0,1,2 \ldots;\ \theta > 0, \alpha > 0 \quad . \quad (5.4.1)$$

Now, the pmf of ZTGTPL distribution may be obtained as,

$$P^{**}(X = x) = p^{**}(x) = \frac{P_0(x;\, \theta,\alpha)}{1-P_0(0;\, \theta,\alpha)} ,$$

where, $P_0(x;\, \theta, \alpha)$ is the pmf of GTPL distribution,

and, $P_0(0;\, \theta,\, \alpha)$ is the pmf of ZTGTPL distribution at point $x = 0$.

Then, the resultant distribution is the ZTGTPL distribution obtained by Shanker and Shukla [91] having the pmf as,

$$P^{**}(X = x) = p^{**}(x) = \frac{\theta^2(x+\alpha(\theta+1)+1)}{(\theta^2\alpha+\theta\alpha+2\theta+1)(\theta+1)^x}, x = 1,2, \ldots;\ \theta > 0, \alpha > 0 \quad (5.4.2)$$

It has been observed that ZTPL distribution obtained by Ghitany et al. [40] is a particular case of ZTGTPL distribution when $\alpha = 1$.

5.4.1 Probability generating function

The probability generating function of ZTGTPPL distribution may be obtained as,

$$g(t) = E(t^x) ,$$

$$= \sum_{x=1}^{\infty} t^x p^{**}(x) ,$$

$$= \frac{\theta^2 t\{(\alpha(\theta+1)+1)(1+\theta-t)+(1+\theta)\}}{(\theta^2\alpha+\theta\alpha+2\theta+1)(1+\theta-t)^2}. \qquad (5.4.3)$$

Differentiating equation (5.4.3) w.r.t $'t'$ and equating the co-efficient of t^r we have obtained the recurrence relation for probabilities as

$$p_{r+1} = \frac{\{3r(1+\theta)^2 p_r - 3(1+\theta)(r-1)p_{r-1} + (r-2)p_{r-2}\}}{(r+1)(1+\theta)^3}, \qquad (5.4.4)$$

where, $p_1 = \frac{\theta^2(2+\alpha(\theta+1))}{(\theta^2\alpha+\theta\alpha+2\theta+1)(1+\theta)}$,

$$p_2 = \frac{\theta^2(3+\alpha(\theta+1))}{(\theta^2\alpha+\theta\alpha+2\theta+1)(1+\theta)^2} .$$

The higher order probabilities of ZTGTPL distribution may be obtained from equation (5.4.4) for $r = 3, 4, \ldots$ etc.

5.4.2 Recursive expression for moment generating function

The moment generating function (mgf) of ZTGPL distribution

$$m(t) = \frac{\theta^2 e^t\{(\alpha(\theta+1)+1)(1+\theta-e^t)+(1+\theta)\}}{(\theta^2\alpha+\theta\alpha+2\theta+1)(1+\theta-e^t)^2} \qquad (5.4.5)$$

Now by differentiating equation (5.4.5) and equating the co-efficient of $\frac{t^r}{r!}$ we obtain the recurrence relation for moment generating function as

$$\mu'_{r+1} = \frac{A\{(1+\theta)(\theta-\alpha(1+\theta)-2^r)-4.3^r(\alpha(1+\theta)+1)\}+\sum_{j=1}^{r}\{3(1+\theta)^2-3(1+\theta)2^{r-j+1}+3^{r-j+1}\}\mu'_j}{\theta^3}, r > 1$$

where, $A = \frac{\theta^3}{(\theta^2\alpha+\theta\alpha+2\theta+1)}$

The first four moments about origin may be obtained from mgf given in equation (5.4.5) as

$$\mu_1' = \frac{(\theta+1)^2(\theta\alpha+2)}{\theta(\theta^2\alpha+\theta\alpha+2\theta+1)},$$

$$\mu_2' = \frac{(\theta+1)^2\{\theta\alpha(\theta+2)+2(\theta+3)\}}{\theta^2(\theta^2\alpha+\theta\alpha+2\theta+1)},$$

$$\mu_3' = \frac{(\theta+1)^2\{\theta\alpha(\theta^2+6\theta+6)+2(\theta^2+9\theta+12)\}}{\theta^3(\theta^2\alpha+\theta\alpha+2\theta+1)},$$

$$\mu_3' = \frac{(\theta+1)^2\{\theta\alpha(\theta^3+14\theta^2+36\theta+24)+2(\theta^3+21\theta^2+72\theta+60)\}}{\theta^4(\theta^2\alpha+\theta\alpha+2\theta+1)}.$$

The variance of ZTGTPL distribution has been obtained as,

$$\mu_2 = \frac{(\theta+1)^2\{\theta^3\alpha^2+\alpha^2\theta^2+5\theta^2\alpha+4\alpha\theta+6\theta+2\}}{\theta^2(\theta^2\alpha+\alpha\theta+2\theta+1)^2}.$$

5.4.3 Recursive expression for factorial moment generating function (fmgf)

The factorial moment generating function may be written as

$$G(t) = g(1+t),$$

$$= \frac{\theta^2(1+t)\{(\alpha(\theta+1)+1)(\theta-t)+(1+\theta)\}}{(\theta^2\alpha+\theta\alpha+2\theta+1)(\theta-t)^2}. \quad (5.4.6)$$

The recursive expression for factorial moment generating function has been obtained as

$$\mu_{(r+1)}' = \frac{r\left[3\theta^2\mu_{(r)}'-3\theta(r-1)\mu_{(r-1)}'+(r-1)(r-2)\mu_{(r-2)}'\right]}{\theta^3}, r > 2$$

where, $\mu_{(1)}' = \frac{(\theta+1)^2(\theta\alpha+2)}{\theta(\theta^2\alpha+\theta\alpha+2\theta+1)}$,

$$\mu_{(2)}' = \frac{(\theta+1)^2(\theta\alpha+2)}{\theta^2(\theta^2\alpha+\theta\alpha+2\theta+1)}.$$

The general expression for factorial moments may be written as,

$$\mu_{(r)}' = \frac{r!(\theta+1)^2(\theta\alpha+r+1)}{\theta^r(\theta^2\alpha+\theta\alpha+2\theta+1)}.$$

5.5 Index of dispersion

The Index of dispersion for ZTGTPL distribution has been obtained as

$$\text{I.D} = \frac{\sigma^2}{\mu} = \frac{\{\theta^3\alpha^2+\alpha^2\theta^2+5\theta^2\alpha+4\alpha\theta+6\theta+2\}}{\theta(\theta^2\alpha+\alpha\theta+2\theta+1)\big((\theta\alpha+2)\big)}.$$

The I.D may be equi-dispersed, over-dispersed or under-dispersed according as $\sigma^2 = \mu$ or $\sigma^2 > \mu$ or $\sigma^2 < \mu$.

5.6 Estimation of parameters

This section is based on the estimation of parameters by the following methods.

5.6.1 Estimation of parameters ZTPL distribution

The parameters of ZTPL distribution may be obtained by the method of maximum likelihood. Supposing $x_1, x_2, \dots, x_n$ to be a sample of size n from ZTPL distribution and f_x be the observed frequency corresponding to $X = x(x = 1,2, \dots k)$ such that $\sum_{x=1}^{k} f_x = n$.

Then the likelihood function of ZTPL distribution may be written as,

$$L = \prod_{x=1}^{k} p(x;\ \theta, \alpha)\ ,$$

$$L = \left(\frac{\theta^{2n}}{(\theta^2+3\theta+1)^n}\right)\frac{1}{(\theta+1)^{\sum_{x=1}^{k} x f_x}}\prod_{x=1}^{k}\{\theta x + \theta + 2\}^n\ .$$

The log-likelihood function is

$$\log L = 2n\log\theta - n\log(\theta^2 + 3\theta + 1) - \sum_{x=1}^{k} x f_x \log(\theta + 1) + \sum_{x=1}^{k} f_x \log(x + \theta + 2)\ .$$

The derivative of log likelihood equations is,

$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{n(2\theta+3)}{(\theta^2+3\theta+1)} - \frac{n\bar{x}}{(\theta+1)} + \sum_{x=1}^{k}\frac{f_x}{(x+\theta+2)} = 0,$ which is a non-linear equation and can be solved by numerical methods.

5.6.2 Estimation of parameters of ZTNGPL distribution

The parameters of ZTNGPL distribution may be obtained the method of maximum likelihood by considering $x_1, x_2, \dots, x_n$ to be a sample of size n from

ZTNGPL distribution and f_x be the observed frequency corresponding to $X = x(x = 1,2, \dots k)$ such that $\sum_{x=1}^{k} f_x = n$.

Then, the likelihood function may be written as

$$L = \prod_{x=1}^{k} p^*(x;\ \theta, \alpha) \ ,$$

$$L = \left(\frac{\theta^{2n}}{(\theta^2+2\theta\alpha+\alpha+\theta))^n}\right) \frac{1}{(\theta+1)^{\sum_{x=1}^{k} x f_x}} \prod_{x=1}^{k} \{\alpha x + \theta + \alpha + 1\}^n \ .$$

The log-likelihood function may be written as,

$$\log L = 2n\log\theta - n\log(\theta^2 + 2\theta\alpha + \theta + \alpha) - \sum_{x=1}^{k} x f_x \log(\theta + 1)$$

$$+ \sum_{x=1}^{k} f_x \log\{\alpha x + \theta + \alpha + 1\} \ .$$

The derivatives of log likelihood equations are,

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{n(2\theta+2\alpha+1)}{(\theta^2+2\theta\alpha+\theta+\alpha)} - \frac{n\bar{x}}{(\theta+1)} + \sum_{x=1}^{k} \frac{f_x}{(\alpha x+\theta+\alpha+1)} = 0 \ ,$$

$$\frac{\partial \log L}{\partial \alpha} = -\frac{n(2\theta+1)}{(\theta^2+2\theta\alpha+\theta+\alpha)} + \sum_{x=1}^{k} \frac{(x+1) f_x}{(\alpha x+\theta+\alpha+1)} = 0 \ ,$$

The second derivatives are

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{2n}{\theta^2} + \frac{n\{2\ (\theta^{\ 2}+2\alpha\theta+\theta+2\alpha^2+\alpha)\ +1\}}{(\theta^2+2\theta\alpha+\theta+\alpha)^2} + \frac{n\bar{x}}{(\theta+1)^2} + \sum_{x=1}^{k} \frac{f_x}{(\alpha x+\theta+\alpha+1)^2} = 0 \ ,$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{n(2\theta+1)^2}{(\theta^2+2\theta\alpha+\theta+\alpha)^2} - \sum_{x=1}^{k} \frac{(x+1)^2 f_x}{(\alpha x+\theta+\alpha+1)^2} = 0 \ ,$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = \frac{n\{2\theta^{\ 2}+2\theta+2\alpha+1\}}{(\theta^2+2\theta\alpha+\theta+\alpha)^2} + \sum_{x=1}^{k} \frac{(x+1) f_x}{(\alpha x+\theta+\alpha+1)^2} = 0 \ .$$

The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial^2 \theta} & \frac{\partial^2 \log L}{\partial \theta \partial \alpha} \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial^2 \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \theta} \\ \frac{\partial \log L}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} ,$$

where θ_0 and α_0 are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently closed values of $\hat{\theta}$ and $\hat{\alpha}$ can be obtained.

5.6.3 Estimation of parameters of ZTGTPL distribution

Suppose $x_1, x_2, \dots, x_n$ be a sample of size n from ZTGTPL distribution and f_x be the observed frequency corresponding to $X = x(x = 1,2, \dots k)$ such that $\sum_{x=1}^{k} f_x = n$.

Then, the likelihood function may be written as

$$L = \prod_{x=1}^{k} p^{**}(x;\ \theta, \alpha) ,$$

$$L = \left(\frac{\theta^{2n}}{(\theta^2\alpha+\theta\alpha+2\theta+1)^n}\right)\frac{1}{(\theta+1)^{\sum_{x=1}^{k} x f_x}}\prod_{x=1}^{k}\{x + 1 + \alpha(\theta + 1)\}^{f_x}.$$

The log likelihood function may be written as

$$\log L = 2n\log\theta - n\log(\theta^2\alpha + \theta\alpha + 2\theta + 1) - \sum_{x=1}^{k} x f_x \log(\theta + 1)$$

$$+ \sum_{x=1}^{k} f_x \log\{x + 1 + \alpha(\theta + 1)\} .$$

The maximum likelihood estimates of θ and α can be solved from the equations,

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{n(2\theta\alpha+\alpha+2)}{(\theta^2\alpha+\theta\alpha+2\theta+1)} - \frac{n\bar{x}}{(\theta+1)} + \sum_{x=1}^{k}\frac{\alpha f_x}{(x+1+\alpha(\theta+1))} = 0 ,$$

$$\frac{\partial \log L}{\partial \alpha} = -\frac{n(\theta^2+\theta)}{(\theta^2\alpha+\theta\alpha+2\theta+1)} + \sum_{x=1}^{k}\frac{(\theta+1)f_x}{(x+1+\alpha(\theta+1))} = 0 .$$

The second derivatives are

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{2n}{\theta^2} - \frac{n\{2\theta^2+2\theta+1\}}{(\theta^2\alpha+\theta\alpha+2\theta+1)^2} + \sum_{x=1}^{k}\frac{(x+1)f_x}{\left(x+1+\alpha(\theta+1)\right)^2} = 0 ,$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{n\left(\theta+\theta^2\right)^2}{(\theta^2\alpha+\theta\alpha+2\theta+1)^2} - \sum_{x=1}^{k}\frac{\theta(\theta+1)^2 f_x}{\left(x+1+\alpha(\theta+1)\right)^2} = 0 ,$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = \frac{n\{2\theta^2+2\theta+1\}}{(\theta^2\alpha+\theta\alpha+2\theta+1)^2} - \sum_{x=1}^{k}\frac{(\theta+1)^2 f_x}{\left(x+1+\alpha(\theta+1)\right)^2} = 0 .$$

The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved by numerical method iteratively till close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial^2 \theta} & \frac{\partial^2 \log L}{\partial \theta \partial \alpha} \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial^2 \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \theta} \\ \frac{\partial \log L}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$

5.7 Goodness of fit

In this section an attempt has been made to test the suitability of ZTPL, ZTNGPL and ZTGTPL distributions. The goodness of fit of the three distributions has been studied for two data sets. The parameters are estimated by the method of maximum likelihood.

Table 5.1 Observed and expected frequencies of Number of flower heads with number of fly eggs. [data by Finney and Varley [36]]

Number of fly eggs	Number of flowers	Expected frequency		
		ZTPL	ZTNGPL	ZTGTPL
1	22	26.8	26.0	25.1
2	18	19.8	19.3	20.3
3	18	13.9	14.1	15.0
4	11	9.5	9.9	10.1
5	9	6.4	7.1	7.0
6	6	4.2	4.0	4.5
7	3	2.7	2.7	3.0
8	0	1.7	1.4	1.6
9	1	1.1	1.0	1.0
Total	88	88	88	88
Parameter estimates		$\hat{\theta} = 0.7185$	$\hat{\theta} = 0.5426$ $\hat{\alpha} = 12.2145$	$\hat{\theta} = 0.0.82$ $\hat{\alpha} = 16.67$
χ^2		5.9901	4.8457	3.9951
d.f		4	3	3
p-value		0.1999	0.1835	0.2620

In table 5.1 the observed and expected frequency of ZTPL, ZTNGPL and ZTGTPL distribution has computed for data sets regarding counts of number of flower having number of fly eggs which is due to Finney and Varley [36]. The expected frequencies have been obtained to calculate the χ^2 values.

Table 5.2 Observed and expected frequencies of number of snowshoe hares counts captured over 7 days. [data by Keith and Meslow [61]]

Number of times hares caught	Observed frequency	Expected frequency		
		ZTPL	ZTNGPL	ZTGTPL
1	184	182.6	183	183.4
2	55	55.3	54.0	54.9
3	14	16.5	16.1	15.2
4	4	5.1	4.8	4.0
5	4	1.6	1.9	2.7
Total	261	261	261	126
Parameter estimates		$\hat{\theta} = 0.7185$	$\hat{\theta} = 2.4080$ $\hat{\alpha} = 17.3677$	$\hat{\theta} = 2.5570$ $\hat{\alpha} = 0.2343$
χ^2		4.7578	2.7517	0.7170
d.f		2	1	1
p-values		0.0927	0.1283	0.2102

In table 5.2 we have considered data set due to Keith and Meslow [61] which is regarding the number of snowshoe hares count captured over 7 days. The expected frequencies and χ^2 values have been obtained.

5.8 Conclusion

In Table 5.1 and Table 5.2 it has been observed that the χ^2 values have been calculated from the observed and observed and expected frequencies. Comparing the values of χ^2 in both the tables we may conclude that of all the three distribution ZTGTPL distribution gives a closer fit than ZTPL and ZTNGPL distribution.