Chapter 6

Poisson-Sushila distribution and its applications

6.1 Introduction

The Poisson-Sushila (PS) distribution which is a special case of Poisson-Lindley distribution has been studied in this chapter. An attempt has been made to investigate certain properties of this distribution. It has been obtained by compounding Poisson distribution with that of Sushila distribution introduced by Shanker et al. [94] having the density function as

$$f(x,\ \theta,\alpha)=\frac{\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x};\ \ x>0,\ \theta,\alpha>0. \tag{6.1.1}$$

The chapter has been organized as follows: In section 6.2 Poisson-Sushila distributions has been proposed. Section 6.3 is based on certain statistical properties of Poisson-Sushila distribution. In section 6.4 zero-truncated version of Poisson-Sushila distribution has been obtained. The estimation of parameters has been studied in section 6.5.

6.2 Proposed Model

The Poisson-Sushila (PS) distribution arises when the Parameter λ of the Poisson distribution having the probability mass function

$$P^{*}(x;\ \theta)=\frac{e^{-\lambda}\lambda^{x}}{x!};x=0,1,2\ldots,\lambda>0$$

follows the Sushila distributions having the probability density function given in equation (6.1.1).

Then, the pmf of Poisson-Sushila distribution may be obtained as,

$$P(X=x)=\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^2}{\alpha(\theta+1)}\left(1+\frac{\lambda}{\alpha}\right)e^{-\frac{\theta}{\alpha}\lambda}d\lambda\ ,$$

$$=\frac{\theta^2}{\alpha(\theta+1)x!}\left[\int_0^\infty e^{-\lambda}\lambda^x\left(1+\frac{\lambda}{\alpha}\right)e^{-\frac{\theta}{\alpha}\lambda}d\lambda\right],$$

$$=\frac{\theta^2}{\alpha(\theta+1)x!}\left[\int_0^\infty e^{-\lambda\left(1+\frac{\theta}{\alpha}\right)}\lambda^x d\lambda+\frac{1}{\alpha}\int_0^\infty e^{-\lambda\left(1+\frac{\theta}{\alpha}\right)}\lambda^{x+1}d\lambda\right],$$

$$=\frac{\theta^2}{\alpha(\theta+1)x!}\left[\frac{\Gamma(x+1)}{\left(1+\frac{\theta}{\alpha}\right)^{x+1}}+\frac{\Gamma(x+2)}{\left(1+\frac{\theta}{\alpha}\right)^{x+2}}\right],$$

$$=\frac{\theta^2\alpha^x}{(\theta+\alpha)^{x+2}}\left(\frac{\theta+\alpha+x+1}{\theta+1}\right),$$

$$=\left(\frac{\theta}{\alpha+\theta}\right)^2\left(\frac{\alpha}{\alpha+\theta}\right)^x\left(1+\frac{\alpha+x}{\theta+1}\right);\ x=0,1,2\ldots,\ \alpha,\theta>0\ . \qquad (6.2.1)$$

Special case: It has been observed that the PL distribution has been obtained as a particular case of PS distribution when $\alpha=1$.

The cumulative distribution function has been obtained as,

$$F_X(x)=P(X\le x)=\sum_{n=0}^{x}P(X=n)\ ,$$

$$=\sum_{n=0}^{x}\left(\frac{\theta}{\theta+\alpha}\right)^2\left(\frac{\alpha}{\theta+\alpha}\right)^n\left(1+\frac{\alpha+n}{\theta+1}\right),$$

$$=1-\frac{\alpha^{x+1}(\theta^2+2\theta+x\theta+\theta\alpha+\alpha)}{(1+\theta)(\alpha+\theta)^{x+2}}\ . \qquad (6.2.2)$$

6.3 Properties of Poisson-Sushila (PS) Distribution

6.3.1 Shape of the probability function

We have,

$$\frac{P(x+1)}{P(x)}=\frac{\alpha}{\alpha+\theta}\left(1+\frac{1}{\alpha+\theta+x+1}\right)\ , \qquad (6.3.1)$$

which is a decreasing function in x. Thus, we may conclude that PS distribution is unimodal. [Johnson et al. [56]]

6.3.2 Probability generating function

The probability generating function (pgf) of PSD has been obtained as

$$g(t) = E(t^x) ,$$

$$= \sum_{x=0}^{\infty} t^x P(x) ,$$

where, $P(x)$ is the pmf of PS distribution.

$$= \sum_{x=0}^{\infty} t^x \left(\frac{\theta}{\alpha+\theta}\right)^2 \left(\frac{\alpha}{\alpha+\theta}\right)^x \left(1 + \frac{\alpha+x}{\theta+1}\right) ,$$

$$= \left(\frac{\theta}{\alpha+\theta}\right)^2 \sum_{x=0}^{\infty} \left(\frac{\alpha t}{\alpha+\theta}\right)^x \left(1 + \frac{\alpha+x}{\theta+1}\right) ,$$

$$= \frac{\theta^2}{\theta+1} \left(\frac{\theta+(\alpha+1-\alpha t)}{(\alpha+\theta-\alpha t)^2}\right) . \tag{6.3.2}$$

It has been observed that the pgf of PS distribution reduces to the pgf of PL distribution for $\alpha = 1$.

Now, from equation (6.3.2) equating the coefficient of t^r on both sides, the probability recurrence relation of PS distribution may be obtained as

$$p_r = \frac{\alpha}{(\alpha+\theta)^2} (2(\alpha + \theta)p_{r-1} - \alpha p_{r-2}) , \; r \geq 2 \tag{6.3.3}$$

where, $p_0 = \left(\frac{\theta}{\alpha+\theta}\right)^2 \left(\frac{\alpha+\theta+1}{\theta+1}\right)$,

$$p_1 = \left(\frac{\theta}{\theta+\alpha}\right)^2 \left(\frac{\alpha}{\alpha+\theta}\right) \left(\frac{\alpha+\theta+2}{\theta+1}\right).$$

The higher probabilities may be obtained from equation (6.3.3).

The moment generating function (mgf) of PS distribution may be obtained as

$$m(t) = \sum_{x=0}^{\infty} e^{tx} P(x) ,$$

$$= \frac{\theta^2}{\theta+1} \left(\frac{\theta+(\alpha+1-\alpha e^t)}{(\alpha+\theta-\alpha e^t)^2}\right). \tag{6.3.4}$$

The recursive expression for mgf may be obtained as,

$$\mu'_{r+1} = \frac{\left[B\{\alpha(\alpha+\theta+2)-2^r\alpha^2\}+\sum_{j=1}^{r}\mu'_j\binom{r}{r+1-j}\{3\alpha(\alpha+\theta)^2-3\alpha^2(\alpha+\theta)+\alpha^3\}\right]}{\theta^3}, r > 1.$$

where, $\mu'_1 = \frac{\alpha(\theta+2)}{\theta(\theta+1)}$.

6.3.5 Recurrence relation for factorial moment generating function

The factorial moment generating function may be obtained from pgf of equation (6.3.2) as

$$g(t) = G(1+t) \,,$$

$$g(t) = \frac{\theta^2}{\theta+1}\left(\frac{\theta+(1-\alpha t)}{(\theta-\alpha t)^2}\right). \qquad (6.3.5)$$

Differentiating equation (6.3.4) we get,

$$g'(t) = \frac{2\alpha g(t)}{(\theta-\alpha t)} - \frac{\theta^3\alpha}{(\theta+1)(\theta-\alpha t)^3}. \qquad (6.3.6)$$

Now expanding equation (6.3.5) and equating the co-efficient of $\frac{t^r}{r!}$ on both sides we have,

$$\mu'_{(r+1)} = \frac{2\alpha\mu'_{(r)}+3\theta\alpha r\left(\theta\mu'_{(r-1)}-\alpha(r-1)\mu'_{(r-2)}\right)+\alpha^3 r(r-1)(r-2)\mu'_{(r-3)}}{\theta^3}, r \geq 3. \qquad (6.3.7)$$

where, $\mu'_{(1)} = \frac{\alpha(\theta+2)}{\theta(\theta+1)}$,

$$\mu'_{(2)} = \frac{2\alpha^2(\theta+2)}{\theta^2(\theta+1)},$$

$$\mu'_{(3)} = \frac{3\alpha^2(\theta+2)}{\theta^3(\theta+1)}.$$

The higher factorial moments may be obtained from the recursive expression (6.3.7) by using the first three factorial moments.

6.3.5 Moments of PS distribution

The raw moments may be obtained from

$$\mu_r' = \int_0^\infty \left(\sum_{x=0}^\infty \frac{x^r e^{-\lambda}\lambda^x}{x!}\right)\frac{\theta^2}{\alpha(\theta+1)}\left(1+\frac{\lambda}{\alpha}\right)e^{-\frac{\theta}{\alpha}\lambda}d\lambda. \tag{6.3.8}$$

The expression within the bracket is the r^{th} moment of the Poisson distribution. On integrating equation (6.3.8) and putting $r = 1, 2, 3$ and 4 respectively with the respective raw moments of Poisson distribution, we have obtained the first four raw moments as,

$$\mu_1' = \frac{\alpha(\theta+2)}{\theta(\theta+1)},$$

$$\mu_2' = \frac{\alpha\theta(\theta+2)+2\alpha^2(\theta+3)}{\theta^2(\theta+1)},$$

$$\mu_3' = \frac{\alpha\theta^2(\theta+2)+6\alpha^2\theta(\theta+3)+6\alpha^3(\theta+4)}{\theta^3(\theta+1)},$$

$$\mu_4' = \frac{\alpha\theta^3(\theta+2)+14\alpha^2\theta^2(\theta+3)+36\alpha^3\theta(\theta+4)+24\alpha^4(\theta+5)}{\theta^4(\theta+1)}.$$

The central moments are

$$\mu_2 = \frac{\alpha(\theta^3+3\theta^2+\alpha\theta^2+4\alpha\theta+2\theta+2\alpha)}{\theta^2(\theta+1)^2},$$

$$\mu_3 = \frac{\theta^5\alpha+\theta^4\alpha(3\alpha+4)+\theta^3\alpha(2\alpha^2+15\alpha+5)+\theta^2\alpha(24\alpha^2+6\alpha+2)+6\theta\alpha^2(2\alpha+1)+4\alpha^2}{\theta^3(\theta+1)^3},$$

$$\mu_4 = \frac{\begin{array}{c}\theta^7\alpha+\theta^6(5\alpha+10\alpha^2)+\theta^5(9\alpha+60\alpha^2+18\alpha^3)+\theta^4(7\alpha+116\alpha^2+126\alpha^3+9\alpha^4)\\+\theta^3(2\alpha+92\alpha^2+240\alpha^3+72\alpha^4)+\theta^2(26\alpha^2+180\alpha^3+132\alpha^4)+\theta(48\alpha^3+96\alpha^4)+24\alpha^4\end{array}}{\theta^4(\theta+1)^4}.$$

6.3.6 Index of Dispersion and Coefficient of Variation

The index of dispersion denoted by $'\gamma'$ may be written as,

$$\text{I.D} = \frac{\sigma^2}{\mu} = 1+\frac{\alpha\theta^2+4\alpha\theta+2\alpha}{\theta^3+3\theta^2+2\theta} > 1,\ (\mu \neq 0).$$

The index of dispersion gives an idea about whether the distribution is over-dispersed (I.D > 1), equi-dispersed (I.D =1) or under-dispersed (I.D < 1).

Here, since I.D >1, therefore we may conclude that the PS distribution is over-dispersed.

and the coefficient of variation is denoted as,

$$\text{C.V} = \frac{\sigma}{\mu} = \frac{\sqrt{\alpha(\theta^3+3\theta^2+\alpha\theta^2+4\alpha\theta+2\theta+2\alpha)}}{\alpha(\theta+2)}.$$

6.3.7 Coefficient of Skewness and Kurtosis

For the Poisson Sushila distribution these indices are given by,

$$\sqrt{\beta_1} = \frac{\theta^5\alpha+\theta^4\alpha(3\alpha+4)+\theta^3\alpha(2\alpha^2+15\alpha+5)+\theta^2\alpha(24\alpha^2+6\alpha+2)+6\theta\alpha^2(2\alpha+1)+4\alpha^2}{\left(\alpha(\theta^3+3\theta^2+\alpha\theta^2+4\alpha\theta+2\theta+2\alpha)\right)^{\frac{3}{2}}},$$

$$\beta_2 = \frac{\begin{array}{c}\theta^7\alpha+\theta^6(5\alpha+10\alpha^2)+\theta^5(9\alpha+60\alpha^2+18\alpha^3)+\theta^4(7\alpha+116\alpha^2+126\alpha^3+9\alpha^4)\\ +\theta^3(2\alpha+92\alpha^2+240\alpha^3+72\alpha^4)+\theta^2(26\alpha^2+180\alpha^3+132\alpha^4)+\theta(48\alpha^3+96\alpha^4)+24\alpha^4\end{array}}{\left(\alpha(\theta^3+3\theta^2+\alpha\theta^2+4\alpha\theta+2\theta+2\alpha)\right)^2}.$$

6.4 Zero-truncated Poisson-Sushila distribution

The zero-truncated distribution as has already been studied in the previous chapter arises when the data to be modeled arises from a mechanism that structurally excludes zero count. It is a certain class of discrete distribution whose support is the set of positive integers.

If $P_0(x;\ \theta)$ is the pmf of the original distribution then the zero truncated version of $P_0(x;\ \theta)$ may be defined as,

$$P^*(x;\ \theta) = \frac{P_0(x;\theta)}{1-P_0(0;\theta)}.$$

Then, the pmf of zero-truncated Poisson-Sushila (PS) distribution may be obtained as,

$$P^*(x;\ \alpha,\theta) = \frac{P_0(x;\alpha,\theta)}{1-P_0(0;\alpha,\theta)},$$

where, $P_0(x;\ \alpha,\ \theta)$ is the p.m.f of PS distribution,

$P_0(0;\ \alpha,\ \theta)$ is the p.m.f of PS distribution at $x = 0$.

Thus, we have,

$$P^*(x;\ \alpha,\theta) = \frac{\left(\frac{\theta}{\alpha+\theta}\right)^2\left(\frac{\alpha}{\alpha+\theta}\right)^x\left(1+\frac{\alpha+x}{\theta+1}\right)}{1-\left(\frac{\theta}{\alpha+\theta}\right)^2\left(1+\frac{\alpha}{\theta+1}\right)},$$

$$= \left(\frac{\alpha}{\alpha+\theta}\right)^x \frac{\theta^2(\alpha+\theta+x+1)}{\alpha^2+\alpha\theta(\alpha+\theta+2)}; x = 1,2,\ldots,\theta > 0, \alpha > 0. \qquad (6.4.1)$$

6.4.1 Recurrence relation for probability generating function

The probability generating function (pgf) of zero-truncated PS distribution may be obtained as,

$$g(t) = E(t^x), |t| < 1,$$

$$= \sum_{x=1}^{\infty} t^x P^*(x;\ \alpha,\theta)\ ,$$

where, $P^*(x;\ \alpha,\theta)$: pmf of zero-truncated PS distribution.

$$= \sum_{x=1}^{\infty} t^x \left(\frac{\alpha}{\alpha+\theta}\right)^x \frac{\theta^2(\alpha+\theta+x+1)}{\alpha^2+\alpha\theta(\alpha+\theta+2)},$$

$$= \sum_{x=1}^{\infty} \left(\frac{\alpha t}{\alpha+\theta}\right)^x \frac{\theta^2(\alpha+\theta+x+1)}{\alpha^2+\alpha\theta(\alpha+\theta+2)},$$

$$= \frac{\theta^2}{\alpha^2+\alpha\theta(\alpha+\theta+2)}\left\{(\alpha+\theta+1)\sum_{x=1}^{\infty}\left(\frac{\alpha t}{\alpha+\theta}\right)^x + \sum_{x=1}^{\infty} x\left(\frac{\alpha t}{\alpha+\theta}\right)^x\right\},$$

$$= \frac{\theta^2}{\alpha^2+\alpha\theta(\alpha+\theta+2)}\left\{(\alpha+\theta+1)\left(\frac{\alpha t}{\alpha+\theta}\right)\left(1-\frac{\alpha t}{\alpha+\theta}\right)^{-1} + \left(\frac{\alpha t}{\alpha+\theta}\right)\left(1-\frac{\alpha t}{\alpha+\theta}\right)^{-2}\right\}$$

$$= \frac{\theta^2\alpha t}{\{\alpha^2+\alpha\theta(\alpha+\theta+2)\}}\left\{\frac{(\alpha+\theta)(\alpha+\theta+2-\alpha t)-\alpha t}{(\alpha+\theta-\alpha t)^2}\right\}. \qquad (6.4.2)$$

Expanding equation (6.4.2) and equating the co-efficient of t^r we have obtained the recursive expression for probability generating function as

$$p_r = \frac{2\alpha(\alpha+\theta)P_{r-1}-\alpha^2 P_{r-2}}{(\alpha+\theta)^2}, r > 1 \qquad (6.4.3)$$

where,

$$p_1 = \left(\frac{\alpha}{\alpha+\theta}\right)\frac{\theta^2(\alpha+\theta+2)}{\alpha^2+\alpha\theta(\alpha+\theta+2)},$$

Higher order probabilities may be obtained from equation (6.4.3) by taking $r = 2, 3$ ....

6.4.2 Factorial moment generating function (fmgf)

The factorial moment generating function (fmgf) may be obtained as,

$$G(t) = g(1+t) ,$$

$$= \frac{\theta^2\alpha(1+t)}{\{\alpha^2+\alpha\theta(\alpha+\theta+2)\}}\left\{\frac{(\alpha+\theta)(\theta+2-\alpha t)-\alpha(1+t)}{(\theta-\alpha t)^2}\right\} . \qquad (6.4.4)$$

Expanding equation (6.4.4) and equating the co-efficient of $\frac{t^r}{r!}$ we have obtained the recursive expression for factorial moment generating function as

$$\mu'_{(r+1)} = \frac{\left[3\theta^2\alpha r\mu'_{(r)}-3\alpha^2 r(r-1)\mu'_{(r-1)}+\alpha^3 r(r-1)(r-2)\mu'_{(r-2)}\right]}{\theta^3}, r > 2$$

6.5 Estimation of Parameters

6.5.1 Method of Moment

Let $x_1,\ x_2,\ \ldots\ldots,\ x_n$ are sample of size n from equation (6.2.1). Then, using the first two moments, the moment estimates $\hat{\alpha}$ and $\hat{\theta}$ of α and θ can be obtained as

$$m_1 = \mu'_1 = \frac{\alpha(\theta+2)}{\theta(\theta+1)}, \qquad (6.5.1)$$

and, $$m_2 = \mu'_2 = \frac{\alpha\theta(\theta+2)+2\alpha^2(\theta+3)}{\theta^2(\theta+1)}. \qquad (6.5.2)$$

From equation (6.5.1) we have obtained

$$\alpha = \frac{m_1\theta(\theta+1)}{\theta+2}.$$

Substituting the value of α in m_2 i.e equation (6.5.2) a quadratic equation in terms of θ has been obtained.

Thus, $$\hat{\theta} = \frac{2(m_1-m_2+2m_1{}^2)\pm\sqrt{4(m_2-m_1-2m_1{}^2)^2+(m_2-m_1-2m_1{}^2)(4m_1-4m_2+3m_1{}^2)}}{(m_2-m_1-2m_1{}^2)} \qquad (6.5.3)$$

We have chosen,

$$\hat{\theta} = \frac{2(m_1 - m_2 + 2m_1{}^2) - \sqrt{4(m_2 - m_1 - 2m_1{}^2)^2 + (m_2 - m_1 - 2m_1{}^2)(4m_1 - 4m_2 + 3m_1{}^2)}}{(m_2 - m_1 - 2m_1{}^2)}.$$

Theorem 6.1 The estimator $\hat{\theta}$ of θ is positively biased for fixed α i.e $E(\hat{\theta}) > \theta$.

Proof: Let us assume,

$$\hat{\theta} = g(\bar{x}), \text{ where } g(t) = \frac{\alpha - t + \sqrt{(t-\alpha)^2 + 8t\alpha}}{2t}, t > 0.$$

Then, $g''(t) = \frac{\alpha}{t^3}\left(1 + \frac{15\alpha t^2 + 9\alpha^2 t + \alpha^3 + 3t^3}{(t-\alpha)^3 + 8\alpha t}\right) > 0.$

Therefore we may conclude that $g(t)$ is strictly convex.

It has found that $g\{E(\bar{X})\} = g\left(\frac{\alpha(\theta+2)}{\theta(\theta+1)}\right) = \theta.$

Thus, by Jensen's inequality, $E\{g(\bar{X})\} > g\{E(\bar{X})\}$.

Hence we have $E(\hat{\theta}) > \theta.$

Theorem 6.2 For fixed value of α, asymptotically normal and distributed as

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \vartheta^2(\theta)),$$

where, $$\vartheta^2(\theta) = \frac{\theta^2(\theta+1)^2(\theta^3 + 3\theta^2 + \alpha\theta^2 + 4\alpha\theta + 2\theta + 2\alpha)}{\alpha(\theta^2 + 4\theta + 2)^2}.$$

Proof: *Consistency:* Since $\sigma^2 < \infty$, $\bar{X} \xrightarrow{P} \mu$

Also, $g(t) = \frac{\alpha - t + \sqrt{(t-\alpha)^2 + 8t\alpha}}{2t}$ is a continuous function at $t = \mu$.

So, we have, $g(\bar{X}) \xrightarrow{P} g(\mu)$ i.e. $\hat{\theta} \xrightarrow{P} \theta$.

Asymptotic normality: Since $\sigma^2 < \infty$, then by central limit theorem, we have,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \sigma^2).$$

Also, since $g(\mu)$ is differentiable and $g'(\mu) \neq 0$,by the delta method, we have,

$$\sqrt{n}(g(\bar{X}) - g(\mu)) \xrightarrow{d} N(0, \vartheta^2(\theta)),$$

where, $\vartheta^2(\theta) = \left[g'\left(\frac{\alpha(\theta+2)}{\theta(\theta+1)}\right)\right]^2 \sigma^2$,

$$= \frac{\theta^2(\theta+1)^2(\theta^3+3\theta^2+\alpha\theta^2+4\alpha\theta+2\theta+2\alpha)}{\alpha(\theta^2+4\theta+2)^2}.$$

(ii) Method of Maximum Likelihood

Suppose $x_1, x_2, \ldots\ldots, x_n$ are sample of size n from Poisson-Sushila distribution.

The Likelihood function

$$L = \prod_{i=1}^{n} f(x_i, \theta, \alpha) ,$$

$$= \prod_{i=1}^{n} \left(\frac{\theta}{\alpha+\theta}\right)^2 \left(\frac{\alpha}{\alpha+\theta}\right)^{x_i} \left(1 + \frac{\alpha+x_i}{\theta+1}\right), \qquad (6.5.4)$$

$$\log L = 2n \log\left(\frac{\theta}{\theta+\alpha}\right) - n\log(\theta+1) + \sum_{i=1}^{n} x_i \log\left(\frac{\alpha}{\theta+\alpha}\right) + \sum_{i=1}^{n} \log(\theta+\alpha+x_i+1).$$

Now,

$$\frac{\partial \log L}{\partial \theta} = \frac{2n\alpha}{\theta(\theta+\alpha)} - \frac{n}{(\theta+1)} - \frac{\theta+\alpha}{\alpha}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{x_i}{\theta+\alpha+x_i+1} = 0 , \qquad (6.5.5)$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{-2n}{\theta+\alpha} + \frac{\theta+\alpha}{\alpha}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{\theta+\alpha+x_i+1} = 0 , \qquad (6.5.6)$$

$$\frac{\partial^2 \log L}{\partial \theta^2} = \frac{-2n\alpha(2\theta+\alpha)}{\theta^2(\theta+\alpha)^2} + \frac{n}{(\theta+1)^2} - \frac{1}{\alpha}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{1}{(\theta+\alpha+x_i+1)^2},$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = \frac{2n}{(\theta+\alpha)^2} + \frac{1}{\alpha}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{1}{(\theta+\alpha+x_i+1)^2} ,$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{2n}{(\theta+\alpha)^2} - \frac{\theta}{\alpha^2}\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{1}{(\theta+\alpha+x_i+1)^2} .$$

The above equations cannot be solved directly. However, Fisher's scoring method can be applied to solve these equations as,

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \theta} & \frac{\partial^2 \log L}{\partial \theta^2} \end{bmatrix} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \alpha} \\ \frac{\partial \log L}{\partial \theta} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} .$$

The equations can be solved for $\hat{\theta}$ and $\hat{\alpha}$ iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained, where $\theta_0$and α_0 are the initials value of θ and α respectively.

6.6 Goodness of fit

To illustrate the applicability of Poisson-Sushila distribution we have considered a number of data sets which were earlier fitted by different authors for different distributions. So, we have tried to fit PS distribution to a number of data-sets and test for its goodness of fit. The fittings of PS distribution to three sets of data have been presented in the following tables.

Table 6.1: Comparison of observed and expected frequencies of mistakes in copying groups of random digits. [data by Kemp and Kemp [62]]

No. of errors per page	Observed frequency	Expected frequency	
		PJ	PS
0	35	32.44	32
1	11	15.85	15.1
2	8	6.96	6.73
3	4	2.88	2.89
4	2	1.86	2.07
Total	60	60	60
Parameters		$\hat{\theta}$ =5.2308 $\hat{\alpha}$ =2.6154	$\hat{\theta}$ =1.8403 $\hat{\alpha}$ =1.1450
	χ^2	2.10	1.8403
	d.f	1	1
	p-value	0.1473	0.1749

The first data set in table 6.1 is due to Kemp and Kemp [62] which is regarding the distribution of mistakes in copying groups of random digits. The expected frequency has been calculated and the χ^2 value has been obtained by comparing the observed and expected frequency.

Table 6.2: Observed and expected frequency of Pyrausta nublilalis in 1937. [data by Beall [7]]

No. of accidents	Observed frequencies	Expected frequencies	
		PJ	PS
0	33	31.90	33.99
1	12	13.80	13.72
2	6	5.92	5.88
3	3	2.53	2.52
4	1	1.07	1.08
5	1	0.78	0.81
Total	56	56	56
Parameters		$\hat{\theta} = 0.3920$ $\hat{\alpha} = 2.2569$	$\hat{\theta} = 26.9754$ $\hat{\alpha} = 19.5333$
χ^2		0.33	0.2964
d.f		2	2
p-value		0.8479	0.8623

In table 6.2 we considered data set due to Beall [7] regarding the distribution of Pyrausta nublilalis in 1937. The observed and expected frequency has been presented in the table and the χ^2 value has been estimated based on the observed and expected frequency.

Table 6.3: Distribution of the number of Haemocytometer yeast cell counts per square observed by students in 1907

No. of yeast cells per square	Observed frequency	Expected frequencies	
		PJ	PS
0	128	125.32	127.62
1	37	43.83	40.89
2	18	13.03	12.82
3	3	3.57	3.95
4	1	0.93	1.20
5	0	0.32	1.51
Total	187	187	187
Parameters		$\hat{\theta} = 29.4097$ $\hat{\alpha} = 8.0208$	$\hat{\theta} = 3.4954$ $\hat{\alpha} = 1.3150$
χ^2		2.0863	1.0418
d.f		1	1
p-value		0.1486	0.3074

The data set in table 6.3 is the students' historic data on haemocytometer count of yeast cells used by Borah [13] for fitting Gegenbauer distribution. The expected frequency for PJ distribution and PS distribution has been calculated and obtained the χ^2 value for both the distributions.

6.7 Conclusion:

The expected frequencies according to the Poisson Janardan (PJ) distribution and PS distribution have been given along with its chi- square values in these tables for ready comparison. The estimates of the parameters have been obtained by the method of moments.

Now, from the Table 6.1, Table 6.2 and Table 6.3 comparing the values of χ^2 and p-value it can be clearly seen that to almost all the data-sets, the PS distribution provides closer fits than those by the PJ distribution.