Chapter 7

Some properties of Poisson Size-biased new quasi Lindley distribution

7.1 Introduction

Size biased distributions as has been discussed in previous chapter arises when observations with unequal probabilities are recorded having probability proportional to some measure of unit size. Size-biased distributions are known to be a special case of the weighted distributions. Fisher [37] first introduce these distributions to model ascertainment biased and was later formalized by Rao [84] in a unifying theory.

In this chapter, Poisson Size-biased new quasi Lindley distribution has been introduced by compounding Poisson distribution with size biased new quasi Lindley distribution. It has been observed that Poisson size biased Lindley distribution introduced by Adhikari and Srivastava [2] is a particular case of Poisson size biased new quasi Lindley distribution. We have studied some statistical properties like probability generating function, recurrence relations, index of dispersion and estimation of parameters.

7.2 Derivation of Poisson size-biased new quasi-Lindley distribution

The Poisson size-biased new quasi-Lindley (PSBNQL) distribution may be obtained by compounding the Poisson distribution with the size biased new quasi-Lindley distribution having the probability density function (pdf),

$$f(x;\ \theta,\alpha) = \frac{x\theta^3(\theta+\alpha x)e^{-\theta x}}{\theta^2+2\alpha};\ x > 0, \alpha > 0, \theta > 0. \tag{7.2.1}$$

Then, the probability mass function (pmf) of PSBNQL distribution may be obtained as

$$P(x;\ \theta,\alpha) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\lambda\theta^3(\theta+\alpha\lambda)e^{-\theta\lambda}}{(\theta^2+2\alpha)}d\lambda\ ,$$

$$= \frac{\theta^3}{(\theta^2+2\alpha)x!}\int_0^\infty e^{-(1+\theta)\lambda}\lambda^{x+1}(\theta+\alpha\lambda)d\lambda\ ,$$

$$= \frac{\theta^3}{(\theta^2+2\alpha)x!}\left(\theta\int_0^\infty \lambda^{x+1}e^{-\lambda(1+\theta)}d\lambda + \alpha\int_0^\infty \lambda^{x+2}e^{-\lambda(1+\theta)}d\lambda\right),$$

$$= \frac{\theta^3}{(\theta^2+2\alpha)x!}\left(\theta\frac{\Gamma(x+2)}{(1+\theta)^{x+2}} + \alpha\frac{\Gamma(x+3)}{(1+\theta)^{x+3}}\right),$$

$$= \frac{\theta^3}{(\theta^2+2\alpha)x!}\left(\frac{\theta(x+1)x!}{(1+\theta)^{x+2}} + \frac{\alpha(x+2)(x+1)x!}{(1+\theta)^{x+3}}\right),$$

$$= \frac{\theta^3(x+1)}{(\theta^2+2\alpha)(1+\theta)^{x+2}}\left(\theta + \frac{\alpha(x+2)}{(1+\theta)}\right), x = 1,2,3\ldots, \alpha > 0, \theta > 0 \tag{7.2.2}$$

Particular Case: Poisson size biased Lindley distribution is a particular case of Poisson size biased new quasi Lindley distribution at $\alpha = \theta$.

7.3 Graphical representation of PSBNQL distribution

To study the behaviour of PSBNQL distribution, the probabilities for possible values of x are computed for different values of parameter α and θ. In figure 7.1 and 7.2, it is clear for fixed α i.e. at $\alpha = 0.03$ and $\alpha = 1.0$ respectively and as θ varies. As θ increases the probability curves shift upward and decreases monotonically becoming unimodal.

In Figure 7.3 it can been seen that for lager value of α i.e $\alpha = 7.0$ and smaller values of θ i.e $\theta = 0.3,\ 0.5,\ 0.9$ the probability curve is positively skewed and shift upward as θ increases reducing its flatness.

In Figure 7.4, for larger values of α and θ i.e for $\alpha = 15$ and $\theta = 5.0, 7.0, 9.0$ the probability curve is decreasing monotonically and tends to zero after a certain point.

Figure 7.1 pmf plot of PSBNQL distribution for $\alpha = 0.03$ and $\theta = 0.3, 3.0, 7.0$.

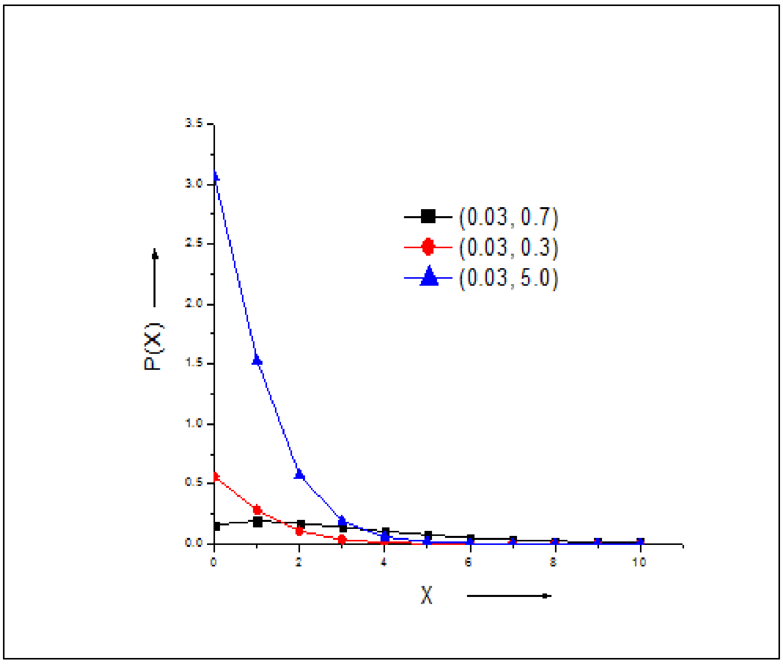


Figure 7.2 pmf plot of PSBNQL distribution for $\alpha = 1.0$ and $\theta = 0.7, 3.0, 7.0$.

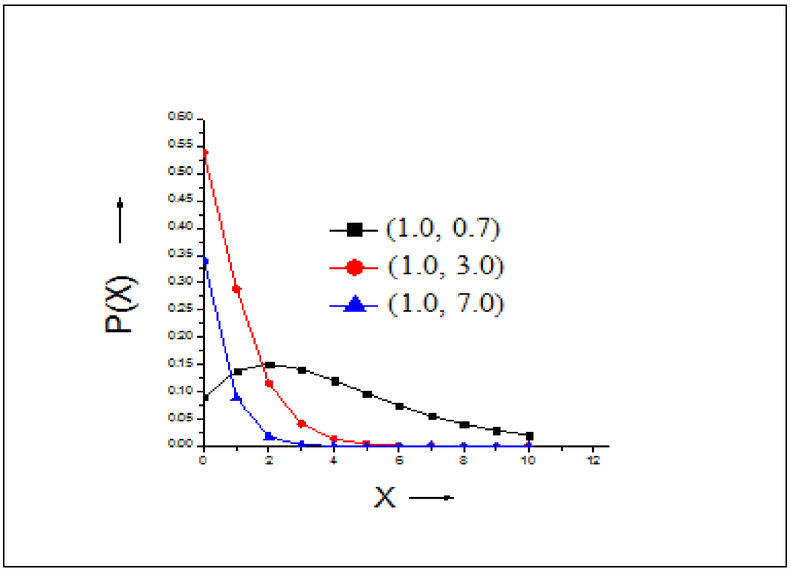


Figure 7.3 pmf plot of PSBNQL distribution for $\alpha = 7.0$ and $\theta = 0.3, 3.0, 7.0$.

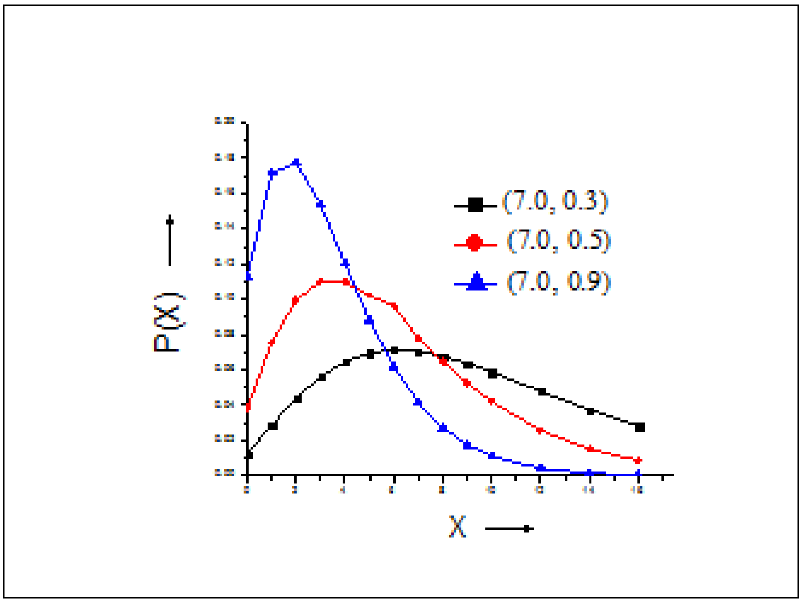
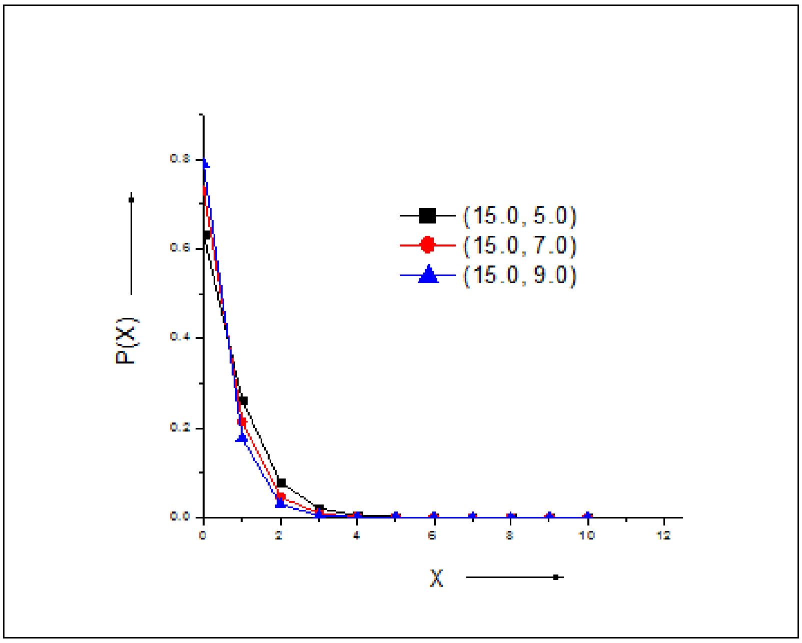


Figure 7.4 pmf plot of PSBNQL distribution for $\alpha = 15$ and $\theta = 5.0,\ 7.0,\ 9.0$



7.4 Distributional properties of PSBNQPL distribution

7.4.1 Shape of the probability function

We have,

$$\frac{P(x+1;\theta,\alpha)}{P(x;\theta,\alpha)} = \frac{1}{(1+\theta)}\left(1 + \frac{\alpha}{\theta(1+\theta)+\alpha(x+2)}\right),$$

which is a decreasing function in $'x'$. Therefore, PSBNQL distribution is unimodal and has an increasing failure rate. [Johnson et al. [56]]

7.4.2 Factorial moments

Let the random variable X~Poisson distribution with parameter λ and λ~SBNQL distribution given in equation (7.2.1). Then, the r^{th} factorial moment of PSBNQL distribution may be obtained as

$$\mu'_{(r)} = E\left[E\left(X^{(r)}|\lambda\right)\right],$$

where, $X^{(r)} = X(X-1)(X-2)\ldots(X-r+1)$,

$$= \int_0^\infty \left[\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}\right] \frac{\lambda\theta^3(\theta+\alpha\lambda)e^{-\theta\lambda}}{(\theta^2+2\alpha)} d\lambda,$$

$$= \int_0^\infty \left[\lambda^r \sum_{x=r}^{\infty} \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right] \frac{\lambda\theta^3(\theta+\alpha\lambda)e^{-\theta\lambda}}{(\theta^2+2\alpha)} d\lambda.$$

Substituting $'x+r'$ by $'x'$ we have,

$$= \int_0^\infty \lambda^r \left[\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}\right] \frac{\lambda\theta^3(\theta+\alpha\lambda)e^{-\theta\lambda}}{(\theta^2+2\alpha)} d\lambda.$$

The expression within the bracket in the above expression is unity as it is the summation of pmf of Poisson distribution.

Thus, we obtain

$$\mu'_{(r)} = \frac{\theta^3}{(\theta^2+2\alpha)} \int_0^\infty \lambda^{r+1}(\theta+\alpha\lambda)e^{-\theta\lambda} d\lambda,$$

$$= \frac{\Gamma(r+2)}{(\theta^2+2\alpha)\theta^r}(\theta^2+2\alpha+\alpha r). \quad (7.4.1)$$

Now for $r = 1, 2, 3$ and 4 in equation (7.4.1) the first four factorial moments may be obtained as

$$\mu'_{(1)} = \frac{2(\theta^2+3\alpha)}{\theta(\theta^2+2\alpha)},$$

$$\mu'_{(2)} = \frac{6(\theta^2+4\alpha)}{\theta^2(\theta^2+2\alpha)},$$

$$\mu'_{(3)} = \frac{24(\theta^2+5\alpha)}{\theta^3(\theta^2+2\alpha)},$$

$$\mu'_{(4)} = \frac{24(\theta^2+6\alpha)}{\theta^4(\theta^2+2\alpha)}.$$

7.4.3 Raw and Central moments

If the random variable (r.v) $X\sim$Poisson distribution with parameter λ and $\lambda\sim$SBNQL distribution given in equation (7.2.1) then the r^{th} moment of PSBNQL distribution may be written as

$$\mu'_r = \int_0^\infty \left\{\sum_{x=0}^\infty x^r \frac{e^{-\lambda}\lambda^x}{x!}\right\} \frac{\lambda\theta^3(\theta+\alpha\lambda)e^{-\theta\lambda}}{\theta^2+2\alpha} d\lambda ,$$

$$= \frac{\theta^3}{(\theta^2+2\alpha)} \left[\int_0^\infty \left(\sum_{x=0}^\infty x^r \frac{e^{-\lambda}\lambda^x}{x!}\right) \lambda(\theta+\alpha\lambda)e^{-\theta\lambda} d\lambda\right]. \qquad (7.4.2)$$

The expression within the bracket is the r^{th} moment about origin of Poisson distribution. For $r = 1$, in equation (7.4.2) and the expression within the bracket as mean of Poisson distribution we get the first moment about origin as,

$$\mu'_1 = \frac{\theta^3}{(\theta^2+2\alpha)} \left[\int_0^\infty \lambda^2(\theta+\alpha\lambda)e^{-\theta\lambda} d\lambda\right],$$

$$= \frac{2(\theta^2+3\alpha)}{\theta(\theta^2+2\alpha)}.$$

Taking $r = 2$, in equation (7.4.2) and the expression within the bracket as the second moment about origin of Poisson distribution we get,

$$\mu'_2 = \frac{\theta^3}{(\theta^2+2\alpha)} \left[\int_0^\infty (\lambda^2+\lambda)\lambda(\theta+\alpha\lambda)e^{-\theta\lambda} d\lambda\right],$$

$$= \frac{2\theta(\theta^2+3\alpha)+6(\theta^2+4\alpha)}{\theta^2(\theta^2+2\alpha)}.$$

For $r = 3$ and 4 in equation (7.4.2) and using the respective moment about origin of Poisson distribution we get,

$$\mu_3' = \frac{2\theta^2(\theta^2+3\alpha)+18\theta(\theta^2+4\alpha)+24(\theta^2+5\alpha)}{\theta^3(\theta^2+2\alpha)} \ ,$$

$$\mu_4' = \frac{2\theta^3(\theta^2+3\alpha)+42\theta^2(\theta^2+4\alpha)+24\theta(\theta^2+5\alpha)+120(\theta^2+6\alpha)}{\theta^4(\theta^2+2\alpha)} \ .$$

The central moments about mean μ_2, μ_3 has been obtained as,

$$\mu_2 = \mu_2' - {\mu_1'}^2.$$

Thus, $\mu_2 = \frac{2\theta^5+2\theta^4+12\alpha^2+12\alpha^2\theta+10\theta^3\alpha+12\alpha^2\theta}{\theta^2(\theta^2+2\alpha)^2}$.

$$\mu_3 = \frac{2\theta^8+6\theta^7+\theta^6(14\alpha+14)+48\theta^5+\theta^4\alpha(40\alpha+36)+36\theta^3\alpha^2+24\theta^2\alpha^2(\alpha+3)+72\theta\alpha^3+48}{\theta^3(\theta^2+2\alpha)^3} \ .$$

7.4.4 Index of dispersion, co-efficient of variation and Skewness

The index of dispersion may be defined as the ratio of variance to mean and may be defined as,

$$\text{I.D} = \frac{\sigma^2}{\mu} \ ,$$

$$\text{I.D} = \frac{2\theta^5+2\theta^4+12\alpha^2+12\alpha^2\theta+10\theta^3\alpha+12\alpha^2\theta}{2\theta(\theta^2+3\alpha)(\theta^2+2\alpha)},$$

$$\text{I.D} = 1 + \frac{2\theta^4+12\theta^2\alpha+12\alpha^2}{2\theta(\theta^2+3\alpha)(\theta^2+2\alpha)} > 1.$$

Thus, it has been observed that I.D>1 which means that PSBNQL distribution is over dispersed.

The coefficient of variation may be defined as,

$$\text{C.V} = \frac{\sigma}{\mu} = \frac{\sqrt{2\theta^5+2\theta^4+12\alpha^2+12\theta^2\alpha+4\theta^3\alpha+6\alpha\theta^3+12\alpha^2\theta}}{2(\theta^2+3\alpha)} \ .$$

The skewness of PSBNQL distribution denoted by $\sqrt{\beta_1}$ may be written as,

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} \ ,$$

$$\sqrt{\beta_1} = \frac{2\theta^8+6\theta^7+\theta^6(14\alpha+14)+48\theta^5+\theta^4\alpha(40\alpha+36)+36\theta^3\alpha^2+24\theta^2\alpha^2(\alpha+3)+72\theta\alpha^3+48}{(2\theta^5+2\theta^4+12\alpha^2+12\alpha^2\theta+10\theta^3\alpha+12\alpha^2\theta)^{3/2}}.$$

7.4.5 Probability generating function

The probability generating function (pgf) of PSBNQL distribution has been obtained as

$$g(t) = E(t^x),$$

$$= \sum_{x=0}^{\infty} t^x P(x),$$

$$= \sum_{x=0}^{\infty} t^x \frac{\theta^3(x+1)}{(\theta^2+2\alpha)(1+\theta)^{x+2}}\left(\theta + \frac{\alpha(x+2)}{(1+\theta)}\right),$$

$$= \frac{\theta^3\{(\theta(1+\theta)+2\alpha)-\theta t\}}{(\theta^2+2\alpha)(1+\theta-t)^3}, |t| < 1.$$

The probability recurrence relation has been obtained as

$$p_r = \frac{3(1+\theta)^2 P_{r-1} - 3(1+\theta)P_{r-2} + P_{r-3}}{(1+\theta)^3}, r \geq 3 \qquad (7.4.3)$$

and, $$p_0 = \frac{\theta^3}{(\theta^2+2\alpha)(1+\theta)^2}\left(\theta + \frac{2\alpha}{(1+\theta)}\right),$$

$$p_1 = \frac{2\theta^3}{(\theta^2+2\alpha)(1+\theta)^3}\left(\theta + \frac{3\alpha}{(1+\theta)}\right),$$

$$p_2 = \frac{3\theta^3}{(\theta^2+2\alpha)(1+\theta)^4}\left(\theta + \frac{4\alpha}{(1+\theta)}\right),$$

$$p_3 = \frac{4\theta^3}{(\theta^2+2\alpha)(1+\theta)^5}\left(\theta + \frac{5\alpha}{(1+\theta)}\right).$$

The higher order probabilities may be obtained from equation (7.4.3).

It has been observed that the pgf of PSBNQL distribution is same as the pgf of PSBL distribution of Adhikari and Srivastava [2] when $\alpha = \theta$.

The moment generating function may be obtained as,

$$m(t) = \frac{\theta^3\{(\theta(1+\theta)+2\alpha)-\theta e^t\}}{(\theta^2+2\alpha)(1+\theta-e^t)^3}.$$

7.4.6 Factorial moment generating function

The factorial moment generating function (fmgf) may be obtained as,

$G(t) = g(1+t)$,

$$G(t) = \frac{\theta^3\{(\theta^2+2\alpha)-\theta t\}}{(\theta^2+2\alpha)(\theta-t)^3} . \qquad (7.4.4)$$

Differentiating equation (7.4.4) w.r.t $'t'$ we have,

$$G'(t) = \frac{\theta^3}{(\theta^2+2\alpha)}\left\{\frac{3(\theta^2+2\alpha-\theta t)-\theta(\theta-t)}{(\theta^2+2\alpha)(\theta-t)^4}\right\} \qquad (7.4.5)$$

Now, expanding equation (7.4.5) and equating the co-efficient of $\frac{t^r}{r!}$ We have obtained the recurrence relation for factorial moment generating function as

$$\mu'_{(r+1)} = \frac{r\left[4\mu'_{(r)}-6(r-1)\mu'_{(r-1)}+(r-1)(r-2)(r-3)\mu'_{(r-2)}-(r-1)(r-2)(r-3)(r-4)\mu'_{(r-3)}\right]}{\theta^4}, r > 3$$

7.5 Methods of estimation of parameters

The estimation of parameters is considered as an important property of a distribution. In order to estimate the parameters of PSBNQL distribution we have discussed the method of moment and method of maximum likelihood.

7.5.1 Method of moment

The first two raw moments are used the estimate the parameters. From the first two moments we have,

$$\mu'_1 = \frac{2(\theta^2+3\alpha)}{\theta(\theta^2+2\alpha)} ,$$

$$\mu'_2 = \frac{2\theta(\theta^2+3\alpha)+6(\theta^2+4\alpha)}{\theta^2(\theta^2+2\alpha)} .$$

Now, let $\frac{\mu'_2-\mu_1}{{\mu'_1}^2} = \frac{6(\theta^2+4\alpha)(\theta^2+2\alpha)}{4(\theta^2+3\alpha)^2} = k(\text{say})$. (7.5.1)

Substituting $\alpha = b\theta^2$ in equation (7.5.1) we have,

$$\frac{\mu'_2-\mu_1}{{\mu'_1}^2} = \frac{3(1+4b)(1+2b)}{2(1+3b)^2} = k(\text{say}) ,$$

we have obtained a quadratic equation in b as,

$$(24-18k)b^2+(18-12k)b+(3-2k)=0\ . \tag{7.5.2}$$

Now, if we replace the first two population moments by the respective sample moments in (7.5.1) an estimate of k may be obtained. Then by substituting the estimate of k in (7.5.2), the estimate $\hat{b}$ can be obtained from the quadratic equation.

Also we have

$$\mu_1' = \bar{x} = \frac{2(\theta^2+3\alpha)}{\theta(\theta^2+\alpha)}, \tag{7.5.3}$$

Substituting $\alpha = b\theta^2$ in the expression (7.5.3) we have

$$\bar{x} = \frac{2(1+3b)}{\theta(1+b)}.$$

Hence $$\hat{\theta} = \left(\frac{2(1+3b)}{1+b}\right)\frac{1}{\bar{X}},$$

and $$\hat{\alpha} = b\hat{\theta}^2 = \frac{4\hat{b}\left(1+3\hat{b}\right)^2}{\left(1+\hat{b}\right)^2(\bar{X})^2}.$$

7.5.2 Method of maximum likelihood

Let, $x_1,\ x_2,\ \ldots\ldots,\ x_n$ are sample of size n from PSBNQL distribution. Then, the Likelihood function may be obtained as

$$L = \prod_{i=1}^{n} f(x_i,\theta,\alpha)\ ,$$

$$= \prod_{i=1}^{n} \frac{\theta^3(x_i+1)}{(\theta^2+2\alpha)(1+\theta)^{x_i+2}}\left(\theta+\frac{\alpha(x_i+2)}{(1+\theta)}\right)\ . \tag{7.5.4}$$

The log-likelihood function may be obtained as,

$$\log L = 3n\log\theta - n\log(\theta^2+2\alpha) + \sum_{i=1}^{n}(x_i+1) - (x_i+3)\log(1+\theta)$$

$$+\sum_{i=1}^{n}\log(\theta(1+\theta)+\alpha(x_i+2). \tag{7.5.5}$$

Now, Differentiating equation (7.5.5) w.r.t θ and α we have obtained

$$\frac{\partial\log L}{\partial\theta} = \frac{3n}{\theta} - \frac{2n\theta}{(\theta^2+2\alpha)} - \frac{(x_i+3)}{1+\theta} + \sum_{i=1}^{n}\frac{(2\theta+1)}{\{\theta(\theta+1)+\alpha(x_i+2)\}} = 0\ ,$$

$$\frac{\partial^2 \log L}{\partial\theta^2} = \frac{-3n}{\theta^2} - \frac{n(4\alpha-2\theta^2)}{(\theta^2+2\alpha)^2} + \frac{(x_i+3)}{(1+\theta)^2} + \sum_{i=1}^{n} \frac{(2\alpha(x_i+2)-2\theta^2-2\theta-1)}{\{\theta(\theta+1)+\alpha(x_i+2)\}^2},$$

$$\frac{\partial^2 \log L}{\partial\alpha^2} = \frac{-4n}{(\theta^2+2\alpha)^2} - \sum_{i=1}^{n} \frac{(x_i+2)^2}{(\theta(\theta+1)+\alpha(x_i+2))^2},$$

$$\frac{\partial^2 \log L}{\partial\theta\partial\alpha} = \frac{-4n}{(\theta^2+2\alpha)^2} - (2\theta+1)\sum_{i=1}^{n} \frac{(x_i+2)}{(\theta(\theta+1)+\alpha(x_i+2))^2}.$$

The above equations cannot be solved directly and so Fisher's scoring method can be applied to solve these equations. The equations can be solved for $\hat{\theta}$ and $\hat{\alpha}$ iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained, where $\theta_0$and α_0 are the initials value of θ and α respectively

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial\alpha^2} & \frac{\partial^2 \log L}{\partial\alpha\partial\theta} \\ \frac{\partial^2 \log L}{\partial\alpha\partial\theta} & \frac{\partial^2 \log L}{\partial\theta^2} \end{bmatrix} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial\alpha} \\ \frac{\partial \log L}{\partial\theta} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$