Chapter 4

Minimal reducing subspaces of operator pseudo shifts of type I

4.1 Introduction

If S is a scalar weighted unilateral shift on $\ell^2_+(\mathbb{C})$, then S is irreducible, as already mentioned earlier. For $N > 1$, if we consider weighted unilateral shifts $S_0, S_1, \ldots, S_{N-1}$ on $\ell^2_+(\mathbb{C})$, then $S_0 \oplus S_1 \oplus \cdots \oplus S_{N-1}$ on the direct sum of N copies of $\ell^2_+(\mathbb{C})$ is unitarily equivalent to M_z^N on $H^2(\beta)$, as discussed in Section 2.3. Therefore, by [50] we get a complete description of the reducing subspaces of $S_0 \oplus S_1 \oplus \cdots \oplus S_{N-1}$. Again, instead of a finite N, if we consider a countable direct sum of weighted unilateral shifts i.e, $S_0 \oplus S_1 \oplus \ldots$, then this operator is unitarily equivalent to the operator weighted shift S on $\ell^2_+(K)$ with weights $\{A_n\}_{n\in\mathbb{N}_0}$, where each A_n is invertible diagonal on K. Now, from [20], we get a description of the reducing subspaces of $S_0 \oplus S_1 \oplus \ldots$.

Thus, we have a fairly good idea of the reducing and minimal reducing subspace of a direct sum of scalar weighted unilateral shifts on $\ell^2_+(\mathbb{C})$. However, we do not know much about the reducing subspaces for a direct sum of operator weighted unilateral shifts on $\ell^2_+(K)$.

Question: "If S_1 and S_2 are operator weighted shifts on $H^2(K)$, what are the reducing subspaces for $S_1 \oplus S_2$?"

To address this question, we first propose the definition of an operator weighted pseudo shift on $\ell^2_+(K)$. The motivation for the definition comes from that of a scalar weighted pseudo shift as given first in [12].

Definition 4.1.1. [12] Let X and Y be topological sequence spaces over I and J respectively. Then a continuous linear operator $T : X \to Y$ is called a weighted pseudo shift if there is a sequence $(b_j)_{j\in J}$ of non-zero scalars and an injective mapping $\varphi : J \to I$ such that

$$T(x_i)_{i\in I} = (b_j x_{\varphi(j)})_{j\in J}$$

for $(x_i) \in X$. T is denoted as $T_{b,\varphi}$ and $(b_j)_{j\in J}$ is called the weight sequence.

Thus, taking $I = J = \mathbb{N}_0$ and $\varphi(j) = j + 1$ for all $j \in \mathbb{N}_0$ in Definition 4.1.1, we get

$$T(x_0, x_1, \dots) = (b_0x_1, b_1x_2, \dots),$$

which is the backward unilateral weighted shift on $\ell^2_+(\mathbb{C})$ with weights $\{b_n\}_{n\in\mathbb{N}_0}$. Similarly, every backward bilateral weighted shift is a weighted pseudo shift. These operators are further studied in [35, 52, 53, 54].

Motivated by Definition 4.1.1, we propose the definition of an operator weighted pseudo shift on $\ell^2_+(K)$ as follows:

Definition 4.1.2. Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be an injective map, and $\{A_n\}_{n\in\mathbb{N}_0}$ be a sequence of bounded linear operators on K such that there exists $m, M > 0$ with $m \le \|A_n\| \le M$ for all $n \in \mathbb{N}_0$. Then the operator $T : \ell^2_+(K) \to \ell^2_+(K)$ defined by

$$T(f_0, f_1, \dots) = (A_0f_{\varphi(0)}, A_1f_{\varphi(1)}, \dots)$$

is called the operator pseudo shift induced by φ, usually denoted by T_φ.

We consider each A_n to be positive invertible and to have a diagonal matrix representation with respect to basis $\{e_i\}_{i\in\mathbb{N}_0}$ of K. Hence, for each $n \in \mathbb{N}_0$ there exists a sequence of positive scalars $\{\alpha_i^{(n)}\}_{i\in\mathbb{N}_0}$ such that $A_n e_i = \alpha_i^{(n)} e_i$.

For $i, j \in \mathbb{N}_0$, let $g_{i,j} := (0, \dots, e_i, 0, \dots)$ with e_i occuring at the jth place. Then $\{g_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthonormal basis for $\ell^2_+(K)$. Thus, if $R(\varphi)$ denotes the range of the injective map φ on $\mathbb{N}_0$, then for each $i, j \in \mathbb{N}_0$,

$$T_\varphi g_{i,j} := \begin{cases} \alpha_i^{(\varphi^{-1}(j))} g_{i,\varphi^{-1}(j)}, & \text{if } j \in R(\varphi); \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.1}$$

In particular,

$$T_\varphi g_{i,\varphi(k)} = \alpha_i^{(k)} g_{i,k} \text{ for all } i, k \in \mathbb{N}_0. \tag{4.1.2}$$

In Theorem 4.3.14 of this chapter, we show that T_φ can be identified with a direct sum of copies of unilateral (backward) operator weighted shifts, circulant operators and bilateral operator weighted shifts.

4.2 Preliminaries

Lemma 4.2.1. *Let T_φ be an operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. Then for $k > 0$ and $i, j \in \mathbb{N}_0$, we have*

$$T_\varphi^k g_{i,j} := \begin{cases} \alpha_i^{(\varphi^{-1}(j))} \alpha_i^{(\varphi^{-2}(j))} \dots \alpha_i^{(\varphi^{-k}(j))} g_{i,\varphi^{-k}(j)}, & \text{if } j \in R(\varphi^k); \\ 0, & \text{otherwise.} \end{cases}$$

The proof follows from repeated applications of Equation 4.1.1.

Lemma 4.2.2. *Let T_φ be an operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. For $i, j \in \mathbb{N}_0$, $T_\varphi^* g_{i,j} = \alpha_i^{(j)} g_{i,\varphi(j)}$.*

Proof. For $f = (f_i) \in \ell^2_+(K)$, we have

$$\begin{aligned}\langle T_\varphi f, g_{i,j}\rangle &= \langle A_j f_{\varphi(j)}, e_i\rangle \\ &= \langle f_{\varphi(j)}, A_j^* e_i\rangle \\ &= \alpha_i^{(j)}\langle f_{\varphi(j)}, e_i\rangle \\ &= \alpha_i^{(j)}\langle f, g_{i,\varphi(j)}\rangle \\ &= \langle f, \alpha_i^{(j)} g_{i,\varphi(j)}\rangle.\end{aligned}$$

Therefore, $T_\varphi^* g_{i,j} = \alpha_i^{(j)} g_{i,\varphi(j)}$. □

Lemma 4.2.3. *Let T_φ be an operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. For $k > 0$ and $i, j \in \mathbb{N}_0$, we have the following:*

(i) $T_\varphi^k (T_\varphi^*)^k g_{i,j} = (\alpha_i^{(j)}\alpha_i^{(\varphi(j))}\dots\alpha_i^{(\varphi^{k-1}(j))})^2 g_{i,j}$, *and*

(ii)$(T_\varphi^*)^k T_\varphi^k g_{i,j} := \begin{cases} (\alpha_i^{(\varphi^{-1}(j))}\alpha_i^{(\varphi^{-2}(j))}\dots\alpha_i^{(\varphi^{-k}(j))})^2 g_{i,j}, & \text{if } j \in R(\varphi^k); \\ 0, & \text{otherwise.}\end{cases}$

The result follows immediately from Lemma 4.2.1 and Lemma 4.2.2.

Lemma 4.2.4. *Let T_φ be an operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. For $f = (f_i) \in \ell^2_+(K)$, the adjoint of T is defined as $T_\varphi^* f = (y_0, y_1, \dots)$ where*

$$y_j := \begin{cases} A^*_{\varphi^{-1}(j)} f_{\varphi^{-1}(j)}, & \text{if } j \in R(\varphi); \\ 0, & \text{otherwise.}\end{cases}$$

Proof. Let $f = (f_i) \in \ell^2_+(K)$ and for each $j \in \mathbb{N}_0$, define

$$y_j := \begin{cases} A^*_{\varphi^{-1}(j)} f_{\varphi^{-1}(j)}, & \text{if } j \in R(\varphi); \\ 0, & \text{otherwise.}\end{cases}$$

Then $y = (y_j) \in \ell^2_+(K)$ and for any $h = (h_i) \in \ell^2_+(K)$, we have

$$\begin{aligned}\langle T_\varphi h, f\rangle &= \sum_{i\in\mathbb{N}_0} \langle A_i h_{\varphi(i)}, f_i\rangle \\ &= \sum_{i\in\mathbb{N}_0} \langle h_{\varphi(i)}, A_i^* f_i\rangle \\ &= \sum_{j\in R(\varphi)} \langle h_j, A^*_{\varphi^{-1}(j)} f_{\varphi^{-1}(j)}\rangle \\ &= \langle h, y\rangle.\end{aligned}$$

Thus, $T^*_\varphi f = y$. □

Lemma 4.2.5. *Let T_φ be an operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. Then T^*_φ is a pseudo shift if and only if φ is bijective.*

Proof. Let φ be bijective. Then $R(\varphi) = \mathbb{N}_0$ and so by Lemma 4.2.4, it follows that for each $f = (f_i) \in \ell^2_+(K)$, the adjoint of T_φ is given as

$$T^*_\varphi f = (A^*_{\varphi^{-1}(0)} f_{\varphi^{-1}(0)}, A^*_{\varphi^{-1}(1)} f_{\varphi^{-1}(1)}, \dots).$$

Let $B_n := A^*_{\varphi^{-1}(n)}$ for all $n \in \mathbb{N}_0$ and $\psi := \varphi^{-1}$. Then T^*_φ is the pseudo shift on $\ell^2_+(K)$ with operator weights $\{B_n\}_{n\in\mathbb{N}_0}$, induced by the injective map ψ.

Conversely, let, if possible, φ is not bijective. Then there exists $j \in \mathbb{N}_0$ which is not in the range of φ. So, by Lemma 4.2.4, $y_j = 0$ where $T^*_\varphi f = (y_0, y_1, \dots)$, and so by definition T^*_φ cannot be a pseudo shift. □

For example, if T is the unilateral backward shift induced by $\varphi(n) = n + 1$ for all $n \in \mathbb{N}_0$, then T^*, which is the unilateral forward shift is not a pseudo shift.

However, the bilateral shift and its adjoint are both pseudo shifts.

4.3 φ induced partition of $\mathbb{N}_0$.

Definition 4.3.1. For an injective map φ on $\mathbb{N}_0$, we define the following subsets of $\mathbb{N}_0$:

(i) $\mathcal{M}_1 = \{n \in \mathbb{N}_0 : n \notin R(\varphi)\}$.

(ii) $\mathcal{M}_2 = \{n \in \mathbb{N}_0 : n = \varphi^k(m) \text{ for some } m \in \mathcal{M}_1, k > 0\}$.

(iii) $\mathcal{M}_3 = \{n \in \mathbb{N}_0 : n = \varphi^k(n) \text{ for some } k > 0\}$.

(iv) $\mathcal{M}_4 = \mathbb{N}_0 - (\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3)$.

Here, $R(\varphi)$ denotes the range of φ.

Remark 4.3.2. (a) For an injective map φ on $\mathbb{N}_0$, $\mathcal{M}_i \cap \mathcal{M}_j = \phi$ if $i \neq j$ and $1 \leq i, j \leq 4$.

(b) If φ is bijective, then $\mathcal{M}_1 = \mathcal{M}_2 = \phi$ and $\mathbb{N}_0 = \mathcal{M}_3 \cup \mathcal{M}_4$.

(c) If φ is injective but not surjective, then $R(\varphi) = \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4 \subsetneq \mathbb{N}_0$.

Example 4.3.3. *Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be defined as follows:*

$\varphi(0) = 1,\ \varphi(1) = 2,\ \varphi(2) = 0,$

$\varphi(2n+1) = 2n+3$ *for all* $n \geq 1,$

$\varphi(4) = 6,\ \varphi(4n) = 4(n-1)$ *for all* $n \geq 2,$

and $\varphi(4n+2) = 4(n+1)+2$ *for all* $n \geq 1.$

Then $R(\varphi) = \mathbb{N}_0 - \{3\}$ *and* φ *is injective. Here,* $\mathcal{M}_1 = \{3\}$, $\mathcal{M}_2 = \{2n+1 : n \geq 2\}$, $\mathcal{M}_3 = \{0, 1, 2\}$ *and* $\mathcal{M}_4 = \{2n : n \geq 2\}$.

Definition 4.3.4. Let φ be an injective map on $\mathbb{N}_0$. For $n \in \mathbb{N}_0$, we define the set $[[n]]$ as follows:

(i) If $n \in \mathcal{M}_1$, then $[[n]] := \{\varphi^k(n) : k \geq 0\}$

(ii) If $n \in \mathcal{M}_2$, then $[[n]] := \{\varphi^k(n) : k \geq -j, \text{ where } n = \varphi^j(m) \text{ for } m \in \mathcal{M}_1\}$

(iii) If $n \in \mathcal{M}_3 \cup \mathcal{M}_4$, then $[[n]] := \{\varphi^k(n) : k \in \mathbb{Z}\}$.

Lemma 4.3.5. *If* $j \in [[n]]$, *then* $[[j]] = [[n]]$ *and vice-versa.*

Proof. Let $j \in [[n]]$. Then $j = \varphi^k(n)$ for some $k \in \mathbb{Z}$.

If $t \in [[j]]$, then $t = \varphi^\eta(j)$ for some $\eta \in \mathbb{Z}$, so that $t = \varphi^{\eta+k}(n)$ which implies that $t \in [[n]]$. Thus, $[[j]] \subseteq [[n]]$.

Again, if $\lambda \in [[n]]$, then $\lambda = \varphi^{\tau}(n)$ for some $\tau \in \mathbb{Z}$ which implies that $\lambda = \varphi^{\tau-k}(j)$, so that $\lambda \in [[j]]$. Thus $[[n]] \subseteq [[j]]$. □

In view of Lemma 4.3.5, we now propose the following definition:

Definition 4.3.6. Let φ be an injective map on $\mathbb{N}_0$. Let $\lambda_0 := 0$ and for $n \in \mathbb{N}$, let λ_n be defined as the smallest positive integer not belonging to $[[\lambda_0]] \cup [[\lambda_1]] \cup \cdots \cup [[\lambda_{n-1}]]$. Then, we have $\lambda_0 < \lambda_1 < \dots$, and $\mathbb{N}_0 = \bigcup_{i \in \mathbb{N}_0} [[\lambda_i]]$. $\Lambda_\varphi := \{\lambda_0, \lambda_1, \dots\}$ is called the φ induced partition of $\mathbb{N}_0$. Note that depending on the map φ, Λ_φ may also be a finite set.

Example 4.3.7. *In Example 4.3.3, we have* $\Lambda_\varphi = \{\lambda_0 = 0, \lambda_1 = 3, \lambda_2 = 4\}$, *where* $\lambda_0 \in \mathcal{M}_3$, $\lambda_1 \in \mathcal{M}_1$, $\lambda_2 \in \mathcal{M}_4$.

Definition 4.3.8. Let φ be an injective map on $\mathbb{N}_0$ and $\Lambda_\varphi = \{\lambda_0, \lambda_1, \cdots\}$ be the φ induced partition of $\mathbb{N}_0$. For $\lambda_n \in \Lambda_\varphi$, we define order of λ_n, denoted by $o(\lambda_n)$ as follows:

(i) If $\lambda_n \in \mathcal{M}_1$, then $o(\lambda_n) := 0$.

(ii) If $\lambda_n \in \mathcal{M}_2$, then $o(\lambda_n) := r$, where $\lambda_n = \varphi^r(m)$ for some $m \in \mathcal{M}_1$.

(iii) If $\lambda_n \in \mathcal{M}_3$, then $o(\lambda_n) := r$, where r is the smallest positive integer such that $\lambda_n = \varphi^r(\lambda_n)$.

(iv) If $\lambda_n \in \mathcal{M}_4$, then $o(\lambda_n) := \infty$.

Theorem 4.3.9. *Let φ be an injective map on $\mathbb{N}_0$ and $\Lambda_\varphi = \{\lambda_0, \lambda_1, \dots\}$ be the φ induced partition of $\mathbb{N}_0$. For $n \in \mathbb{N}_0$, let K_n be the closed linear span of $\{g_{i,j} : i \in \mathbb{N}_0, j \in [[\lambda_n]]\}$. If T_φ is the operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n \in \mathbb{N}_0}$, then the following must hold:*

(a) T_φ reduces each K_n.

(b) $\ell^2_+(K) = \sum_{n \in \mathbb{N}_0} \oplus K_n$.

Proof. For $n \in \mathbb{N}_0$, it follows from Lemmas 4.2.1, 4.2.2 and 4.3.5 that T_φ reduces K_n. Now (b) follows from (a) together with Definition 4.3.6. □

Theorem 4.3.10. *Let φ be an injective map on $\mathbb{N}_0$ and $\Lambda_\varphi = \{\lambda_0, \lambda_1, \dots\}$ be the φ induced partition of $\mathbb{N}_0$. Let T_φ be the operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. If $\lambda_n \in \mathcal{M}_1 \cup \mathcal{M}_2$, then $T_\varphi|_{K_n}$ is unitarily equivalent to a unilateral backward operator weighted shift on $\ell^2_+(K)$.*

Proof. Let $\lambda_n \in \mathcal{M}_1 \cup \mathcal{M}_2$ for some $n \in \mathbb{N}_0$, arbitrarily fixed. If $r = o(\lambda_n)$, then $[[\lambda_n]] = \{\varphi^k(\lambda_n) : k \geq -r\}$. As K_n is the closed linear span of $\{g_{i,j} : i \in \mathbb{N}_0, j \in [[\lambda_n]]\}$, so for $x = (x_0, x_1, \dots) \in K_n$ we have $x_j = 0$ for all $j \in \mathbb{N}_0 \setminus [[\lambda_n]]$.

For $i \in \mathbb{N}_0$, let $P_i : K_n \to K$ be defined as $P_i x = x_{\varphi^{i-r}(\lambda_n)}$ for $x = (x_0, x_1, \dots) \in K_n$. If $H_n := \{(P_0x, P_1x, \dots) : x \in K_n\}$, then H_n is isomorphic to K_n.

Let W be the backward shift on H_n with operator weights $\{W_i\}_{i\in\mathbb{N}_0}$, where $W_i := A_{\varphi^{i-r}(\lambda_n)}$ for all $i \in \mathbb{N}_0$ i.e, $W(P_0x, P_1x, \dots) = (W_0P_1x, W_1P_2x, \dots)$ for all $x \in K_n$. Then W is unitarily equivalent to $T_\varphi|_{K_n}$. □

Definition 4.3.11. Let K be a separable complex Hilbert space, and for $n \in \mathbb{N}$, let $H_n := K \oplus \cdots \oplus K$ (n copies). For bounded linear operators $\{W_i\}_{i=0}^{n-1}$ on K, we define $\tilde{W} : H_n \to H_n$ as

$$\tilde{W}(y_0, y_1, \dots, y_{n-1}) = (W_0y_1, W_1y_2, \dots, W_{n-2}y_{n-1}, W_{n-1}y_0).$$

The operator $\tilde{W}$ is called a weighted circulant operator on H_n.

Note: For $n = 1$, $\tilde{W}y_0 := W_0y_0$ for all $y_0 \in H_0 = K$.

Theorem 4.3.12. *Let φ be an injective map on $\mathbb{N}_0$ and $\Lambda_\varphi = \{\lambda_0, \lambda_1, \dots\}$ be the φ induced partition of $\mathbb{N}_0$. Let T_φ be the operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. If $\lambda_n \in \mathcal{M}_3$, then $T_\varphi|_{K_n}$ is unitarily equivalent to a weighted circulant operator on H_r, where $r = o(\lambda_n)$.*

Proof. Let $r = o(\lambda_n)$. Then r is the smallest positive integer such that $\varphi^r(\lambda_n) = \lambda_n$, so that $[[\lambda_n]] = \{\lambda_n, \varphi(\lambda_n), \ldots, \varphi^{r-1}(\lambda_n)\}$. For $i = 0, 1, \ldots, r-1$, define $P_i : K_n \to K$ as $P_i x = x_{\varphi^i(\lambda_n)}$. If $H_r := \{(P_0 x, P_1 x, \ldots, P_{r-1} x) : x \in K_n\}$, then H_r is isomorphic to K_n. Also if $W_i := A_{\varphi^i(\lambda_n)}$ for all $0 \leq i \leq r-1$ and W be the circulant operator on H_r defined as $W(y_0, y_1, \ldots, y_{r-1}) = (W_0 y_1, W_1 y_2, \ldots, W_{r-2} y_{r-1}, W_{r-1} y_0)$, then W on H_r is unitarily equivalent to $T_\varphi|_{K_n}$. □

Theorem 4.3.13. *Let φ be an injective map on $\mathbb{N}_0$ and $\Lambda_\varphi = \{\lambda_0, \lambda_1, \ldots\}$ be the φ induced partition of $\mathbb{N}_0$. Let T_φ be the operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. If $\lambda_n \in \mathcal{M}_4$, then $T_\varphi|_{K_n}$ is unitarily equivalent to a bilateral backward operator weighted shift on $\ell^2(K)$.*

Proof. Let $\lambda_n \in \mathcal{M}_4$ for some $n \in \mathbb{N}_0$, arbitrarily fixed. For $i \in \mathbb{Z}$, let $P_i x := x_{\varphi^i(\lambda_n)}$ for $x = (x_0, x_1, \ldots) \in K_n$, and let $H_n := \{(\ldots, P_{-1}x, [P_0 x], P_1 x, \ldots) : x \in K_n\}$. Then H_n is isomorhic to K_n. Also if $W_i := A_{\varphi^i(\lambda_n)}$ for all $i \in \mathbb{Z}$ and W be the bilateral (backward) operator weighted shift on H_n with weight sequence $\{W_i\}_{i\in\mathbb{Z}}$ i.e, $W(\ldots, y_{-1}, [y_0], y_1, \ldots) = (\ldots, W_{-1} y_0, [W_0 y_1], W_1 y_2, \ldots)$, then W is unitarily equivalent to $T_\varphi|_{K_n}$. □

From Theorems 4.3.9, 4.3.10, 4.3.12 and 4.3.13 we can thus conclude the following:

Theorem 4.3.14. *Let T_φ be an operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ and induced by an injective map φ on $\mathbb{N}_0$. Then T_φ is a countable (or finite) direct sum of unilateral backward shift, circulant operators and bilateral shifts.*

Definition 4.3.15. The operator weighted shift T_φ on $\ell^2_+(K)$ is classified as follows:

(i) T_φ is of type I if $\mathcal{M}_3$ and $\mathcal{M}_4$ are empty.

(ii) T_φ is of type II if $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are empty.

(iii) T_φ is of type III if $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_4$ are empty.

In view of Theorem 4.3.14, we can say that if T_φ is of type I, then it is a countable (or finite) direct sum of unilateral backward operator weighted shifts. We now proceed to determine the necessary and sufficient conditions for a reducing subspace X_F of T_φ to be minimal.

4.4 Transparent sequences.

Definition 4.4.1. Let φ be an injective map on $\mathbb{N}_0$ and $\{A_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of invertible operators on K, such that $A_n e_i = \alpha_i^{(n)} e_i$ for all $i, n \in \mathbb{N}_0$. Let $T = T_\varphi$ be the operator pseudo shift on $\ell_+^2(K)$ with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$. For each $n \in \mathbb{N}_0$, two non-negative integers p and q are said to be R_n^T-related, denoted as $pR_n^T q$, if $\alpha_p^{(j)} = \alpha_q^{(j)}$ for all $j \in [[n]]$.

Remark 4.4.2. (i) For each $n \in \mathbb{N}_0$, R_n^T is an equivalence relation on $\mathbb{N}_0$. For $p \in \mathbb{N}_0$, we denote the equivalence class of p as $[p]_n$. Thus $[p]_n = \{q \in \mathbb{N}_0 : pR_n^T q\}$. For each $n \in \mathbb{N}_0$, we define $\Omega_0^{(n)} = [0]_n$, and for $m > 0$, $\Omega_m^{(n)} := [p]_n$, where p is the smallest positive integer such that $p \notin \bigcup_{j=0}^{m-1} \Omega_j^{(n)}$.
(ii) For $n \in \mathbb{N}_0$, let $\omega_n := \{k \in \mathbb{N}_0 : \Omega_k^{(n)} \neq \phi\}$.
(iii) If $j \in [[n]]$, then by Lemma 4.3.5, the set of equivalence classes of R_n^T and R_j^T are identical. Hence for $j \in [[n]]$, we have $\omega_j = \omega_n$, and $\Omega_k^{(j)} = \Omega_k^{(n)}$ for all $k \in \omega_n$.

Definition 4.4.3. For an injective map φ on $\mathbb{N}_0$, we define a relation $\sim^\varphi$ on $\mathbb{N}_0$ as follows:

$$\text{For } p, q \in \mathbb{N}_0,\ p \sim^\varphi q \text{ if } \omega_p = \omega_q \text{ and } \Omega_k^{(p)} = \Omega_k^{(q)} \text{ for all } k \in \omega_p.$$

Remark 4.4.4. If φ be an injective map on $\mathbb{N}_0$ and $\lambda_n \sim^\varphi \lambda_m$ for all $\lambda_n, \lambda_m \in \Lambda_\varphi$, then $\Omega_k^{(\lambda_n)} = \Omega_k^{(\lambda_0)}$ for all $k \in \omega_{\lambda_0}$. Hence, in this case, we denote $\Omega_k^{(\lambda_0)}$ simply as Ω_k.

Definition 4.4.5. Let $f = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$ be a non-zero linear expression in K. Let r be the smallest non-negative integer such that $\alpha_r \neq 0$. The order of f is defined to be r and denoted as $o(f) = r$.

Definition 4.4.6. Let $F = (f_0, f_1, \dots)$ be a nonzero element in $\ell^2_+(K)$. If there exists a nonnegative integer m such that
(i) $o(f_i) \geq m$ for each nonzero f_i, and
(ii) there exists at least one f_i such that $o(f_i) = m$
then m is defined to be the order of F, denoted as $o(F)$.

Definition 4.4.7. Let Y be a nonzero nonempty subset of any separable Hilbert space H. Then order of Y, denoted as $o(Y)$, is defined to be the nonnegative integer m satisfying the following conditions:
(i) $o(f) \geq m$ for all nonzero f in Y, and
(ii) there exists $\tilde{f} \in Y$ such that $o(\tilde{f}) = m$.

Definition 4.4.8. Let X be a nonzero subset of $\ell^2_+(K)$. Then for each $j \in \mathbb{N}_0$, define X_j to be the set $\{f_j : (f_0, f_1, \dots) \in X\}$.

Remark 4.4.9. X is a non zero subset implies that the set X_j is also non zero for some $j \in \mathbb{N}_0$.

Lemma 4.4.10. *Let T_φ be the operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ induced by the injective map φ on $\mathbb{N}_0$. If T_φ is of type I and X is a nonzero reducing subspace of T_φ, then there exists $j \in \mathcal{M}_1$ such that $X_j \neq 0$.*

Proof. Let, if possible, $X_j = 0$ for all $j \in \mathcal{M}_1$. Thus $X \neq 0$ implies that there exists $j \in \mathcal{M}_2$ with $X_j \neq 0$. This in turn implies that there exists $r > 0$ such that $\varphi^{-r}(j) \in \mathcal{M}_1$.
Let $f_j \in X_j$, $f_j \neq 0$ and $F = (f_0, f_1, \dots) \in X$. Suppose $f_j = \sum_{i\in\mathbb{N}_0} \beta_i e_i$. Then $f_j \neq 0$ implies there exist at least one β_i which is not zero. Now by Lemma 4.2.1, we have $T^r_\varphi F = (y_0, y_1, \dots) \in X$ where

$$y_{\varphi^{-r}(j)} = \sum_{i\in\mathbb{N}_0} \beta_i(\alpha_i^{(\varphi^{-1}(j))}\alpha_i^{(\varphi^{-2}(j))}\dots\alpha_i^{(\varphi^{-r}(j))})e_i \neq 0.$$

This means $X_{\varphi^{-r}(j)} \neq 0$ which implies $\varphi^{-r}(j) \in \mathcal{M}_2$. But this is a contradiction as $\mathcal{M}_1 \cap \mathcal{M}_2 = \phi$. Thus, $X \neq 0$ implies that there exists $j \in \mathcal{M}_1$ such that $X_j \neq 0$. $\square$

Definition 4.4.11. If X is a nonzero reducing subspace of T_φ, then

$$(\mathcal{M}_1)_X := \{j \in \mathcal{M}_1 : X_j \neq 0\}.$$

Definition 4.4.12. Let X be a nonzero reducing subspace of the operator pseudo shift T_φ. If $(\mathcal{M}_1)_X \neq \phi$, then X is said to be an $\mathcal{M}_1$-reducing subspace of T_φ.

Lemma 4.4.13. *Let T_φ be the operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ and induced by the injective map φ on $\mathbb{N}_0$. Let T_φ be of type I and X be a non zero reducing subspace of T_φ with $o(X) = r$. Then X is $\mathcal{M}_1$ reducing subspace of T_φ and there exists $j \in (\mathcal{M}_1)_X$ such that $o(X_j) = r$.*

Proof. By Lemma 4.4.10, $(\mathcal{M}_1)_X \neq \phi$ if T_φ is of type I. Hence, X is an $\mathcal{M}_1$ reducing subspace of T_φ. Clearly, $o(X) = r$ implies $o(X_j) \geq r$ for all $j \in (\mathcal{M}_1)_X$. Also $o(X) = r$ implies there exists $f = (f_0, f_1, \dots) \in X$ such that $o(f) = r$. This in turn implies that there exists $j \in \mathbb{N}_0$ with $o(f_j) = r$. Thus if $f_j = \sum_{i\in\mathbb{N}_0} a_{i,j} e_i$, then $a_{r,j} \neq 0$ and $a_{i,j} = 0$ for all $i < r$.

If $j \in \mathcal{M}_1$, then $f_j \in X_j$ with $o(f_j) = r$. Thus $o(X_j) = r = o(X)$ and we are done. If $j \notin \mathcal{M}_1$, then since T_φ be of type I, we must have $j \in \mathcal{M}_2$, so that $\varphi^k(n) = j$ for some $n \in \mathcal{M}_1$ and $k \in \mathbb{N}$.

If $T_\varphi^k f = (g_0, g_1, \dots)$, then $g_t = A_t A_{\varphi(t)} \dots A_{\varphi^{k-1}(t)} f_{\varphi^k(t)}$ for all $t \in \mathbb{N}_0$. In particular,

$$\begin{aligned} g_n &= A_n A_{\varphi(n)} \dots A_{\varphi^{k-1}(n)} f_{\varphi^k(n)} \\ &= A_{\varphi^{-k}(j)} \dots A_{\varphi^{-1}(j)} f_j \\ &= \sum_{i\in\mathbb{N}_0} a_{i,j} \alpha_i^{(\varphi^{-k}(j))} \dots \alpha_i^{(\varphi^{-1}(j))} e_i \end{aligned}$$

with $a_{r,j} \neq 0$ and $a_{i,j} = 0$ for all $i < r$. Thus, $o(g_n) = r$ where $g_n \in X_n$. This implies $n \in (\mathcal{M}_1)_X$ and $o(X_n) = r$. $\square$

Definition 4.4.14. Let T_φ be the operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ and induced by the injective map φ on $\mathbb{N}_0$. Let T_φ be of type I and X be a nonzero reducing subspace of T_φ with $o(X) = r$. Then $o_1(X) := \inf\{j \in (\mathcal{M}_1)_X : o(X_j) = r\}$.

Definition 4.4.15. Let T_φ be the operator pseudo shift on $\ell^2_+(K)$ with uniformly bounded invertible positive diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ and induced by the injective map φ on $\mathbb{N}_0$. Let $\Lambda_\varphi = \{\lambda_0, \lambda_1, \dots\}$ be the φ induced partition of $\mathbb{N}_0$, and consider a non-zero linear expression $f = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$ in K such that $o(f) = r$. If there exists $\lambda_n \in \Lambda_\varphi$ such that $i \in [r]_{\lambda_n}$ for each non-zero α_i in f, then f is said to be n-transparent in K of order r.

Example 4.4.16. *Consider the injective map φ given by $\varphi(2n) = 2(n+1)$ and $\varphi(2n+1) = 2n+3$ for all $n \in \mathbb{N}_0$. Then, $\mathcal{M}_1 = \{0,1\}$, $\mathcal{M}_2 = \{2,3,\dots\}$, $\mathcal{M}_3 = \mathcal{M}_4 = \phi$ and $\Lambda_\varphi = \{\lambda_0, \lambda_1\}$, where $\lambda_0 = 0$ and $\lambda_1 = 1$. Let $f = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 \in K$ and $\alpha_i \neq 0$ for all $i \in \{3,4,5,6\}$. For each $n \in \mathbb{N}_0$, let $A_n e_i = \alpha_i^{(n)} e_i$, where the scalars $\alpha_i^{(n)}$ take the following values:*

$$\alpha_i^{(2n)} = \begin{cases} 1, & \text{if } 0 \le i \le 6; \\ 2, & \text{if } i > 6. \end{cases} \quad \text{and } \alpha_i^{(2n+1)} = \begin{cases} 3, & \text{if } 0 \le i \le 6; \\ 4, & \text{if } i > 6. \end{cases}$$

Let $T = T_\varphi$ be the operator pseudo shift on $\ell^2_+(K)$ with operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ induced by the injective map φ on $\mathbb{N}_0$. Then, the equivalence classes of $R^T_{\lambda_0}$ are

$$\Omega_0^{(\lambda_0)} = [0]_{\lambda_0} = \{0,1,\dots,6\}, \text{ and } \Omega_1^{(\lambda_0)} = [7]_{\lambda_0} = \{7,8,\dots\}.$$

Also, the equivalence classes of $R^T_{\lambda_1}$ are

$$\Omega_0^{(\lambda_1)} = [0]_{\lambda_1} = \{0,1,\dots,6\}, \text{ and } \Omega_1^{(\lambda_1)} = [7]_{\lambda_1} = \{7,8,\dots\}.$$

Clearly, in this case we have $\lambda_0 \sim^\varphi \lambda_1$. Now, if $r = o(f)$, then $r = 3$. Here,

$$[r]_{\lambda_0} = [r]_{\lambda_1} = \{0,1,\dots,6\}.$$

Therefore, f is 0-transparent as well as 1-transparent in K of order 3.

Theorem 4.4.17. *Let f be non-zero n-transparent in K of order r. If $m \in \mathbb{N}_0$ is such that $\lambda_n \sim^{\varphi} \lambda_m$, then f is m-transparent in K of order r.*

Proof. Let $f = \sum_{i \in \mathbb{N}_0} \alpha_i e_i$ with $o(f) = r$. Then $i \in [r]_{\lambda_n}$ for all $i \in \mathbb{N}_0$ such that $\alpha_i \neq 0$. Now $\lambda_n \sim^{\varphi} \lambda_m$ implies $[r]_{\lambda_n} = [r]_{\lambda_m}$. Therefore, $i \in [r]_{\lambda_m}$ for all $i \in \mathbb{N}_0$ such that $\alpha_i \neq 0$. Hence, f is m-transparent in K. □

Remark 4.4.18. Converse of the above is not true. Suppose $f = \sum_{i \in \mathbb{N}_0} \alpha_i e_i$ such that $o(f) = r$ and f is both n-transparent and m-transparent in K. This means $i \in [r]_{\lambda_n}$ and $i \in [r]_{\lambda_m}$ for all $i \in \mathbb{N}_0$ such that $\alpha_i \neq 0$. However, this does not necessarily imply that $[r]_{\lambda_n} = [r]_{\lambda_m}$. In fact, $\lambda_m \sim^{\varphi} \lambda_n$ if and only if $[p]_{\lambda_n} = [p]_{\lambda_m}$ for all $p \in \mathbb{N}_0$.

Example 4.4.19. *Consider φ and f in K as defined in Example 4.4.16. For each $n \in \mathbb{N}_0$, let $A_n e_i = \alpha_i^{(n)} e_i$, where the scalars $\alpha_i^{(n)}$ take the following values:*

$$\alpha_i^{(2n)} = \begin{cases} 1, & \text{if } 0 \leq i \leq 6; \\ 2, & \text{if } i > 6. \end{cases} \quad \text{and } \alpha_i^{(2n+1)} = \begin{cases} 3, & \text{if } 0 \leq i \leq 7; \\ 4, & \text{if } i > 7. \end{cases}$$

Then, the equivalence classes of $R^T_{\lambda_0}$ are

$$\Omega_0^{(\lambda_0)} = [0]_{\lambda_0} = \{0, 1, \dots, 6\}, \text{ and } \Omega_1^{(\lambda_0)} = [7]_{\lambda_0} = \{7, 8, \dots\}.$$

Also, the equivalence classes of $R^T_{\lambda_1}$ are

$$\Omega_0^{(\lambda_1)} = [0]_{\lambda_1} = \{0, 1, \dots, 7\}, \text{ and } \Omega_1^{(\lambda_1)} = [8]_{\lambda_1} = \{8, 9, \dots\}.$$

Here, $\Omega_k^{(\lambda_0)} \neq \Omega_k^{(\lambda_1)}$ for $k = 0, 1$ and so $\lambda_0 \nsim^{\varphi} \lambda_1$. However, f is 0-transparent since $3, 4, 5, 6 \in [r]_{\lambda_0} = \Omega_0^{(\lambda_0)}$ and f is 1-transparent since $3, 4, 5, 6 \in [r]_{\lambda_1} = \Omega_0^{(\lambda_1)}$.

Definition 4.4.20. For $j \in \mathbb{N}_0$, let $F = \sum_{i \in \mathbb{N}_0} \alpha_i g_{i,j}$ be a non-zero linear expression in $\ell^2_+(K)$. Let φ be an injective map on $\mathbb{N}_0$ and $\Lambda_\varphi = \{\lambda_0, \lambda_1, \dots\}$ be the φ-induced partition of $\mathbb{N}_0$. If $f = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$ is n-transparent in K of order r and $j \in [[\lambda_n]]$, then F is said to be n-transparent in $l^2_+(K)$. In view of Definition 4.4.6, $o(F) = o(f)$.

Lemma 4.4.21. *If $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,j}$ is n-transparent of order r for $j \in \mathbb{N}_0$, then $\sum_{i\in\mathbb{N}_0} \alpha_i g_{i,t}$ is n-transparent of order r for all $t \in [[j]]$.*

The proof follows immediately from Lemma 4.3.5.

Lemma 4.4.22. *Let φ be an injective map on $\mathbb{N}_0$ and $\Lambda_\varphi = \{\lambda_0, \lambda_1, \dots\}$ be the φ induced partition of $\mathbb{N}_0$. Also let $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,j}$ be n-transparent with $o(F) = r$. If $t \sim^\varphi j$ and $t \in [[\lambda_m]]$, then $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,t}$ is m-transparent with $o(G) = r$.*

Proof. Since $t \sim^\varphi j$, so $\lambda_n \sim^\varphi \lambda_m$. This implies $[r]_{\lambda_n} = [r]_{\lambda_m}$. Now, since F is n-transparent in $\ell^2_+(K)$, so $f = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$ is n-transparent in K. Therefore, $i \in [r]_{\lambda_n} = [r]_{\lambda_m}$ for each non-zero α_i in f. This means f is m-transparent in K which in turn implies $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,t}$ is m-transparent. □

Definition 4.4.23. Let the operator pseudo shift T_φ be of type I. Then $F = \sum_{j\in\mathbb{N}_0} \sum_{i\in\mathbb{N}_0} \alpha_{i,j} g_{i,j}$ is said to be transparent if for each $j \in \mathbb{N}_0$, $\sum_{i\in\mathbb{N}_0} \alpha_{i,j} g_{i,j}$ is n-transparent for some $n \in \mathbb{N}_0$ depending on j. If there exists $n \in \mathbb{N}_0$ such that $\sum_{i\in\mathbb{N}_0} \alpha_{i,j} g_{i,j}$ is n-transparent for all $j \in \mathbb{N}_0$, then F is said to be jointly n-transparent.

Remark 4.4.24. Let $t \in \mathbb{N}_0$ and $F = \sum_{j\in[[t]]} \sum_{i\in\mathbb{N}_0} \alpha_{i,j} g_{i,j}$. If F is transparent, then it is jointly n-transparent, where $t \in [[\lambda_n]]$ for $\lambda_n \in \Lambda_\varphi$.

Definition 4.4.25. Let T_φ be an operator pseudo shift on $\ell^2_+(K)$, and let $\mathcal{S}$ be a vector space consisting of all finite linear combinations of finite products of the operators T_φ and T^*_φ. For any non zero $F \in \ell^2_+(K)$, $\mathcal{S}F := \{\tilde{T}F : \tilde{T} \in \mathcal{S}\}$. The closure of $\mathcal{S}F$ in $\ell^2_+(K)$ is a reducing subspace of T_φ and is denoted by X_F. X_F is called the subspace generated by F. Clearly, X_F is the smallest reducing subspace of T_φ containing F.

Definition 4.4.26. Let T_φ be an operator pseudo shift of type I and and let the function $F = \sum_{i\in\mathbb{N}_0}\sum_{j\in\mathbb{N}_0}\alpha_{i,j}g_{i,j}$ be transparent in $\ell^2_+(K)$. For $j \in \mathcal{M}_1$, let f_j be defined as $f_j := \sum_{i\in\mathbb{N}_0}\sum_{t\in[[j]]}\alpha_{i,t}g_{i,t}$. Then each f_j is jointly n-transparent in $\ell^2_+(K)$, for some n depending on j. Dropping those f_j's which are zero, we list the remaining ones as $f_{\hat{0}}, f_{\hat{1}}, \dots$ in such a way that $f_{\hat{j}}$ is jointly g_j-transparent with $g_0 < g_1 < g_2 < \dots$. Then $F = \sum_{j\in\mathbb{N}_0} f_{\hat{j}}$ is called the transparent decomposition of F.

Definition 4.4.27. Let φ be an injective map on $\mathbb{N}_0$. Also let $\Lambda_\varphi = \{\lambda_0, \lambda_1, \dots\}$ be the φ induced partition of $\mathbb{N}_0$ such that $\lambda_n \sim^\varphi \lambda_m$ for all $n, m \in \mathbb{N}_0$. Consider $F = \sum_{i\in\mathbb{N}_0}\sum_{j\in\mathbb{N}_0}\alpha_{i,j}g_{i,j}$. For $k \in \omega_{\lambda_0}$, let q_k be defined as $q_k := \sum_{i\in\Omega_k}\sum_{j\in\mathbb{N}_0}\alpha_{i,j}g_{i,j}$. Dropping those q_k's which are zero, the remaining q_k's are arranged as $f^{(1)}, f^{(2)}, \dots$ in such a way that $o(f^{(k)}) < o(f^{(k+1)})$. The resulting decomposition $F = f^{(1)} + f^{(2)} + \dots$ is called the *canonical decomposition* of F with respect to T_φ. Clearly each $f^{(k)}$ is transparent in $\ell^2_+(K)$. If there exists a finite positive integer n such that $F = f^{(1)} + f^{(2)} + \cdots + f^{(n)}$, then F is said to have a *finite canonical decomposition.*

In the above definition, we observe that $o(f^{(k_1)}) = o(f^{(k_2)})$ is not possible for distinct elements $k_1, k_2 \in \omega_{\lambda_0}$ since $\Omega_{k_1} \cap \Omega_{k_2} = \phi$ in this case.

Lemma 4.4.28. *Let T_φ be the operator pseudo shift induced by the injective map φ and with uniformly bounded invertible diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. Let T_φ be of type I and $\lambda_n \sim^\varphi \lambda_m$ for all $\lambda_n, \lambda_m \in \Lambda_\varphi$. Let $\Omega_1, \Omega_2, \dots$ be the disjoint equivalence classes under $R^T_{\lambda_0}$. Let $F = \sum_{i\in\mathbb{N}_0}\sum_{j\in\mathcal{M}_1}\alpha_{i,j}g_{i,j} \in \ell^2_+(K)$ have a finite canonical decomposition $F = f^{(1)} + f^{(2)} + \cdots + f^{(n)}$ where $o(f^{(i)}) = r_i$ for all $1 \le i \le n$. If for each $k \in \mathbb{N}$ and every $\xi, \eta \in \mathcal{M}_1$, $\alpha_i^{(\varphi^t(\xi))} = \alpha_i^{(\varphi^t(\eta))}$ for all $i \in \Omega_k$ and for all $t \ge 0$, then $f^{(i)} \in X_F$ for all $1 \le i \le n$.*

Proof. Without loss of generality, we may assume that for each $k = 1, 2, \dots, n$,

$f^{(k)} = \sum_{i\in\Omega_k}\sum_{j\in\mathcal{M}_1}\alpha_{i,j}g_{i,j}$. Let $m = \min_{t\in\mathcal{M}_1} t$ and $w_k = \min_{t\in\Omega_k} t$.

Step I. As $\Omega_1\cap\Omega_n = \phi$, so for $\xi \in \mathcal{M}_1$ we can choose the smallest nonnegative integer t_1 such that $\alpha_{w_1}^{(\varphi^{t_1}(\xi))} \neq \alpha_{w_n}^{(\varphi^{t_1}(\xi))}$. As $\alpha_{w_n}^{(\varphi^{t_1}(\xi))} = \alpha_{w_n}^{(\varphi^{t_1}(m))}$ and $\alpha_{w_1}^{(\varphi^{t_1}(\xi))} = \alpha_{w_1}^{(\varphi^{t_1}(m))}$, so we have $\alpha_{w_1}^{(\varphi^{t_1}(m))} \neq \alpha_{w_n}^{(\varphi^{t_1}(m))}$. For $\eta \geq 0$ and $i \in \Omega_k$ $(k = 1, 2, \ldots, n)$, we have

$$\begin{aligned}T_\varphi^{\eta+1}(T_\varphi^*)^{\eta+1}g_{i,j} &= \big(\alpha_i^{(j)}\alpha_i^{(\varphi(j))}\ldots\alpha_i^{(\varphi^\eta(j))}\big)^2 g_{i,j}\\ &= \big(\alpha_{w_k}^{(j)}\alpha_{w_k}^{(\varphi(j))}\ldots\alpha_{w_k}^{(\varphi^\eta(j))}\big)^2 g_{i,j}.\end{aligned}$$

Hence,

$$\begin{aligned}T_\varphi^{\eta+1}(T_\varphi^*)^{\eta+1}f^{(k)} &= \sum_{i\in\Omega_k}\sum_{j\in\mathcal{M}_1}\big(\alpha_{w_k}^{(j)}\alpha_{w_k}^{(\varphi(j))}\ldots\alpha_{w_k}^{(\varphi^\eta(j))}\big)^2\alpha_{i,j}g_{i,j}\\ &= \big(\alpha_{w_k}^{(m)}\alpha_{w_k}^{(\varphi(m))}\ldots\alpha_{w_k}^{(\varphi^\eta(m))}\big)^2 f^{(k)}.\end{aligned}$$

Therefore, we get

$$T_\varphi^{\eta+1}(T_\varphi^*)^{\eta+1}F = \big(\alpha_{w_1}^{(m)}\alpha_{w_1}^{(\varphi(m))}\ldots\alpha_{w_1}^{(\varphi^\eta(m))}\big)^2 f^{(1)} + \cdots + \big(\alpha_{w_n}^{(m)}\alpha_{w_n}^{(\varphi(m))}\ldots\alpha_{w_n}^{(\varphi^\eta(m))}\big)^2 f^{(n)}.$$

As $\alpha_{w_1}^{(\varphi^{t_1}(m))} \neq \alpha_{w_n}^{(\varphi^{t_1}(m))}$, and $\alpha_{w_1}^{(\varphi^{t}(m))} = \alpha_{w_n}^{(\varphi^{t}(m))}$ for all $0 \leq t < t_1$, so

$$\begin{aligned}F_1 &= \Big[\big(\alpha_{w_n}^{(m)}\alpha_{w_n}^{(\varphi(m))}\ldots\alpha_{w_n}^{(\varphi^{t_1}(m))}\big)^2 - T_\varphi^{t_1+1}(T_\varphi^*)^{t_1+1}\Big]F\\ &= \sum_{k=1}^{n-1}\Big[\big(\alpha_{w_n}^{(m)}\alpha_{w_n}^{(\varphi(m))}\ldots\alpha_{w_n}^{(\varphi^{t_1}(m))}\big)^2 - \big(\alpha_{w_k}^{(m)}\alpha_{w_k}^{(\varphi(m))}\ldots\alpha_{w_k}^{(\varphi^{t_1}(m))}\big)^2\Big]f^{(k)} \in X_F.\end{aligned}$$

So if $\beta_k^{(1)} := \big(\alpha_{w_n}^{(m)}\alpha_{w_n}^{(\varphi(m))}\ldots\alpha_{w_n}^{(\varphi^{t_1}(m))}\big)^2 - \big(\alpha_{w_k}^{(m)}\alpha_{w_k}^{(\varphi(m))}\ldots\alpha_{w_k}^{(\varphi^{t_1}(m))}\big)^2$ for $1 \leq k < n$, then $F_1 = \sum_{k=1}^{n-1}\beta_k^{(1)}f^{(k)} \in X_F$ where $\beta_1^{(1)} \neq 0$.

Step II. As $\Omega_1 \cap \Omega_{n-1} = \phi$, so there exists a smallest nonnegative integer t_2 such that $\varphi_{w_1}^{(t_2(m))} \neq \varphi_{w_{n-1}}^{(t_2(m))}$. Therefore

$$\begin{aligned}F_2 &= \Big[\big(\alpha_{w_{n-1}}^{(m)}\alpha_{w_{n-1}}^{(\varphi(m))}\ldots\alpha_{w_{n-1}}^{(\varphi^{t_2}(m))}\big)^2 - T_\varphi^{t_2+1}(T_\varphi^*)^{t_2+1}\Big]F_1\\ &= \sum_{k=1}^{n-2}\beta_k^{(2)}\beta_k^{(1)}f^{(k)} \in X_F,\end{aligned}$$

where $\beta_k^{(2)} = \left(\alpha_{w_{n-1}}^{(m)} \alpha_{w_{n-1}}^{(\varphi(m))} \ldots \alpha_{w_{n-1}}^{(\varphi^{t_2}(m))}\right)^2 - \left(\alpha_{w_k}^{(m)} \alpha_{w_k}^{(\varphi(m))} \ldots \alpha_{w_k}^{(\varphi^{t_2}(m))}\right)^2$ and $\beta_1^{(2)} \neq 0$.

Repeating the above argument $n-1$ times, we get $F_{n-1} = \beta_1^{(1)} \beta_1^{(2)} \ldots \beta_1^{(n-1)} f^{(1)} \in X_F$ with $\beta_1^{(k)} \neq 0$ for all $1 \leq k \leq n-1$. This implies that $f^{(1)} \in X_F$. Similarly, $f^{(i)} \in X_F$ for $1 < i \leq n$. □

Theorem 4.4.29. ***Extremal Theorem for T_φ of type I .***

Let T_φ be the operator pseudo shift induced by the injective map φ and with uniformly bounded invertible diagonal operator weights $\{A_n\}_{n \in \mathbb{N}_0}$. Let T_φ be of type I and X be a nonzero reducing subspace of T_φ with $o(X) = m$, and $o_1(X) = \tilde{j}$. Then the extremal problem

$$\sup\{Re\ \alpha_{m,\tilde{j}} : F = (f_0, f_1, \ldots) \in X, \|F\| \leq 1, f_{\tilde{j}} = \sum_{i \in \mathbb{N}_0} \alpha_{i,\tilde{j}} e_i\}$$

has a unique solution $G = \sum_{j \in (\mathcal{M}_1)_X} \sum_{i \in \mathbb{N}_0} \beta_{i,j} g_{i,j}$ with $\|G\| = 1$ and $o(G) = m = o(g_{\tilde{j}})$, where $g_{\tilde{j}} = \sum_{i \in \mathbb{N}_0} \beta_{i,\tilde{j}} e_i$.

Proof. Note that as T_φ is of type I, so $\mathcal{M}_1$ is a nonempty set. Define $\eta : X \to \mathbb{C}$ as $\eta(F) = \alpha_{m,\tilde{j}}$ where $F = (f_0, f_1, \ldots)$ and $f_{\tilde{j}} = \sum_{i \in \mathbb{N}_0} \alpha_{i,\tilde{j}} e_i$ with $\tilde{j} = o_1(X) = \min\{j \in (\mathcal{M}_1)_X : o(X_j) = m\}$.
$o(X_{\tilde{j}}) = m$ and so there exists $0 \neq \tilde{F} = (f_0, f_1, \ldots) \in X$ such that $f_{\tilde{j}} \neq 0$, $o(f_{\tilde{j}}) = m$. Thus η is a nonzero bounded linear functional on X. Hence, from [8] there exists a unique $G \in X$ such that $\eta(G) > 0$, $\|G\| = 1$ and

$$\begin{aligned}\eta(G) &= \sup\{\text{Re } \eta(F) : F \in X, \|F\| \leq 1\} \\ &= \sup\{\text{Re } \alpha_{m,\tilde{j}} : F = (f_0, f_1, \ldots) \in X, \|F\| \leq 1, f_{\tilde{j}} = \sum_{i \in \mathbb{N}_0} \alpha_{i,\tilde{j}} e_i\}.\end{aligned}$$

We will show that $G = \sum_{j \in (\mathcal{M}_1)_X} \sum_{i \in \mathbb{N}_0} \beta_{i,j} g_{i,j}$ and $o(G) = m$. For this we consider $G = (g_0, g_1, \ldots)$ with $g_j = \sum_{i \in \mathbb{N}_0} \beta_{i,j} e_i$.

Claim I. If $F \in X$ and $\|F\| < 1$, then Re $\eta(F) < \eta(G)$. Let, if possible, Re $\eta(F) = \eta(G)$. If $H := \frac{F}{\|F\|}$, then $H \in X$, $\|H\| = 1$ and Re $\eta(H) = \frac{\text{Re }\eta(F)}{\|F\|} = \frac{\eta(G)}{\|F\|} > \eta(G)$, contradicting the maximality of G. Hence, claim I holds.

Now for each $F \in X$, Re $\eta(G + T_\varphi^* F) = \eta(G)$, and so by claim I, we must have $\|G + T_\varphi^* F\| \geq 1$ which implies $G \perp T_\varphi^* F$. In particular, $\langle G, T_\varphi^* T_\varphi G\rangle = 0$. Since $G = (g_0, g_1, \dots)$, so $T_\varphi^* T_\varphi G = (y_0, y_1, \dots)$ where

$$y_j := \begin{cases} A^*_{\varphi^{-1}(j)} A_{\varphi^{-1}(j)} g_j, & \text{if } j \in R(\varphi); \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $\langle G, T_\varphi^* T_\varphi G\rangle = 0$ implies $A_{\varphi^{-1}(j)} g_j = 0$ for all $j \in R(\varphi)$. As $A_{\varphi^{-1}(j)}$ is invertible, so $g_j = 0$ for all $j \in R(\varphi)$. Equivalently, we must have $g_j = 0$ for all $j \notin \mathcal{M}_1$. Hence, we have $G = \sum_{j \in (\mathcal{M}_1)_X} \sum_{i \in \mathbb{N}_0} \beta_{i,j} g_{i,j}$.

By Lemma 4.4.13, $o(X_{\tilde{j}}) = m$ and so $o(g_{\tilde{j}}) \geq m$. As $g_{\tilde{j}} = \sum_{i \in \mathbb{N}_0} \beta_{i,\tilde{j}} e_i$, so $o(g_{\tilde{j}}) \geq m$ gives $\beta_{i,\tilde{j}} = 0$ for all $i < m$. Again $\eta(G) > 0$ implies $\beta_{m,\tilde{j}} \neq 0$. Thus, $o(g_{\tilde{j}}) = m$. Also, $o(X_j) \geq m$ for all $j \in (\mathcal{M}_1)_X$. This implies that $o(g_j) \geq m$ for all $j \in (\mathcal{M}_1)_X$. Hence, $o(G) = m$. □

Remark 4.4.30. The function G in Theorem 4.4.29 is called the *extremal function* of the nonzero reducing subspace X of T_φ.

Theorem 4.4.31. *Let T_φ be the operator pseudo shift induced by the injective map φ and with uniformly bounded invertible diagonal operator weights $\{A_n\}_{n \in \mathbb{N}_0}$. Let T_φ be of type I and $\lambda_n \sim^\varphi \lambda_m$ for all $n, m \in \mathbb{N}_0$. Let $\Omega_1, \Omega_2, \dots$ are the disjoint equivalence classes of $R^T_{\lambda_0}$ and let for each $k \in \mathbb{N}$, $\alpha_i^{(\varphi^t(\xi))} = \alpha_i^{(\varphi^t(\eta))}$ for all $i \in \Omega_k$ and for all $t \geq 0$, $\xi, \eta \in \mathcal{M}_1$. Let X be a non zero reducing subspace of T_φ with $o(X) = m$. If the extremal function of X has a finite canonical decomposition, then it must be transparent.*

Proof. Let $\tilde{j} = o_1(X) = \min\{j \in (\mathcal{M}_1)_X : o(X_j) = m\}$ and G be the extremal function of X with finite canonical decomposition $G = g^{(1)} + g^{(2)} + \cdots + g^{(n)}$. Then by Lemma 4.4.28, $g^{(i)} \in X$ for all $1 \leq i \leq n$.

By Lemma 4.4.29, we have $G = \sum_{j\in(\mathcal{M}_1)_X} g_j$ with $o(G) = m = o(g_{\tilde{j}})$, where $g_j = \sum_{i\geq m} \beta_{i,j} g_{i,j}$ for all $j \in (\mathcal{M}_1)_X$. As $o(g_{\tilde{j}}) = m$, so $\beta_{m,\tilde{j}} \neq 0$.

For each $g^{(k)}$, there exists Ω_{τ_k} such that $g^{(k)} = \sum_{i\in\Omega_{\tau_k}} \sum_{j\in(\mathcal{M}_1)_X} \alpha_{i,j} g_{i,j}$ and $o(g^{(k)}) < o(g^{(k+1)})$ for all $k = 1, 2, \ldots, n-1$.

As $G = g^{(1)}+g^{(2)}+\cdots+g^{(n)}$ with $o(g^{(i)}) < o(g^{(i+1)})$, so $o(G) = m$ implies $o(g^{(1)}) = m$ which gives us $m \in \Omega_{\tau_1}$ and $\alpha_{m,\tilde{j}} = \beta_{m,\tilde{j}} \neq 0$. Also, $\|g^{(1)}\| \leq \|G\| = 1$. So, by extremality of G, we must have $G = g^{(1)}$. As $g^{(1)}$ by definition is transparent, so G is transparent. □

4.5 Minimal reducing subspaces

Lemma 4.5.1. *Let $T = T_\varphi$ be the operator pseudo shift induced by the injective map φ and with uniformly bounded invertible diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. Let T_φ be of type I and $\lambda_n \sim^\varphi \lambda_m$ for all $\lambda_n, \lambda_m \in \Lambda_\varphi$. Let $\Omega_1, \Omega_2, \ldots$ be the disjoint equivalence classes under $R^T_{\lambda_0}$. Assume that for each $k \in \mathbb{N}$ and every $\xi, \eta \in \mathcal{M}_1$, $\alpha_i^{(\varphi^t(\xi))} = \alpha_i^{(\varphi^t(\eta))}$ for all $i \in \Omega_k$ and for all $t \geq 0$. Let F be a transparent function in $\ell^2_+(K)$ of the form $F = \sum_{j\in\mathcal{M}_1} \sum_{i\in\mathbb{N}_0} \alpha_{i,j} g_{i,j}$. If $G \in X_F$ such that $G \neq 0$ and is of the form $G = \sum_{j\in\mathcal{M}_1} \sum_{i\in\mathbb{N}_0} \beta_{i,j} g_{i,j}$, then $G = \lambda F$ for some nonzero scalar λ.*

Proof. As $0 \neq G \in X_F$, so by Definition 4.4.25 we have $G = \sum_{k\in\mathbb{N}_0} \lambda_k T^k_\varphi (T^*_\varphi)^k F$ for scalars λ_k, not all zero. As $F = \sum_{j\in\mathcal{M}_1} \sum_{i\in\mathbb{N}_0} \alpha_{i,j} g_{i,j}$, so it can be written as $F = \sum_{j\in\mathcal{M}_1} f_j$ where $f_j = \sum_{i\in\mathbb{N}_0} \alpha_{i,j} g_{i,j}$. For each $j \in \mathcal{M}_1$ and $\eta > 0$, let $\alpha_p^{(\varphi^\eta(j))} = \beta_{\varphi^\eta(j)}$ for all p such that $\alpha_{p,j} \neq 0$. Now,

$$T^k_\varphi (T^*_\varphi)^k f_j := \begin{cases} \sum_{i\in\mathbb{N}_0} \alpha_{i,j} \big(\beta_j \beta_{\varphi(j)} \ldots \beta_{\varphi^{k-1}(j)}\big)^2 g_{i,j}, & \text{if } k > 0; \\ f_j, & \text{if } k = 0. \end{cases}$$

Since $\lambda_n \sim^{\varphi} \lambda_m$ for all $n, m \geq 0$, so $\beta_{\varphi^{\eta}(j)} = \beta_{\varphi^{\eta}(\tau)} = \gamma_n$(say) for all $j, \tau \in \mathcal{M}_1$. Therefore for all $j \in \mathcal{M}_1$, $k > 1$, $T^k(T^*)^k f_j = \big(\gamma_0\gamma_1 \ldots \gamma_{k-1}\big)^2 f_j$ and so

$$T_{\varphi}^k (T_{\varphi}^*)^k F := \begin{cases} \big(\gamma_0\gamma_1 \cdots \gamma_{k-1}\big)^2 F, & \text{for } k > 0; \\ F, & \text{for } k = 0. \end{cases}$$

$$\begin{aligned} \therefore G &= \sum_{k \in \mathbb{N}_0} \lambda_k T_{\varphi}^k (T_{\varphi}^*)^k F \\ &= (\lambda_0 + \lambda_1 \gamma_0^2 + \lambda_2 (\gamma_0\gamma_1)^2 + \ldots) F \\ &= \lambda F, \end{aligned}$$

where $\lambda = \lambda_0 + \lambda_1 \gamma_0^2 + \lambda_2 (\gamma_0\gamma_1)^2 + \ldots$. □

Lemma 4.5.2. *Let T_{φ} be the operator pseudo shift induced by the injective map φ and with uniformly bounded invertible diagonal operator weights $\{A_n\}_{n \in \mathbb{N}_0}$. Let T_{φ} be of type I and $\lambda_n \sim^{\varphi} \lambda_m$ for all $\lambda_n, \lambda_m \in \Lambda_{\varphi}$. Let $\Omega_1, \Omega_2, \ldots$ be the disjoint equivalence classes under $R_{\lambda_0}^T$. Assume that for each $k \in \mathbb{N}$ and every $\xi, \eta \in \mathcal{M}_1$, $\alpha_i^{(\varphi^t(\xi))} = \alpha_i^{(\varphi^t(\eta))}$ for all $i \in \Omega_k$ and for all $t \geq 0$. Let $F = \sum_{j \in \mathcal{M}_1} \sum_{i \in \mathbb{N}_0} \alpha_{i,j} g_{i,j}$ with $o(F) = m_1$. If $G \in X_F$ such that $G \neq 0$ and $G = \sum_{j \in \mathcal{M}_1} \sum_{i \in \mathbb{N}_0} \beta_{i,j} g_{i,j}$, then $o(G) \geq m_1$.*

Proof. Let $F = f^{(1)} + f^{(2)} + \ldots$ be the canonical decomposition of F. Then, as in Definition 4.4.27, $o(f^{(i)}) < o(f^{(i+1)})$ for all $i \in \mathbb{N}$, and $f^{(i)} = \sum_{t \in \Omega_{g_i}} \sum_{j \in \mathcal{M}_1} \alpha_{t,j} g_{t,j}$. Let $m_i = o(f^{(i)})$ so that $m_i \in \Omega_{g_i}$ for all $i \in \mathbb{N}$; and for $j \in \mathcal{M}_1$, $\alpha_{t,j} = 0$ for all $t \in \Omega_{g_i}$, $t < m_i$. For $j \in \mathcal{M}_1$ and $i \in \mathbb{N}$, if $\alpha_{t,j} \neq 0$ for $t \in \Omega_{g_i}$, then $\alpha_t^{(\varphi^k(j))} = \alpha_{m_i}^{(\varphi^k(j))}$ for all $k \geq 0$.

Also by assumption, $\alpha_t^{(\varphi^k(j))} = \alpha_t^{(\varphi^k(m))}$ for all $t \in \mathbb{N}_0$ and $j \in \mathcal{M}_1$, where $m = \inf\{\xi : \xi \in (\mathcal{M}_1)_X\}$. Thus $\alpha_t^{(\varphi^k(j))} = \alpha_{m_i}^{(\varphi^k(m))}$ for all $t \in \Omega_{g_i}$ and $j \in \mathcal{M}_1$.

For $t \in \Omega_{g_i}$ and $j \in \mathcal{M}_1$, $k \geq 1$,

$$\begin{aligned} T_{\varphi}^k (T_{\varphi}^*)^k g_{t,j} &= \big(\alpha_t^{(j)} \alpha_t^{(\varphi(j))} \ldots \alpha_t^{(\varphi^{k-1}(j))}\big)^2 g_{t,j} \\ &= \big(\alpha_{m_i}^{(m)} \alpha_{m_i}^{(\varphi(m))} \ldots \alpha_{m_i}^{(\varphi^{k-1}(m))}\big)^2 g_{t,j} \end{aligned}$$

and so $T_\varphi^k(T_\varphi^*)^k f^{(i)} = \big(\alpha_{m_i}^{(m)}\alpha_{m_i}^{(\varphi(m))}\dots\alpha_{m_i}^{(\varphi^{k-1}(m))}\big)^2 f^{(i)}$.

For $i \in \mathbb{N}$ and $k \in \mathbb{N}_0$, let

$$\gamma_{i,k} := \begin{cases} \big(\alpha_{m_i}^{(m)}\alpha_{m_i}^{(\varphi(m))}\dots\alpha_{m_i}^{(\varphi^{k-1}(m))}\big)^2, & \text{for } k > 0; \\ 1, & \text{for } k = 0. \end{cases}$$

So $T_\varphi^k(T_\varphi^*)^k f^{(i)} = \gamma_{i,k} f^{(i)}$ for all $k \in \mathbb{N}_0$, and $i \in \mathbb{N}$. Now $G \in X_F$ implies that there exist λ_k's, not all zero, such that $G = \sum_{k\in\mathbb{N}_0}\lambda_k T_\varphi^k(T_\varphi^*)^k F$.

$$\begin{aligned}\therefore G &= \sum_{k\in\mathbb{N}_0}\lambda_k\Big(\sum_{i\in\mathbb{N}} T_\varphi^k(T_\varphi^*)^k f^{(i)}\Big)\\ &= \sum_{k\in\mathbb{N}_0}\lambda_k\Big(\sum_{i\in\mathbb{N}}\gamma_{i,k}f^{(i)}\Big)\\ &= \sum_{i\in\mathbb{N}}\Big(\sum_{k\in\mathbb{N}_0}\lambda_k\gamma_{i,k}\Big)f^{(i)}.\end{aligned}$$

Thus, $o(G) \geq o(f^{(1)}) = o(F)$. □

Theorem 4.5.3. *Let $T = T_\varphi$ be the operator pseudo shift induced by the injective map φ and with uniformly bounded invertible diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. Let T_φ be of type I and $\lambda_n \sim^\varphi \lambda_m$ for all $\lambda_n, \lambda_m \in \Lambda_\varphi$. Let $\Omega_1, \Omega_2, \dots$ be the disjoint equivalence classes under $R_{\lambda_0}^T$. Assume that for each $k \in \mathbb{N}$ and every $\xi, \eta \in \mathcal{M}_1$, $\alpha_i^{(\varphi^t(\xi))} = \alpha_i^{(\varphi^t(\eta))}$ for all $i \in \Omega_k$ and for all $t \geq 0$. Let X be a minimal reducing subspace of T_φ. If $F = \sum_{j\in\mathcal{M}_1}\sum_{i\in\mathbb{N}_0}\alpha_{i,j}g_{i,j} \in X$, then F must be transparent.*

Proof. Let, if possible, F is not transparent. Then the canonical decomposition of $F = f^{(1)} + f^{(2)} + \dots$ will have at least two components $f^{(1)}$ and $f^{(2)}$. Let $o(f^{(i)}) = m_i$. Then $o(F) = m_1$ and $m_1 \in \Omega_{g_1}$, $m_2 \in \Omega_{g_2}$.

As $\Omega_1 \cap \Omega_2 = \phi$, so there exists the smallest nonnegative integer k such that $\alpha_{m_1}^{(\varphi^k(m))} \neq \alpha_{m_2}^{(\varphi^k(m))}$, where $m = \min\{t : t \in (\mathcal{M}_1)_X\}$. For $i \in \mathbb{N}$ and $k \in \mathbb{N}_0$, let

$$\gamma_{i,k} := \begin{cases} \big(\alpha_{m_i}^{(m)}\alpha_{m_i}^{(\varphi(m))}\dots\alpha_{m_i}^{(\varphi^{k-1}(m))}\big)^2, & \text{for } k > 0; \\ 1, & \text{for } k = 0. \end{cases}$$

Then for $k \in \mathbb{N}_0$,

$$G := T_{\varphi}^{k+1}(T_{\varphi}^{*})^{k+1}F - \gamma_{1,k+1}F$$
$$= \sum_{i=2}^{\infty}(\gamma_{i,k+1} - \gamma_{1,k+1})f^{(i)} \in X.$$

Since $\gamma_{2,k+1} - \gamma_{1,k+1} \neq 0$, so $o(G) = o(f^{(2)}) = m_2$. Thus, there exists $0 \neq G \in X$ such that $o(F) < o(G)$. Also X_G is a nonzero reducing subspace of T_{φ} contained in X. So by minimality of X, we must have $X_G = X$. But this implies $F \in X_G$ so that by Lemma 4.5.2, we must have $o(F) \geq o(G)$, which is a contradiction. Thus, F must be transparent. □

Corollary 4.5.4. *Let $T = T_{\varphi}$ be the operator pseudo shift induced by the injective map φ and with uniformly bounded invertible diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. Let T_{φ} be of type I and $\lambda_n \sim^{\varphi} \lambda_m$ for all $\lambda_n, \lambda_m \in \Lambda_{\varphi}$. Let $\Omega_1, \Omega_2, \ldots$ be the disjoint equivalence classes under $R^T_{\lambda_0}$. Assume that for each $k \in \mathbb{N}$ and every $\xi, \eta \in \mathcal{M}_1$, $\alpha_i^{(\varphi^t(\xi))} = \alpha_i^{(\varphi^t(\eta))}$ for all $i \in \Omega_k$ and for all $t \geq 0$. Then the extremal function of a minimal reducing subspace of T is always transparent.*

Theorem 4.5.5. *Let $T = T_{\varphi}$ be the operator pseudo shift induced by the injective map φ and with uniformly bounded invertible diagonal operator weights $\{A_n\}_{n\in\mathbb{N}_0}$. Let T_{φ} be of type I and $\lambda_n \sim^{\varphi} \lambda_m$ for all $\lambda_n, \lambda_m \in \Lambda_{\varphi}$. Let $\Omega_1, \Omega_2, \ldots$ be the disjoint equivalence classes under $R^T_{\lambda_0}$. Assume that for each $k \in \mathbb{N}$ and every $\xi, \eta \in \mathcal{M}_1$, $\alpha_i^{(\varphi^t(\xi))} = \alpha_i^{(\varphi^t(\eta))}$ for all $i \in \Omega_k$ and for all $t \geq 0$. Let X be a nonzero reducing subspace of T_{φ}. Then X is minimal if and only if $X = X_F$, where $F \in X$ is transparent and is of the form $F = \sum_{j\in\mathcal{M}_1}\sum_{i\in\mathbb{N}_0}\alpha_{i,j}g_{i,j}$.*

Proof. Let X be minimal. Then by Corollary 4.5.4, the extremal function G of X is transparent and by minimality of X, we must have $X = X_G$. Also G has the form $G = \sum_{j\in(\mathcal{M}_1)_X}\sum_{i\in\mathbb{N}_0}\beta_{i,j}g_{i,j}$ as shown in Theorem 4.4.29.

Conversely, let $X = X_F$. Here $F = \sum_{j\in\mathcal{M}_1}\sum_{i\in\mathbb{N}_0}\alpha_{i,j}g_{i,j}$ is a transparent function.

Since X_F is a reducing subspace of T_φ, we only need to show that X_F is minimal reducing.

Let, if possible, Y be a non-zero reducing subspace of T contained in X_F. If G is the extremal function of Y, then $G \in X_F$ and so by Lemma 4.5.1, $G = \lambda F$ for some non zero scalar λ. This implies that $F \in Y$. Therefore $Y = X_F$, which shows that X_F is minimal. □

4.6 Necessary and sufficient conditions for minimality.

Theorem 4.6.1. *Let T_φ be an operator pseudo shift of type I with $\lambda_n \sim^\varphi \lambda_m$ for all $n, m \in \mathbb{N}_0$, and $F \in \ell^2_+(K)$ be transparent. Let $F = \sum_{k\in\mathbb{N}_0} f_{\hat{k}}$ be the transparent decomposition of F so that each $f_{\hat{k}}$ is jointly n_k-transparent with $n_0 < n_1 < \dots$. If for each $k \in \mathbb{N}_0$, we have $f_{\hat{k}} = \sum_{i\in\mathbb{N}_0} \beta_{i,j_k} g_{i,j_k}$ with $j_k \in \mathcal{M}_1$ and $o(f_{\hat{k}}) = r_k$, then X_F is a minimal reducing subspace of T_φ if and only if we have $\alpha_{r_k}^{(\varphi^t(j_k))} = \alpha_{r_0}^{(\varphi^t(j_0))}$ for all $t, k \in \mathbb{N}_0$.*

Proof. Let X_F be a minimal reducing subspace of T_φ. By Lemma 4.2.1 and Lemma 4.2.2, for any $t > 0$, we have

$$T_\varphi^t (T_\varphi^*)^t f_{\hat{k}} = \big(\alpha_{r_k}^{(j_k)} \alpha_{r_k}^{(\varphi(j_k))} \dots \alpha_{r_k}^{(\varphi^{t-1}(j_k))}\big)^2 f_{\hat{k}}. \tag{4.6.1}$$

To show $\alpha_{r_k}^{(\varphi^t(j_k))} = \alpha_{r_0}^{(\varphi^t(j_0))}$ for all $t, k \in \mathbb{N}_0$, we apply induction to t.

Taking $t = 1$ in Equation 4.6.1 we get $T_\varphi T_\varphi^* f_{\hat{k}} = (\alpha_{r_k}^{(j_k)})^2 f_{\hat{k}}$, and so

$$T_\varphi T_\varphi^* F - (\alpha_{r_0}^{(j_0)})^2 F = \sum_{k\in\mathbb{N}} \big[(\alpha_{r_k}^{(j_k)})^2 - (\alpha_{r_0}^{(j_0)})^2\big] f_{\hat{k}} \in X_F$$

Thus, for X_F to be a minimal reducing subspace, we must have $\alpha_{r_k}^{(j_k)} = \alpha_{r_0}^{(j_0)}$ for all $k \in \mathbb{N}_0$, showing that the result holds for $t = 0$.

Suppose the result holds for $t \le N$, that is $\alpha_{r_k}^{(\varphi^t(j_k))} = \alpha_{r_0}^{(\varphi^t(j_0))}$ for all $k \in \mathbb{N}_0$ and

$0 \leq t \leq N$. We will show that it holds for $t = N+1$.

We have

$$\begin{aligned} & T_{\varphi}^{N+2}(T_{\varphi}^{*})^{N+2}F - \big(\alpha_{r_0}^{(j_0)}\alpha_{r_0}^{(\varphi(j_0))}\dots\alpha_{r_0}^{(\varphi^{N+1}(j_0))}\big)^2 F \\ =& \sum_{k\in\mathbb{N}} \big[\big(\alpha_{r_k}^{(j_k)}\alpha_{r_k}^{(\varphi(j_k))}\dots\alpha_{r_k}^{(\varphi^{N+1}(j_k))}\big)^2 - \big(\alpha_{r_0}^{(j_0)}\alpha_{r_0}^{(\varphi(j_0))}\dots\alpha_{r_0}^{(\varphi^{N+1}(j_0))}\big)^2\big] f_{\hat{k}} \\ =& \big(\alpha_{r_0}^{(j_0)}\alpha_{r_0}^{(\varphi(j_0))}\dots\alpha_{r_0}^{(\varphi^{N}(j_0))}\big)^2 \sum_{k\in\mathbb{N}} \big[\big(\alpha_{r_k}^{\varphi^{N+1}(j_k)}\big)^2 - \big(\alpha_{r_0}^{\varphi^{N+1}(j_0)}\big)^2\big] f_{\hat{k}}. \end{aligned}$$

which is in X_F. So, for X_F to be minimal we must have $\alpha_{r_k}^{(\varphi^{N+1}(j_k))} = \alpha_{r_0}^{(\varphi^{N+1}(j_0))}$ for all $k \in \mathbb{N}_0$.

Thus by induction on t we can conclude that $\alpha_{r_k}^{(\varphi^t(j_k))} = \alpha_{r_0}^{(\varphi^t(j_0))}$ for all $t, k \in \mathbb{N}_0$. Converse follows immediately from Theorem 4.5.5. □

Theorem 4.6.2. *Let T_φ be an operator weighted pseudo shift of type I with $\lambda_n \sim^{\varphi} \lambda_m$ for all $n, m \in \mathbb{N}_0$, and $F \in \ell_+^2(K)$ be transparent. Let $F = \sum_{k\in\mathbb{N}_0} f_{\hat{k}}$ be the transparent decomposition of F so that each $f_{\hat{k}}$ is jointly n_k-transparent with $n_0 < n_1 < \dots$. If for each $k \in \mathbb{N}_0$, we have $f_{\hat{k}} = \sum_{i\in\mathbb{N}_0} \beta_{i,j_k} g_{i,j_k}$ with $j_k \in \mathcal{M}_2$ and $o(f_{\hat{k}}) = r_k$, then X_F is a minimal reducing subspace of T_φ if and only if the following conditions hold*

(i) *there exists $\mu > 0$ and $t_k \in \mathcal{M}_1$ such that $\varphi^\mu(t_k) = j_k$ for all k*

(ii) $\alpha_{r_k}^{(\varphi^t(j_k))} = \alpha_{r_0}^{(\varphi^t(j_0))}$ *for all $k \geq 0$ and $t \geq -\mu$.*

Proof. Let, if possible, there exist $\mu > \gamma > 0$ such that $\varphi^{-\gamma}(j_0) \in \mathcal{M}_1$ and $\varphi^{-\mu}(j_1) \in \mathcal{M}_1$. Then $T_\varphi^{\gamma+1} f_{\hat{0}} = 0$ and $T_\varphi^{\gamma+1} f_{\hat{1}} \neq 0$. Thus, $G = T^{\gamma+1}F$ is a linear combination of $f_{\hat{k}}$'s for $k \geq 1$. Clearly, $X_G \subseteq X_F$. Since $f_{\hat{0}} \notin X_G$, so $F \notin X_G$ and consequently X_G is a non-zero reducing subspace properly contained in X_F. Hence, in this case X_F cannot be a minimal reducing subspace. Therefore, we must have a unique $\mu > 0$ such that $\varphi^{-\mu}(j_k) \in \mathcal{M}_1$ for all $k \geq 0$.

Next we show that $\alpha_{r_k}^{(\varphi^t(j_k))} = \alpha_{r_0}^{(\varphi^t(j_0))}$ for all $k \geq 0$ and $t \geq -\mu$.

For $k, t \in \mathbb{N}_0$, the result follows exactly as in Theorem 4.6.1. To show that it holds for $-\mu \leq t < 0$, we proceed as follows:

Since by (i) there exists $\mu > 0$ and $t_k \in \mathcal{M}_1$ such that $\varphi^\mu(t_k) = j_k$ for all k, so for $0 < t \leq \mu$, by Theorem 4.2.3, we have

$$(T_\varphi^*)^t T_\varphi^t f_{\hat{k}} = (\alpha_{r_k}^{(\varphi^{-1}(j_k))} \alpha_{r_k}^{(\varphi^{-2}(j_k))} \ldots \alpha_{r_k}^{(\varphi^{-t}(j_k))})^2 f_{\hat{k}}. \tag{4.6.2}$$

Using Equation 4.6.2 we get

$$T_\varphi^* T_\varphi F - \left(\alpha_{r_0}^{\varphi^{-1}(j_0)}\right)^2 F = \sum_{k \in \mathbb{N}} \left((\alpha_{r_k}^{\varphi^{-1}(j_k)})^2 - (\alpha_{r_0}^{\varphi^{-1}(j_0)})^2\right) f_{\hat{k}} \in X_F$$

and so for X_F to be minimal we must have $\alpha_{r_k}^{\varphi^{-1}(j_k)} = \alpha_{r_0}^{\varphi^{-1}(j_0)}$ for all $k \in \mathbb{N}$.

Repeating this argument successively for $t = 2, \ldots, \mu$, we get

$$\begin{aligned} &\alpha_{r_k}^{(\varphi^{-t}(j_k))} = \alpha_{r_0}^{(\varphi^{-t}(j_0))} \text{ for all } k \in \mathbb{N}_0,\ 0 < t \leq \mu \\ \Rightarrow &\alpha_{r_k}^{(\varphi^{t}(j_k))} = \alpha_{r_0}^{(\varphi^{t}(j_0))} \text{ for all } k \in \mathbb{N}_0,\ 0 > t \geq -\mu. \end{aligned} \tag{4.6.3}$$

Conversely, we have to show that X_F is minimal reducing. Now, for each $k \in \mathbb{N}_0$,

$$\begin{aligned} T_\varphi^\mu f_{\hat{k}} &= \sum_{i \in \mathbb{N}_0} \beta_{i,j_k} T_\varphi^\mu g_{i,j_k} \\ &= \alpha_{r_0}^{(\varphi^{-1}(j_0))} \alpha_{r_0}^{(\varphi^{-2}(j_0))} \ldots \alpha_{r_0}^{(\varphi^{-\mu}(j_0))} \sum_{i \in \mathbb{N}_0} \beta_{i,\varphi^\mu(t_k)} g_{i,t_k}. \end{aligned}$$

Therefore

$$T_\varphi^\mu F = \alpha_{r_0}^{(\varphi^{-1}(j_0))} \alpha_{r_0}^{(\varphi^{-2}(j_0))} \ldots \alpha_{r_0}^{(\varphi^{-\mu}(j_0))} \sum_{t_k \in \mathcal{M}_1} \sum_{i \in \mathbb{N}_0} \beta_{i,\varphi^\mu(t_k)} g_{i,t_k}.$$

So if $F_1 := \sum_{t_k \in \mathcal{M}_1} \sum_{i \in \mathbb{N}_0} \beta_{i,\varphi^\mu(t_k)} g_{i,t_k}$ and $\delta = \alpha_{r_0}^{(\varphi^{-1}(j_0))} \alpha_{r_0}^{(\varphi^{-2}(j_0))} \ldots \alpha_{r_0}^{(\varphi^{-\mu}(j_0))}$, then $T_\varphi^\mu F = \delta F_1$ where $\delta \neq 0$. Hence $F_1 \in X_F$ which implies that $X_{F_1} \subseteq X_F$.

Similarly, we can show that $(T_\varphi^*)^\mu T_\varphi^\mu F = \delta^2 F$ which implies that $(T_\varphi^*)^\mu F_1 = \delta F$. So, $F \in X_{F_1}$ and $X_F \subseteq X_{F_1}$. Thus, $X_F = X_{F_1}$.

Let Y be a nonzero reducing subspace contained in X_F. If G is the extremal function of Y, then $G \in X_F$, which implies that $G \in X_{F_1}$. So by Lemma 4.5.1, $G = \lambda F_1$ for some non-zero scalar λ. This implies $F_1 \in Y$ which gives $Y = X_{F_1} = X_F$. Thus X_F is a minimal reducing subspace of T_φ. □

Theorem 4.6.3. *Let T_φ be an operator pseudo shift of type I with $\lambda_n \sim^{\varphi} \lambda_m$ for all $n, m \in \mathbb{N}_0$, and $F \in \ell^2_+(K)$ be transparent. Let $F = \sum_{k \in \mathbb{N}_0} f_{\hat{k}}$ be the transparent decomposition of F so that each $f_{\hat{k}}$ is jointly n_k-transparent with $n_0 < n_1 < \dots$. For each $k \in \mathbb{N}_0$ let $f_{\hat{k}} = \sum_{i \in \mathbb{N}_0} \beta_{i,j_k} g_{i,j_k}$ for some $j_k \in \mathbb{N}_0$. If there exist distinct $k_1, k_2 \in \mathbb{N}_0$ such that $j_{k_1} \in \mathcal{M}_1$ and $j_{k_2} \in \mathcal{M}_2$, then X_F cannot be a minimal reducing subspace of T_φ.*

Proof. Without loss of generality, we assume that $F = f_{\hat{k}_1} + f_{\hat{k}_2}$, where $j_{k_1} \in \mathcal{M}_1, j_{k_2} \in \mathcal{M}_2$ and $o(f_{\hat{k}_1}) = r_{k_1}, o(f_{\hat{k}_2}) = r_{k_2}$. As $j_{k_2} \in \mathcal{M}_2$, so there exists $\mu > 0$ and $t_2 \in \mathcal{M}_1$ such that $\varphi^\mu(t_2) = j_{k_2}$. Now as $T_\varphi f_{\hat{k}_1} = 0$, so $T_\varphi F = T_\varphi f_{\hat{k}_2} = \sum_{i \in \mathbb{N}_0} \beta_{i,j_{k_2}} \alpha_i^{(\varphi^{-1}(j_{k_2}))} g_{i,\varphi^{-1}(j_{k_2})}$.
Therefore, if $F_1 := T_\varphi F$, then we have

$$\begin{aligned} T_\varphi^* F_1 &= \sum_{i \in \mathbb{N}_0} \beta_{i,j_{k_2}} \big[\alpha_i^{(\varphi^{-1}(j_{k_2}))}\big]^2 g_{i,j_{k_2}} \\ &= (\alpha_{r_{k_2}}^{(\varphi^{-1}(j_{k_2}))})^2 f_{\hat{k}_2} \\ &= \delta f_{\hat{k}_2}. \end{aligned}$$

Thus we have $F_1 \in X_F$ such that $f_{\hat{k}_2} \in X_{F_1}$ and $f_{\hat{k}_1} \notin X_{F_1}$. Therefore, $F \notin X_{F_1}$ so that X_{F_1} is a proper reducing subspace of X_F. Hence, though X_F is a reducing subspace of T_φ, it cannot be a minimal reducing subspace of T_φ. □

4.7 Conclusion

Theorems 4.6.1, 4.6.2 and 4.6.3, can be summarized as the following result:

Theorem 4.7.1. *Let φ be an injective map on $\mathbb{N}_0$. Also let $\Lambda_\varphi = \{\lambda_0, \lambda_1, \dots\}$ be the φ induced partition of $\mathbb{N}_0$ such that $\lambda_n \sim^{\varphi} \lambda_m$ for all $n, m \in \mathbb{N}_0$. Let T_φ be an operator pseudo shift of type I with uniformly bounded invertible operator weights $\{A_n\}_{n \in \mathbb{N}_0}$ given by $A_n e_i = \alpha_i^{(n)} e_i$ for all $i \in \mathbb{N}_0$. Let $F \in \ell^2_+(K)$ be transparent and $F = \sum_{k \in \mathbb{N}_0} f_{\hat{k}}$ be the transparent decomposition of F so that each $F_{\hat{k}}$ is jointly*

n_k-transparent with $n_0 < n_1 < \ldots$ If $f_{\hat{k}} = \sum_{i \in \mathbb{N}_0} \beta_{i,j_k} g_{i,j_k}$ with $o(f_{\hat{k}}) = r_k$, then X_F is a minimal reducing subspace of T_φ if and only if one of the following sets of conditions hold:

(I) (i) *$j_k \in \mathcal{M}_1$ for all $k \in \mathbb{N}_0$,*

(ii) *$\alpha_{r_k}^{(\varphi^t(j_k))} = \alpha_{r_0}^{(\varphi^t(j_0))}$ for all $t, k \in \mathbb{N}_0$.*

(II)(i) *$j_k \in \mathcal{M}_2$ for all $k \in \mathbb{N}_0$,*

(ii) *there exists $\mu > 0$ and $t_k \in \mathcal{M}_1$ such that $\varphi^\mu(t_k) = j_k$ for all $k \in \mathbb{N}_0$,*

(iii) *$\alpha_{r_k}^{(\varphi^t(j_k))} = \alpha_{r_0}^{(\varphi^t(j_0))}$ for all $k \in \mathbb{N}_0$ and $t \geq -\mu$.*