Chapter 5

Operator pseudo shifts of types II and III

5.1 Introduction

In Chapter 4, we have discussed the pseudo shift operator $T_{\varphi,\{A_n\}}$ of type I on $\ell^2_+(K)$, wherein each A_n is a positive, invertible, diagonal operator on K. In the first part of this chapter, we will show that the conclusions of Theorem 4.7.1 holds even if we consider each A_n to be invertible, diagonal and not necessarily positive. In fact, we prove the following theorem:

Theorem 5.1.1. *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of invertible diagonal operators on K. Let φ be an injective map on $\mathbb{N}_0$, and $T_{\varphi,\{A_n\}}$ be the weighted pseudo shift operator on $\ell^2_+(K)$ with weights $\{A_n\}_{n\in\mathbb{N}_0}$. Then there exists a sequence of positive invertible diagonal operators $\{B_n\}_{n\in\mathbb{N}_0}$ on K such that $T_{\varphi,\{A_n\}}$ is unitarily equivalent to $T_{\varphi,\{B_n\}}$ provided the following condition holds:*
If $j \in \mathcal{M}_3$ and r is the smallest positive integer such that $\varphi^r(j) = j$, then we must have $U_jU_{\varphi(j)}\dots U_{\varphi^{r-1}(j)} = I$, where $A_k = U_kP_k$ is the polar decomposition of A_k as the product of unitary and positive operators.

In the later part of the chapter, we discuss about the reducing and minimal reducing subspaces of $T_{\varphi,\{A_n\}}$ of types II and III.

5.2 Unitary equivalence

Lemma 5.2.1. *Let U_k be a sequence of unitary operators. Let $\lambda_n \in \Lambda_\varphi$ such that $\lambda_n \in \mathcal{M}_1 \cup \mathcal{M}_2$ and $o(\lambda_n) = r$. For $j \in [[\lambda_n]]$, let*

$$V_j := \begin{cases} I, & \text{if } j = \lambda_n; \\ U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^{k-1}(\lambda_n)}, & \text{if } j = \varphi^k(\lambda_n), k > 0; \\ U^{-1}_{\varphi^{-1}(\lambda_n)} U^{-1}_{\varphi^{-2}(\lambda_n)} \dots U^{-1}_{\varphi^{k}(\lambda_n)}, & \text{if } j = \varphi^k(\lambda_n), -r \leq k < 0. \end{cases}$$

Then $V_{\varphi(j)} = V_j U_j$.

Proof. Since $j \in [[\lambda_n]]$ and $o(\lambda_n) = r$, so there exists $k \geq -r$ such that $j = \varphi^k(\lambda_n)$. Let, $\tilde{j} := \varphi(j)$. Then $\tilde{j} \in [[\lambda_n]]$ and $\tilde{j} = \varphi^{k+1}(\lambda_n)$ where $k + 1 \geq -r + 1$. Therefore

$$V_{\tilde{j}} := \begin{cases} I, & \text{if } \tilde{j} = \lambda_n; \\ U_{\lambda_n} U_{\varphi(\lambda_n)} \cdots U_{\varphi^{k}(\lambda_n)}, & \text{if } k+1 > 0; \\ U^{-1}_{\varphi^{-1}(\lambda_n)} U^{-1}_{\varphi^{-2}(\lambda_n)} \dots U^{-1}_{\varphi^{k+1}(\lambda_n)}, & \text{if } -r+1 \leq k+1 < 0. \end{cases}$$

i.e,

$$V_{\varphi(j)} := \begin{cases} I, & \text{if } k = -1; \\ U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^{k}(\lambda_n)}, & \text{if } k > -1; \\ U^{-1}_{\varphi^{-1}(\lambda_n)} U^{-1}_{\varphi^{-2}(\lambda_n)} \dots U^{-1}_{\varphi^{k+1}(\lambda_n)}, & \text{if } -r \leq k < -1. \end{cases}$$

To show $V_{\varphi(j)} = V_j U_j$.

Case I: For $-r \leq k < -1$, we have

$$\begin{aligned} V_j &= [U^{-1}_{\varphi^{-1}(\lambda_n)} U^{-1}_{\varphi^{-2}(\lambda_n)} \dots U^{-1}_{\varphi^{k+1}(\lambda_n)}] U^{-1}_{\varphi^{k}(\lambda_n)} \\ \Rightarrow V_j &= V_{\varphi(j)} U_j^{-1} \\ \Rightarrow V_{\varphi(j)} &= V_j U_j. \end{aligned}$$

Case II: For $k = -1$, we have $j = \varphi^{-1}(\lambda_n)$, $V_j = U^{-1}_{\varphi^{-1}(\lambda_n)} = U_j^{-1}$ and $V_{\varphi(j)} = I$. Therefore $V_{\varphi(j)} = I = V_j U_j$.

Case III: For $k = 0$, we have $j = \lambda_n$, $V_j = I$ and $V_{\varphi(j)} = U_{\lambda_n}$. Therefore $V_{\varphi(j)} = I U_{\lambda_n} = V_j U_j$.

Case IV: For $k > 0$, we have

$$V_{\varphi(j)} = [U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^{k-1}(\lambda_n)}] U_{\varphi^{k}(\lambda_n)} = V_j U_j.$$

Thus for all $j \in [[\lambda_n]]$, we have $V_{\varphi(j)} = V_j U_j$. □

Lemma 5.2.2. *Let $\{U_k\}$ be a sequence of unitary operators on K, and let $\lambda_n \in \Lambda_\varphi \cap \mathcal{M}_4$. For $j \in [[\lambda_n]]$, let*

$$V_j := \begin{cases} I, & \text{if } j = \lambda_n; \\ U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^{k-1}(\lambda_n)}, & \text{if } j = \varphi^k(\lambda_n), k > 0; \\ U^{-1}_{\varphi^{-1}(\lambda_n)} U^{-1}_{\varphi^{-2}(\lambda_n)} \dots U^{-1}_{\varphi^{k}(\lambda_n)}, & \text{if } j = \varphi^k(\lambda_n), k < 0. \end{cases}$$

Then $V_{\varphi(j)} = V_j U_j$ for all $j \in [[\lambda_n]]$.

The proof being similar to that of Lemma 5.2.1 is omitted.

Lemma 5.2.3. *Let $\{U_k\}$ be a sequence of unitary operators on K, and let $\lambda_n \in \Lambda_\varphi \cap \mathcal{M}_3$ with $o(\lambda_n) = r$. Let $j \in [[\lambda_n]] = \{\lambda_n \varphi(\lambda_n) \dots \varphi^{r-1}(\lambda_n)\}$. Let*

$$V_j := \begin{cases} I, & \text{if } j = \lambda_n; \\ U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^{k-1}(\lambda_n)}, & \text{if } j = \varphi^k(\lambda_n), 0 < k < r; \end{cases}$$

Then $V_{\varphi(j)} = V_j U_j$ if and only if $U_j U_{\varphi(j)} \dots U_{\varphi^{r-1}(j)} = I$ for all $j \in [[\lambda_n]]$.

Proof. If $r = 1$, then $[[\lambda_n]] = \{\lambda_n\}$. So $j \in [[\lambda_n]]$ implies that $\varphi(j) = j = \lambda_n$ and hence $V_{\varphi(j)} = V_j U_j$ if and only if $U_j = I$.

Now suppose $r > 1$. Then for $j \in [[\lambda_n]]$, there exists an integer k where $0 \leq k \leq r-1$ such that $j = \varphi^k(\lambda_n)$. Let $\tilde{j} := \varphi(j)$ so that $\tilde{j} = \varphi^{k+1}(\lambda_n)$.

Case I: For $0 \leq k < r-1$, we have $\tilde{j} = \varphi^{k+1}(\lambda_n)$ where $1 \leq k+1 \leq r-1$ and so $V_{\tilde{j}} = U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^k(\lambda_n)}$. i.e,

$$V_{\varphi(j)} = [U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^{k-1}(\lambda_n)}] U_{\varphi^k(\lambda_n)} = V_j U_j.$$

Case II: For $k = r-1$, we have $j = \varphi^{r-1}(\lambda_n)$ and $\tilde{j} = \varphi(j) = \varphi^r(\lambda_n) = \lambda_n$ (since $o(\lambda_n) = r$.) Therefore $V_j = U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^{r-2}(\lambda_n)}$ and $V_{\tilde{j}} = I$. Therefore $V_{\tilde{j}} = V_j U_j$ if and only if $I = U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^{r-2}(\lambda_n)} U_{\varphi^{r-1}(\lambda_n)}$. □

The proof of Theorem 5.1.1:

Proof. Let us denote $T_{\varphi,\{A_n\}}$ simply as T_φ. Then

$$\begin{aligned} T_\varphi(x_0, x_1, \cdots) &= (A_0 x_{\varphi(0)}, A_1 x_{\varphi(1)}, \dots) \\ &= (A_{\varphi^{-1}(\varphi(0))} x_{\varphi(0)}, A_{\varphi^{-1}(\varphi(1))} x_{\varphi(1)}, \dots). \end{aligned}$$

For each $k \in \mathbb{N}_0$, let $A_k = U_k P_k$ be the polar decomposition of A_k as unitary operator U_k and positive operator P_k respectively. Also for $j \in \mathbb{N}_0$, let

$$\tilde{A}_j := \begin{cases} A_{\varphi^{-1}(j)}, & \text{if } j \in R(\varphi); \\ I, & \text{otherwise.} \end{cases}$$

$$\tilde{U}_j := \begin{cases} U_{\varphi^{-1}(j)}, & \text{if } j \in R(\varphi); \\ I, & \text{otherwise.} \end{cases}$$

$$\tilde{P}_j := \begin{cases} P_{\varphi^{-1}(j)}, & \text{if } j \in R(\varphi); \\ I, & \text{otherwise.} \end{cases}$$

Then for each $j \in \mathbb{N}_0$, $\tilde{A}_{\varphi(j)} = A_j$, $\tilde{U}_{\varphi(j)} = U_j$, and $\tilde{P}_{\varphi(j)} = P_j$.

$$\tilde{U}_j \tilde{P}_j := \begin{cases} A_{\varphi^{-1}(j)}, & \text{if } j \in R(\varphi); \\ I, & \text{otherwise,} \end{cases}$$

which implies $\tilde{U}_j \tilde{P}_j = \tilde{A}_j$.

Define $\tilde{W}_+$ on $\ell^2_+(K)$ as $\tilde{W}_+(x_0, x_1, \cdots) = (x_{\varphi(0)}, x_{\varphi(1)}, \dots)$. Then

$$\begin{aligned} \tilde{W}_+(\tilde{A}_0 x_0, \tilde{A}_1 x_1, \dots) &= (\tilde{A}_{\varphi(0)} x_{\varphi(0)}, \tilde{A}_{\varphi(1)} x_{\varphi(1)}, \dots) \\ &= (A_0 x_{\varphi(0)}, A_1 x_{\varphi(1)}, \dots) \\ &= T_\varphi(x_0, x_1, \dots). \end{aligned}$$

If $U(x_0, x_1, \dots) = (\tilde{U}_0 x_0, \tilde{U}_1 x_1, \dots)$, and $P(x_0, x_1, \dots) = (\tilde{P}_0 x_0, \tilde{P}_1 x_1, \dots)$, then

$$\begin{aligned} UP(x_0, x_1, \dots) &= (\tilde{U}_0 \tilde{P}_0 x_0, \tilde{U}_1 \tilde{P}_1 x_1, \dots) \\ &= (\tilde{A}_0 x_0, \tilde{A}_1 x_1, \dots). \end{aligned}$$

Therefore $\tilde{W}_+ UP = T_\varphi$.

Let $j \in \mathbb{N}_0$. Then there exists $\lambda_n \in \Lambda_\varphi$ such that $j \in [[\lambda_n]]$.

(i) If $\lambda_n \in \mathcal{M}_1 \cup \mathcal{M}_2$ with $o(\lambda_n) = r$, then there exists $k \geq -r$ such that $\varphi^k(\lambda_n) = j$.

Let

$$V_j := \begin{cases} I, & \text{if } k = 0; \\ U_{\lambda_n} U_{\varphi(\lambda_n)} \cdots U_{\varphi^{k-1}(\lambda_n)}, & \text{if } k > 0; \\ U^{-1}_{\varphi^{-1}(\lambda_n)} U^{-1}_{\varphi^{-2}(\lambda_n)} \dots U^{-1}_{\varphi^{k}(\lambda_n)}, & \text{if } -r \leq k < 0. \end{cases}$$

(ii) If $\lambda_n \in \mathcal{M}_3$ with $o(\lambda_n) = r$, then $j \in \{\lambda_n, \varphi(\lambda_n), \dots, \varphi^{r-1}(\lambda_n)\}$. Let

$$V_j := \begin{cases} I, & \text{if } j = \lambda_n; \\ U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^{k-1}(\lambda_n)}, & \text{if } j = \varphi^k(\lambda_n) \text{ for } 0 < k < r; \end{cases}$$

(iii) If $\lambda_n \in \mathcal{M}_4$, then $o(\lambda_n) = \infty$. So there exists $k \in \mathbb{Z}$ such that $\varphi^k(\lambda_n) = j$.

$$V_j := \begin{cases} I, & \text{if } k = 0; \\ U_{\lambda_n} U_{\varphi(\lambda_n)} \dots U_{\varphi^{k-1}(\lambda_n)}, & \text{if } k > 0; \\ U^{-1}_{\varphi^{-1}(\lambda_n)} U^{-1}_{\varphi^{-2}(\lambda_n)} \dots U^{-1}_{\varphi^{k}(\lambda_n)}, & \text{if } k < 0. \end{cases}$$

Clearly, $\{V_n\}_{n \in \mathbb{N}_0}$ is a sequence of unitary operators on K and the operator V defined as $V(x_0, x_1, \dots) := (V_0 x_0, V_1 x_1, \dots)$ is a unitary operator on $\ell^2_+(K)$. Now

$$\begin{aligned} \tilde{W}_+ U(x_0, x_1, \dots) &= \tilde{W}_+(\tilde{U}_0 x_0, \tilde{U}_1 x_1, \dots) \\ &= (\tilde{U}_{\varphi(0)} x_{\varphi(0)}, \tilde{U}_{\varphi(1)} x_{\varphi(1)}, \dots) \\ &= (U_0 x_{\varphi(0)}, U_1 x_{\varphi(1)}, \dots), \text{and} \end{aligned}$$

$$\begin{aligned} V^* \tilde{W}_+ V(x_0, x_1, \dots) &= V^*(V_{\varphi(0)} x_{\varphi(0)}, V_{\varphi(1)} x_{\varphi(1)}, \dots) \\ &= (V_0^* V_{\varphi(0)} x_{\varphi(0)}, V_1^* V_{\varphi(1)} x_{\varphi(1)}, \dots). \end{aligned}$$

Thus, $V^* \tilde{W}_+ V = \tilde{W}_+ U$ if and only if $V_k^* V_{\varphi(k)} = U_k$ for all $k \in \mathbb{N}_0$ i.e, if $V_{\varphi(k)} = V_k U_k$ for all $k \in \mathbb{N}_0$.

By Lemmas 5.2.1 and 5.2.2, we have $V_{\varphi(k)} = V_k U_k$ holds for all $k \in \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_4$, and for $k \in \mathcal{M}_3$, by Lemma 5.2.3, $V_{\varphi(k)} = V_k U_k$ holds if and only if $U_k U_{\varphi(k)} \dots U_{\varphi^{r-1}(k)} = I$, where r is the smallest positive integer such that $\varphi^r(k) = k$.

For $n \in \mathbb{N}_0$, let $D_n := V_n P_n V_n^*$. Then $\langle D_n x, x\rangle = \langle P_n V_n^* x, V_n^* x\rangle \geq 0$ for all $x \in K$. This implies $D_n \geq 0$. Also P_n is invertible diagonal and V_n is unitary implies each D_n is diagonal and invertible. Let $T = \tilde{W}_+ V P V^*$. Then

$$\begin{aligned} T(x_0, x_1, \dots) &= \tilde{W}_+(D_0 x_0, D_1 x_1, \dots) \\ &= (D_{\varphi(0)} x_{\varphi(0)}, D_{\varphi(1)} x_{\varphi(1)}, \dots). \end{aligned}$$

So if $B_n := D_{\varphi(n)}$, then $\{B_n\}_{n\in\mathbb{N}_0}$ is a sequence of positive invertible diagonal operators and $T = T_{\varphi,\{B_n\}}$. Also,

$$T_\varphi = (\tilde{W}_+ U)P = (V^*\tilde{W}_+ V)P = V^*(\tilde{W}_+ VPV^*)V = V^*TV.$$

As V is unitary, so T_φ is unitarily equivalent to $T = T_{\varphi,\{B_n\}}$. □

5.3 Reducing subspace of bilateral weighted shift W on $\ell^2(\mathbb{C})$

Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a bounded sequence of non-zero scalars, and W be the bilateral shift on $\ell^2(\mathbb{C})$ with weight sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$. Then for $x = (\dots, x_{-1}, [x_0], x_1, \dots) \in \ell^2(\mathbb{C})$, $Wx := (\dots, \lambda_{-2}x_{-2}, [\lambda_{-1}x_{-1}], \lambda_0 x_0, \dots)$, and so $W = U\Lambda$, where U is the unweighted bilateral shift and Λ is the diagonal operator with diagonal entries $\{\lambda_n\}_{n\in\mathbb{Z}}$.

The reducing subspaces of W have been studied in [40]. In Theorem 5.3.1 and Theorem 5.3.3 below, we restate Theorem 4 of [40] for future reference.

Theorem 5.3.1. *Let W be the scalar weighted bilateral shift on $\ell^2(\mathbb{C})$ with non-zero weight sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$. Then the following are equivalent:*
(i) *W has non-trivial reducing subspaces.*
(ii) *The set $\mathbb{Z}$ divides up into finitely many arithmetic progressions $\mathbb{Z}_1, \mathbb{Z}_2, \dots, \mathbb{Z}_n$, on each of which $|\lambda_p|$ ($p \in \mathbb{Z}_i$, $i = 1, 2, \dots, n$) is constant.*
(iii) *There exists a natural number m such that $|T|^m = rU^m$, $r > 0$, where $|T| = U|\Lambda|$, and $|\Lambda|$ is the operator of multiplication by the sequence $\{|\lambda_n|\}_{n\in\mathbb{Z}}$.*

Remark 5.3.2. To say $\mathbb{Z}$ divides up into finitely many arithmetic progressions $\mathbb{Z}_1, \mathbb{Z}_2, \dots, \mathbb{Z}_n$ means $\mathbb{Z} = \bigcup_{i=1}^n \mathbb{Z}_i$, where $\mathbb{Z}_i = \{kn + i : k \in \mathbb{Z}\}$.

Theorem 5.3.3. *Let $W = U\Lambda$ be the scalar weighted bilateral shift on $\ell^2(\mathbb{C})$ with weight sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$. Let $\mathbb{Z}_1, \mathbb{Z}_2, \dots, \mathbb{Z}_n$ be disjoint arithmetic progressions with* *difference n and let $\mathbb{Z} = \bigcup_{i=1}^{n} \mathbb{Z}_i$. If $|\lambda_k| = r_i$ for all $k \in \mathbb{Z}_i$, $r_i > 0$, $i = 1, 2, \ldots, n$, then the following are equivalent:*

(i) *The space $H \subset \ell^2(\mathbb{C})$ reduces the operator W.*

(ii) *$H = MH_0$, where $H_0 = \sum_{i=1}^{n} P_{\mathbb{Z}_i} H_0 \equiv \sum_{i=1}^{n} \oplus H_i$ and $UH_i = H_{i+1}$ $(1 \leq i \leq n-1)$, $UH_n = H_1$, $H_1 = \{\{a_i\}_{i\in\mathbb{Z}} : a_{nk} = \tilde{a}_k, a_i = 0 \text{ if } i \neq nk, k \in \mathbb{Z}, \text{ and } \{\tilde{a}_k\}_{k\in\mathbb{Z}} \in \tilde{H}\}$ and $\tilde{H} \subset \ell^2(\mathbb{C})$, $U\tilde{H} = \tilde{H}$.*

Remark 5.3.4. $P_{\mathbb{Z}_i}$ is the projection of $\ell^2(\mathbb{C})$ onto H_i, which is the closed linear span of $\{e_k\}_{k\in\mathbb{Z}_i}$, where $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $\ell^2(\mathbb{C})$. Also M is the operator of multiplication by the sequence $\{m_k\}_{k\in\mathbb{Z}}$ in $\ell^2(\mathbb{C})$, where $m_0 := 1$,

$$m_k := \frac{\lambda_0\lambda_1 \ldots \lambda_{k-1}}{|\lambda_0\lambda_1 \ldots \lambda_{k-1}|} \text{ if } k \geq 1, \text{ and}$$
$$m_k := \frac{\lambda_{-1}\lambda_{-2} \ldots \lambda_k}{|\lambda_{-1}\lambda_{-2} \ldots \lambda_k|} \text{ if } k < 0.$$

Definition 5.3.5. Let $\mathcal{S}$ be the vector space of all finite linear combinations of finite products of the operators W and W^*. For any non-zero $x \in \ell^2(\mathbb{C})$, $\mathcal{S}x := \{Tx : T \in \mathcal{S}\}$. Then the closure of $\mathcal{S}x$ in $\ell^2(\mathbb{C})$ is a reducing subspace of W, and is denoted by X_x. X_x is called the subspace generated by x. Clearly, it is the smallest reducing subspace of W containing x.

Definition 5.3.6. Let $x \in \ell^2(\mathbb{C})$ and $\{e_j\}_{j\in\mathbb{Z}}$ be an orthonormal basis for $\ell^2(\mathbb{C})$. Then there exists scalars $\{\alpha_j\}_{j\in\mathbb{Z}}$ such that $x = \sum_{j\in\mathbb{Z}} \alpha_j e_j$. If there exists integers n_1 and n_2, $n_1 \leq n_2$ such that $\alpha_{n_1} \neq 0$, $\alpha_{n_2} \neq 0$ and $\alpha_j = 0$ for $j < n_1$ and $j > n_2$, then we define the length of x as $n_2 - n_1 + 1$. Otherwise length of x is defined as ∞. Length of x is denoted as $l(x)$.

Now, let us consider a bilateral shift on $\ell^2(\mathbb{C})$ with positive weight sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$. In view of Theorems 5.3.1 and 5.3.3, W has a proper reducing subspace if and only if there exists a positive integer n such that $\mathbb{Z} = \cup_{i=1}^{n} \mathbb{Z}_i$ and $\mathbb{Z}_i = \{kn + i : k \in \mathbb{Z}\}$ for all $1 \leq i \leq n$, and there also exists $r_i > 0$ $(1 \leq i \leq n)$ such that $\lambda_p = r_i$ for all $p \in \mathbb{Z}_i$.

In this case, we have $W^n = rU^n$ where $r = r_1 \dots r_n$. Moreover, a subspace M of $\ell^2(\mathbb{C})$ is reducing for W if and only if $M = M_1 \oplus \cdots \oplus M_n$, where $M_i = P_{\mathbb{Z}_i} M$ for all i.

We make the following observations:

(1) $U(M_i) = M_{i+1}$ for $1 \leq i \leq n-1$ and $U(M_n) = M_1$.

(2) If $x \in M_i$, then

$$\begin{aligned} &Wx = r_i U x \in M_{i+1}, \\ &W^2 x = r_i r_{i+1} U^2 x \in M_{i+2}, \\ &\cdot \quad \cdot \quad \cdot \\ &W^n x = r U^n x \in M_i. \text{ Similarly,} \end{aligned}$$

$$\begin{aligned} &W^* x = r_{i-1} U^* x \in M_{i-1}, \\ &(W^*)^2 x = r_{i-1} r_{i-2} (U^*)^2 x \in M_{i-2}, \\ &\cdot \quad \cdot \quad \cdot \\ &(W^*)^n x = r (U^*)^n x \in M_i. \end{aligned}$$

Thus, $(W^*)^n W^n x = r^2 x = W^n (W^*)^n x$ for $x \in M_i$ $(1 \leq i \leq n)$, and $(W^*)^{kn} W^{kn} x = r^{2k} x = W^{kn} (W^*)^{kn} x$ for all $k \in \mathbb{Z}$, $x \in M_i$ $(1 \leq i \leq n)$.

(3) If $r_i = \lambda$ for all $i \in \mathbb{Z}$, i.e, if $We_i = \lambda e_{i+1}$ for all $i \in \mathbb{Z}$, then W will have no eigen values.

To show this, let if possible, μ be an eigen value of W. Then, there exists a non zero vector x in $\ell^2(\mathbb{C})$ such that $Wx = \mu x$. So if $x = (\dots, x_{-1}, [x_0], x_1, \dots)$, then $Wx = \mu x$ implies $x_{i+1} = \frac{\lambda}{\mu} x_i$ for all $i \in \mathbb{Z}$. Without loss of generality, we can assume

$x_0 \neq 0$. Therefore

$$\|x\|^2 = \sum_{i=-\infty}^{\infty} |x_i|^2 = |x_0|^2 (\sum_{i=0}^{\infty} |\frac{\lambda}{\mu}|^{2i} + \sum_{i=1}^{\infty} |\frac{\mu}{\lambda}|^{2i}) \not< \infty.$$

This gives us a contradiction.

(4) As above, for any positive integer k, it can be shown that W^k also does not have any eigen value. This also means that for non zero $x \in \ell^2(\mathbb{C})$, $W^k x$ and x are always linearly independent.

(5) If $x = x_1 \oplus x_2 \oplus \cdots \oplus x_n \in M$ with $x_i \in M_i$ for all $1 \leq i \leq n$, then $x_i \in M$ for each i.

If all the r_i's are equal then $n = 1$ and so M cannot be reducing for W. Hence, we cannot have all r_i's equal.

If $n = 2$, then we must have $r_1 \neq r_2$. Therefore, $r_2^2 x - W^*Wx = (r_2^2 - r_1^2)x_1 \in M$, so that $x_1 \in M$. Similarly, $r_1^2 x - W^*Wx = (r_1^2 - r_2^2)x_2$ implies $x_2 \in M$.

If $n = 3$, then we may have two situations:

Case I. $r_i \neq r_j$ if $i \neq j$. Then

$$y := r_3^2 x - W^*Wx = (r_3^2 - r_1^2)x_1 + (r_3^2 - r_2^2)x_2 \in M, \text{ and}$$
$$r_2^2 y - W^*Wy = (r_2^2 - r_1^2)(r_3^2 - r_1^2)x_1 \in M.$$

This implies $x_1 \in M$. Similarly, $x_2, x_3 \in M$.

Case II. $r_2 = r_3$ and $r_1 \neq r_3$. Then $y = r_3^2 x - W^*Wx = (r_3^2 - r_1^2)x_1 \in H$. This implies $x_1 \in H$. Therefore $x - x_1 = x_2 + x_3 \in H$. Again,

$$\begin{aligned}(W^*)^2 W^2 (x_2 + x_3) &= r_2^2 r_3^2 x_2 + r_3^2 r_1^2 x_3 \\ &= r_2^2 (r_3^2 x_2 + r_1^2 x_3) \in H.\end{aligned}$$

Therefore, $r_2^2 r_3^2 (x_2 + x_3) - (W^*)^2 W^2 (x_2 + x_3) = r_2^2 (r_3^2 - r_1^2) x_3 \in H$. This implies $x_3 \in H$ and so $x_2 \in H$. Thus each $x_i \in H$. The cases for ($r_3 = r_1$ and $r_1 \neq r_2$) and

($r_1 = r_2$ and $r_1 \neq r_3$) can be similarly settled.

The cases for $n > 3$ can be similarly proved.

(6) If $x \in M_1$ such that $l(x) < \infty$, then

(i) M_1 is not minimal reducing for W^n.

(ii) M_1 does not contain any minimal reducing subspace for W^n.

To show this, let $y := x + U^n x$. Also,

$$\tilde{M}_x := span\{W^{kn}(W^*)^{tn}x, (W^*)^{kn}W^{tn}x : t, k \in \mathbb{N}_0\}, \text{ and}$$

$$\tilde{M}_y := span\{W^{kn}(W^*)^{tn}y, (W^*)^{kn}W^{tn}y : t, k \in \mathbb{N}_0\}.$$

Then $y \in \tilde{M}_x$ which implies $\tilde{M}_y \subseteq \tilde{M}_x$.

Again, $l(y) = l(x) + n$ and for each $z \in \tilde{M}_y$, $l(z) \geq l(y)$. Therefore, $x \notin \tilde{M}_y$, which implies $\tilde{M}_y \subsetneqq \tilde{M}_x \subseteq M_1$.

In this context, we raise the following question:

Question: Let $x \in \ell^2(\mathbb{C})$ such that $l(x) = \infty$. If $y := x + Ux$, then $X_y \subseteq X_x$. Is $x \in X_y$?

An answer to this question would enable us to comment on the existence of minimal reducing subspaces of W on $\ell^2(\mathbb{C})$. If the answer is "no", i.e, if $x \notin X_y$, then X_x is not a minimal reducing subspace of W. This together with the observations made above will imply that W does not have any minimal reducing subspace in $\ell^2(\mathbb{C})$. However, if the answer to the above question is "yes", i.e, if $x \in X_y$, then X_x could possibly be a minimal reducing subspace of W. It is to be mentioned that we could not answer the above question. Hence, it remains open.

5.4 Minimal reducing subspaces for T_φ of type II.

Let $\lambda_n \in \Lambda_\varphi \cap \mathcal{M}_4$ and $K_n := span\{g_{i,j} : i \in \mathbb{N}_0, j \in [[\lambda_n]]\}$. By Theorem 4.3.13, $T_\varphi|_{K_n}$ is unitarily equivalent to the bilateral backward operator weighted shift on $\ell^2(K)$. Let us denote $T_\varphi|_{K_n}$ as $W^{[\lambda_n]}$.

The reducing subspaces of the operator weighted bilateral shift W on $\ell^2(K)$ are discussed in [44] and [17]. Guyker proved the following result:

Theorem 5.4.1. *Let $\{A_n\}_{n\in\mathbb{Z}}$ be a commuting family of compact, normal operators with dense range. Let W be the bilateral shift on $\ell^2(K)$ with operator weights $\{A_n\}_{n\in\mathbb{Z}}$. Then W is unitarily equivalent to a countable direct sum $\sum_{n\in\mathbb{N}_0} \oplus W_n$ of bilateral weighted shifts W_n on $\ell^2(\mathbb{C})$ with non-zero scalar weights. Moreover a subspace M of $\ell^2(K)$ reduces $\sum_{n\in\mathbb{N}_0} \oplus W_n$ if and only if $M = \sum_{n\in\mathbb{N}_0} \oplus M_n$, where M_n reduces W_n for every $n \in \mathbb{N}_0$.*

So, by the above theorem, $W^{[\lambda_n]}$ is unitarily equivalent to a countable direct sum $\sum_{i\in\mathbb{N}_0} \oplus W_i^{[\lambda_n]}$ of bilateral weighted shifts $W_i^{[\lambda_n]}$ on $\ell^2(\mathbb{C})$ with non-zero scalar weights. Also, a subspace M of $\ell^2(K)$ reduces $\sum_{i\in\mathbb{N}_0} \oplus W_i^{[\lambda_n]}$ if and only if $M = \sum_{i\in\mathbb{N}_0} \oplus M_i$, where M_i reduces $W_i^{[\lambda_n]}$ for every $i \in \mathbb{N}_0$.

Following the above notation, we now propose the following theorem:

Theorem 5.4.2. *Let, $M = \sum_{i\in\mathbb{N}_0} \oplus M_i$ be a reducing subspace of $W^{[\lambda_n]}$. Also, let $x = \sum_{i\in\mathbb{N}_0} x_i$ be in M, where each $x_i \in M_i$, and there exists some $i \in \mathbb{N}_0$ such that $l(x_i) < \infty$. Then, M cannot be minimal.*

Proof. By Theorem 5.4.1, each $M_i \subseteq \ell^2(\mathbb{C})$ is a reducing subspace for $W_i^{[\lambda_n]}$. Suppose, there exists some $i \in \mathbb{N}_0$ such that $l(x_i) < \infty$. Then from the observation 6 made in Section 5.3, we can say that the reducing subspace M_i is not minimal reducing. So there exists a reducing subspace $N_i \subsetneqq M_i$. Also, let $N_j = M_j$ for all

$j \in \mathbb{N}_0 - \{i\}$, and $N = \sum_{j\in\mathbb{N}_0} \oplus N_j$. Clearly, $N \subsetneq M$, and N is reducing for $W^{[\lambda_n]}$. Hence, M cannot be a minimal reducing subspace.

□

Remark 5.4.3. In the above theorem, if for each $x = \sum_{i\in\mathbb{N}_0} x_i$ in M, we have $l(x_i) = \infty$ for all $i \in \mathbb{N}_0$, then whether M is a minimal reducing subspace or not still remains unresolved.

5.5 Minimal reducing subspaces for T_φ of type III.

Let $\lambda \in \Lambda_\varphi \cap \mathcal{M}_3$, and $o(\lambda) = r$. Also, let K_λ be the closed linear span of $\{g_{i,j} : i \in \mathbb{N}_0, j \in [[\lambda]]\}$. Then, by Theorem 4.3.12, we have $T_\varphi|_{K_\lambda}$ is a weighted circulant operator on $H_r = K \oplus \cdots \oplus K$ (r copies).

Again, for $i \in \mathbb{N}_0$, let $K_\lambda^{(i)}$ be the closed linear span of $\{g_{i,j} : j \in [[\lambda]]\}$. Then, $K_\lambda = \sum_{i\in\mathbb{N}_0} \oplus K_\lambda^{(i)}$. As $[[\lambda]] = \{\lambda, \varphi(\lambda), \varphi^2(\lambda), \ldots, \varphi^{r-1}(\lambda)\}$ with $\varphi^r(j) = j$ for all $j \in [[\lambda]]$, and $A_n = (\alpha_i^{(n)})_{i\in\mathbb{N}_0}$, so we have $T_\varphi g_{i,j} = \alpha_i^{(\varphi^{-1}(j))} g_{i,\varphi^{-1}(j)}$ for all $j \in [[\lambda]]$.

If for $i \in \mathbb{N}_0$, we define $C_i : K_\lambda^{(i)} \to K_\lambda^{(i)}$ as $C_i g_{i,j} = \alpha_i^{(\varphi^{-1}(j))} g_{i,\varphi^{-1}(j)}$ for all $j \in [[\lambda]]$, then each C_i is a weighted circulant operator on $K_\lambda^{(i)}$. Also, then $T_\varphi|_{K_\lambda}$ is unitarily equivalent to the countable direct sum $\sum_{i\in\mathbb{N}_0} \oplus C_i$.

5.5.1 For $\lambda \in \mathcal{M}_3$, $o(\lambda) = 1$

If $o(\lambda) = 1$, then $\varphi(\lambda) = \lambda$, and so for each $i \in \mathbb{N}_0$, we have $C_i g_{i,\lambda} = \alpha_i^{(\lambda)} g_{i,\lambda}$, so that C_i is irreducible, since the only reducing subspaces of C_i are the trivial ones.

Theorem 5.5.1. *Let $f \in K_\lambda$ such that $f = \sum_{i\in\Lambda} \xi_i g_{i,\lambda}$ for a subset Λ of $\mathbb{N}_0$. Then, X_f is minimal reducing for T_φ if and only if $\alpha_i^{(\lambda)} = \alpha_j^{(\lambda)}$ for all $i, j \in \Lambda$.*

Proof. We have, $T_\varphi f = \sum_{i\in\Lambda} \xi_i \alpha_i^{(\lambda)} g_{i,\lambda} = T_\varphi^* f$.
Condition is sufficient: If $\alpha_i^{(\lambda)} = \alpha_j^{(\lambda)} = \eta$ for all $i, j \in \Lambda$, then $T_\varphi f = \eta f = T_\varphi^* f$, and so $X_f = span\{f\}$. Let Y be a non zero reducing subspace of T_φ such that $Y \subseteq X_f$. Then, for $0 \neq y \in Y$, there exists a scalar ξ such that $y = \xi f$. This implies $f = \frac{1}{\xi} y \in Y$, and hence $Y = X_f$. Thus, X_f is minimal reducing for T_φ.
Condition is necessary: If $i, j \in \Lambda$ such that $\alpha_i^{(\lambda)} \neq \alpha_j^{(\lambda)}$, then let $\Lambda^{(i)} = \{t \in \Lambda : \alpha_i^{(\lambda)} \neq \alpha_t^{(\lambda)}\}$. Thus,

$$\alpha_i^{(\lambda)} f - T_\varphi f = \sum_{t\in\Lambda^{(i)}} (\alpha_i^{(\lambda)} - \alpha_t^{(\lambda)}) g_{t,\lambda} = h \,(\text{say}).$$

Thus, $X_h \subsetneqq X_f$, and so X_f is not minimal for T_φ. □

5.5.2 For $\lambda \in \mathcal{M}_3$, $o(\lambda) = 2$

In this case, $[[\lambda]] = \{\lambda, \varphi(\lambda)\}$.

Lemma 5.5.2. *Let $\lambda \in \mathcal{M}_3$ with $o(\lambda) = 2$. For $i \in \mathbb{N}_0$, let $x = ag_{i,\lambda} + bg_{i,\varphi(\lambda)}$ with $a, b \neq 0$. Then X_x is a minimal reducing subspace for C_i if and only if $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))}$ and $a^2 = b^2$.*

Proof. We have the following relations:

$$\begin{aligned} C_i x =& a\alpha_i^{(\varphi^{-1}(\lambda))} g_{i,\varphi^{-1}(\lambda)} + b\alpha_i^{(\lambda)} g_{i,\lambda} \\ =& a\alpha_i^{(\varphi(\lambda))} g_{i,\varphi(\lambda)} + b\alpha_i^{(\lambda)} g_{i,\lambda}. \end{aligned} \tag{5.5.1}$$

$$C_i^* x = a\alpha_i^{(\lambda)} g_{i,\varphi(\lambda)} + b\alpha_i^{(\varphi(\lambda))} g_{i,\lambda}. \tag{5.5.2}$$

$$C_i^* C_i x = a(\alpha_i^{(\varphi(\lambda))})^2 g_{i,\lambda} + b(\alpha_i^{(\lambda)})^2 g_{i,\varphi(\lambda)}. \tag{5.5.3}$$

Case I. Let $\alpha_i^{(\lambda)} \neq \alpha_i^{(\varphi(\lambda))}$. Then

$$(\alpha_i^{(\lambda)})^2 x - C_i^* C_i x = a((\alpha_i^{(\varphi(\lambda))})^2 - (\alpha_i^{(\lambda)})^2) g_{i,\lambda}.$$

This implies $g_{i,\lambda} \in X_x$. Therefore $g_{i,\varphi(\lambda)} \in X_x$ so that $K_\lambda^{(i)} = X_x$, and hence X_x is not a proper reducing subspace for C_i.

Case II. Let $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))}$, but $a^2 \neq b^2$. Then,

$$x - \frac{b}{a}[\alpha_i^{(\lambda)}\alpha_i^{(\varphi(\lambda))}]^{-\frac{1}{2}}C_i x = (\frac{a^2-b^2}{a})g_{i,\lambda},$$

which implies $g_{i,\lambda} \in X_x$, and hence again we have $K_\lambda^{(i)} = X_x$.

Case III. Let $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))}$, and $a^2 = b^2$. Then,

$$x - \frac{b}{a}[\alpha_i^{(\lambda)}\alpha_i^{(\varphi(\lambda))}]^{-\frac{1}{2}}C_i x = (\frac{a^2-b^2}{a})g_{i,\lambda} = 0$$
$$\Rightarrow C_i x = \frac{a}{b}[\alpha_i^{(\lambda)}\alpha_i^{(\varphi(\lambda))}]^{\frac{1}{2}}x.$$

Similarly, $C_i^* x = \frac{a}{b}[\alpha_i^{(\lambda)}\alpha_i^{(\varphi(\lambda))}]^{\frac{1}{2}}x$. Hence, $X_x = span\{x\}$ and so X_x is a minimal reducing subspace of C_i. □

Corollary 5.5.3. *If $i \in \mathbb{N}_0$ is such that $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))}$, then C_i will have a minimal reducing subspace X_x for $x = ag_{i,\lambda} + bg_{i,\varphi(\lambda)}$ with $a, b \neq 0$ and $a^2 = b^2$. Hence, $T_\varphi|_{K_\lambda}$ has a minimal reducing subspace X_x.*

Theorem 5.5.4. *(Sufficiency condition for minimality.)*

Let Λ be a subset of $\mathbb{N}_0$, and a, b be non zero scalars such that $f = \sum_{i\in\Lambda}[ag_{i,\lambda} + bg_{i,\varphi(\lambda)}] \in K_\lambda$, where $\lambda \in \mathcal{M}_3$ and $o(\lambda) = 2$. Then X_f is a minimal reducing subspace of $T_\varphi|_{K_\lambda}$ if

(i) *$a^2 = b^2$, and*

(ii) *there exists some $\mu > 0$ such that $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))} = \mu$ for all $i \in \Lambda$.*

Proof. For $i \in \Lambda$, let $\delta_i = ag_{i,\lambda} + bg_{i,\varphi(\lambda)}$. Then, $f = \sum_{i\in\Lambda}\delta_i$, and since $a^2 = b^2$, so

$$T_\varphi f = \frac{b}{a}(\sum_{i\in\Lambda}\alpha_i^{(\lambda)}\delta_i) = (\frac{b}{a}\mu)f.$$

Similarly, $T_\varphi^* f = (\frac{a}{b}\mu)f = T_\varphi f$ (since, $a^2 = b^2 \Rightarrow \frac{a}{b} = \frac{b}{a}$). Thus, $X_f = span\{f\}$ and so X_f is a minimal reducing subspace of T_φ. □

Remark 5.5.5. Suppose in Theorem 5.5.4, the conditions $a^2 = b^2$ and $\alpha_t^{(\lambda)} = \alpha_t^{(\varphi(\lambda))}$ for all $t \in \Lambda$ holds. We show below that the condition $\alpha_i^{(\lambda)} = \alpha_j^{(\lambda)}$ for all $i, j \in \Lambda$ is necessary for X_f to be a minimal reducing subspace for T_φ.

To show this, suppose $i, j \in \Lambda$ such that $\alpha_i^{(\lambda)} \neq \alpha_j^{(\lambda)}$. For simplicity, we assume that $\Lambda = \Lambda^{(i)} + \Lambda^{(j)}$, where

$$\Lambda^{(i)} = \{t \in \Lambda : \alpha_i^{(\lambda)} = \alpha_t^{(\lambda)}\}$$
$$\Lambda^{(j)} = \{t \in \Lambda : \alpha_j^{(\lambda)} = \alpha_t^{(\lambda)}\}.$$

Then, $T_\varphi f = \frac{b}{a}(\alpha_i^{(\lambda)} \sum_{t\in\Lambda^{(i)}} \delta_t + \alpha_j^{(\lambda)} \sum_{t\in\Lambda^{(j)}} \delta_t)$, and $f = \sum_{t\in\Lambda^{(i)}} \delta_t + \sum_{t\in\Lambda^{(j)}} \delta_t$. Therefore,

$$\alpha_j^{(\lambda)} f - \frac{a}{b} T_\varphi f = (\alpha_j^{(\lambda)} - \alpha_i^{(\lambda)}) \sum_{t\in\Lambda^{(i)}} \delta_t = h \text{ (say)}.$$

So, $h \in X_f$ which implies $X_h \subseteq X_f$. Also, $\sum_{t\in\Lambda^{(j)}} \delta_t \notin X_h$ implies $f \notin X_h$. Thus, $X_h \subsetneqq X_f$, and so X_f cannot be minimal.

Remark 5.5.6. All other conditions in Theorem 5.5.4 remaining same, the condition $a^2 = b^2$ is necessary for X_f to be minimal reducing. This can be shown by a method similar to case II of Lemma 5.5.2.

5.5.3 For $\lambda \in \mathcal{M}_3$, $o(\lambda) = 3$

Theorem 5.5.7. *For $i \in \mathbb{N}_0$, let $x = ag_{i,\lambda} + bg_{i,\varphi(\lambda)}$ with $a, b \neq 0$, and $\lambda \in \mathcal{M}_3$ with $o(\lambda) = 3$. If X_x is a proper minimal reducing subspace of C_i in $K_\lambda^{(i)}$, then there must exist $\mu > 0$ such that $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))} = \alpha_i^{(\varphi^2(\lambda))} = \mu$.*

Proof. Here, we have the following relations:

$$x = ag_{i,\lambda} + bg_{i,\varphi(\lambda)} \tag{5.5.4}$$

$$C_i^* x = a\alpha_i^{(\lambda)} g_{i,\varphi(\lambda)} + b\alpha_i^{(\varphi(\lambda))} g_{i,\varphi^2(\lambda)} \tag{5.5.5}$$

$$\begin{aligned} C_i x &= a\alpha_i^{(\varphi^{-1}(\lambda))} g_{i,\varphi^{-1}(\lambda)} + b\alpha_i^{(\lambda)} g_{i,\lambda} \\ &= a\alpha_i^{(\varphi^2(\lambda))} g_{i,\varphi^2(\lambda)} + b\alpha_i^{(\lambda)} g_{i,\lambda} \end{aligned} \tag{5.5.6}$$

$$C_iC_i^*x = a(\alpha_i^{(\lambda)})^2 g_{i,\lambda} + b(\alpha_i^{(\varphi(\lambda))})^2 g_{i,\varphi(\lambda)} \tag{5.5.7}$$

$$\begin{aligned} C_i^*C_ix &= a(\alpha_i^{(\varphi^{-1}(\lambda))})^2 g_{i,\lambda} + b(\alpha_i^{(\lambda)})^2 g_{i,\varphi(\lambda)} \\ &= a(\alpha_i^{(\varphi^2(\lambda))})^2 g_{i,\lambda} + b(\alpha_i^{(\lambda)})^2 g_{i,\varphi(\lambda)} \end{aligned} \tag{5.5.8}$$

Now, $(\alpha_i^{(\lambda)})^2 \times (5.5.4) - (5.5.7)$ gives

$$(\alpha_i^{(\lambda)})^2 x - C_iC_i^*x = b[(\alpha_i^{(\lambda)})^2 - (\alpha_i^{(\varphi(\lambda))})^2] g_{i,\varphi(\lambda)}.$$

This implies $g_{i,\varphi(\lambda)} \in X_x$ if $\alpha_i^{(\lambda)} \neq \alpha_i^{(\varphi(\lambda))}$.

Similarly, $(\alpha_i^{(\varphi^2(\lambda))})^2 \times (5.5.4) - (5.5.8)$ gives

$$(\alpha_i^{(\varphi^2(\lambda))})^2 x - C_i^*C_ix = b[(\alpha_i^{(\varphi^2(\lambda))})^2 - (\alpha_i^{(\lambda)})^2] g_{i,\varphi(\lambda)},$$

and so, $g_{i,\varphi(\lambda)} \in X_x$ if $\alpha_i^{(\lambda)} \neq \alpha_i^{(\varphi^2(\lambda))}$.

Thus, if either $\alpha_i^{(\lambda)} \neq \alpha_i^{(\varphi(\lambda))}$ or $\alpha_i^{(\lambda)} \neq \alpha_i^{(\varphi^2(\lambda))}$, then $g_{i,\varphi(\lambda)} \in X_x$. This implies $X_x = span\{g_{i,j} : j \in [[\lambda]]\} = K_\lambda^{(i)}$, and hence X_x is not a proper reducing subspace for C_i. Thus, if X_x is minimal reducing, we must have $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))} = \alpha_i^{(\varphi^2(\lambda))} = \mu$. □

Theorem 5.5.8. *For $i \in \mathbb{N}_0$, let $x = ag_{i,\lambda} + bg_{i,\varphi(\lambda)}$ with $a, b \neq 0$, and $\lambda \in \mathcal{M}_3$, $o(\lambda) = 3$. Also, suppose there exists $\mu > 0$ such that $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))} = \alpha_i^{(\varphi^2(\lambda))} = \mu$. Then X_x is a proper reducing subspace for C_i in $K_\lambda^{(i)}$ if and only if $a^3 + b^3 = 0$.*

Proof. We have the following relations:

$$\begin{aligned} ag_{i,\lambda} + bg_{i,\varphi(\lambda)} = x &= \frac{1}{\mu^3}C_i^3x = \frac{1}{\mu^3}(C_i^*)^3x = \frac{1}{\mu^4}C_i^2(C_i^*)^2x = \frac{1}{\mu^4}(C_i^*)^2C_i^2x \\ &= \frac{1}{\mu^2}C_i^*C_ix = \frac{1}{\mu^2}C_iC_i^*x. \end{aligned} \tag{5.5.9}$$

$$ag_{i,\varphi(\lambda)} + bg_{i,\varphi^2(\lambda)} = \frac{1}{\mu}C_i^*x = \frac{1}{\mu^2}C_i^2x \tag{5.5.10}$$

$$ag_{i,\varphi^2(\lambda)} + bg_{i,\lambda} = \frac{1}{\mu^2}(C_i^*)^2x = \frac{1}{\mu}C_ix \tag{5.5.11}$$

Claim: $g_{i,\lambda} \notin X_x$ if $a^3 + b^3 = 0$. From the above relations, we see that all elements in X_x are finite linear combinations of the functions $ag_{i,\lambda} + bg_{i,\varphi(\lambda)}$, $ag_{i,\varphi(\lambda)} + bg_{i,\varphi^2(\lambda)}$

and $ag_{i,\varphi^2(\lambda)} + bg_{i,\lambda}$. Let, if possible, $g_{i,\lambda} \in X_x$. So, there exists A, B, C such that

$$\begin{aligned} g_{i,\lambda} &= A(ag_{i,\lambda} + bg_{i,\varphi(\lambda)}) + B(ag_{i,\varphi(\lambda)} + bg_{i,\varphi^2(\lambda)}) + C(ag_{i,\varphi^2(\lambda)} + bg_{i,\lambda}) \\ \Rightarrow & aA + bC - 1 = 0 \\ & bA + aB = 0 \\ & bB + aC = 0. \end{aligned}$$

This implies $PX = Q$, where $P = \begin{pmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{pmatrix}$, $X = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ and $Q = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

To solve this system, we consider the augmented matrix

$$[P|Q] = \left[\begin{array}{ccc|c} a & 0 & b & 1 \\ b & a & 0 & 0 \\ 0 & b & a & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} a & 0 & b & 1 \\ 0 & a^2 & -b^2 & -b \\ 0 & b & a & 0 \end{array}\right] \quad R_2 \to aR_2 - bR_1$$

$$\sim \left[\begin{array}{ccc|c} a & 0 & b & 1 \\ 0 & a^2 & -b^2 & -b \\ 0 & 0 & a^3 + b^3 & b^2 \end{array}\right] \quad R_3 \to a^2 R_3 - bR_2.$$

By assumption, we have $a \neq 0$, $b \neq 0$. So, if $a^3 + b^3 = 0$, then the system becomes inconsistent and therefore has no solution. In other words, if $a^3 + b^3 = 0$, then $g_{i,\lambda} \notin X_x$. Hence, $X_x \subsetneqq K_\lambda^{(i)}$ so that X_x is a proper reducing subspace of C_i.

Conversely, suppose $a^3 + b^3 = 0$. Then, for $A = \frac{a^2}{a^3+b^3}$, $B = -\frac{ab}{a^3+b^3}$, $C = \frac{b^2}{a^3+b^3}$, we get

$$g_{i,\lambda} = Ax + \frac{B}{\mu} C_i^* x + \frac{C}{\mu^2} (C_i^*)^2 x,$$

so that $g_{i,\lambda} \in X_x$. This implies that $g_{i,\varphi(\lambda)}$ and $g_{i,\varphi^2(\lambda)}$ are also in X_x, and so $X_x = K_\lambda^{(i)}$. Thus, X_x is not a proper reducing subspace for C_i if $a^3 + b^3 \neq 0$. □

Remark 5.5.9. Analogous results can be found if we consider $x = ag_{i,\lambda} + bg_{i,\varphi^2(\lambda)}$ and $x = ag_{i,\varphi(\lambda)} + bg_{i,\varphi^2(\lambda)}$.

Remark 5.5.10. In Theorem 5.5.8, as X_x is a proper reducing subspace for C_i in $K_\lambda^{(i)}$, and $\dim K_\lambda^{(i)} = 3$, so $\dim X_x$ is either 1 or 2. However from (5.5.9) and (5.5.10), we see that $h_1 = ag_{i,\lambda} + bg_{i,\varphi(\lambda)}$, and $h_2 = ag_{i,\varphi(\lambda)} + bg_{i,\varphi^2(\lambda)}$, are in X_x, where h_1 and h_2 are linearly independent in $K_\lambda^{(i)}$.
Also, $ag_{i,\varphi^2(\lambda)} + bg_{i,\lambda} = -\frac{a^2}{b}(bh_2 - ah_1)$. Hence, $X_x = span\{h_1, h_2\}$, i.e, $\dim X_x = 2$. Thus, if at all we have a minimal reducing subspace for C_i, then it will be of the form X_y, where $y = ag_{i,\lambda} + bg_{i,\varphi(\lambda)} + cg_{i,\varphi^2(\lambda)}$ with scalars a, b, c non zero.

Theorem 5.5.11. *Let $\lambda \in \mathcal{M}_3$ with $o(\lambda) = 3$. For $i \in \mathbb{N}_0$, let $x = ag_{i,\lambda} + bg_{i,\varphi(\lambda)} + cg_{i,\varphi^2(\lambda)}$ with $a, b, c \neq 0$. Then X_x is minimal reducing for C_i if*
(i) $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$, *and*
(ii) *there exists $\mu > 0$ such that $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))} = \alpha_i^{(\varphi^2(\lambda))} = \mu$.*

Proof. We have $C_i^* x = \mu(ag_{i,\varphi(\lambda)} + bg_{i,\varphi^2(\lambda)} + cg_{i,\lambda})$. Therefore,

$$(c\mu)x - aC_i^* x = \mu[(bc - a^2)g_{i,\varphi(\lambda)} + (c^2 - ab)g_{i,\varphi^2(\lambda)}].$$

As, $a^2 = bc$ and $c^2 = ab$, so $C_i^* x = (\frac{c\mu}{a})x$. Similarly, $C_i x = (\frac{b\mu}{a})x$. Hence, $X_x = span\{x\}$, and hence X_x is a minimal reducing subspace. □

Theorem 5.5.12. *Let $\lambda \in \mathcal{M}_3$ with $o(\lambda) = 3$. For $i \in \mathbb{N}_0$, let $x = ag_{i,\lambda} + bg_{i,\varphi(\lambda)} + cg_{i,\varphi^2(\lambda)}$ with $a, b, c \neq 0$. Suppose X_x is a proper minimal reducing subspace for C_i in $K_\lambda^{(i)}$. If two of the values $\alpha_i^{(\lambda)}, \alpha_i^{(\varphi(\lambda))}, \alpha_i^{(\varphi^2(\lambda))}$ are equal, then all three must be equal.*

Proof. Here, we have the following relations:

$$x = ag_{i,\lambda} + bg_{i,\varphi(\lambda)} + cg_{i,\varphi^2(\lambda)}. \tag{5.5.12}$$

$$C_i C_i^* x = a(\alpha_i^{(\lambda)})^2 g_{i,\lambda} + b(\alpha_i^{(\varphi(\lambda))})^2 g_{i,\varphi(\lambda)} + c(\alpha_i^{(\varphi^2(\lambda))})^2 g_{i,\varphi^2(\lambda)}. \tag{5.5.13}$$

$$C_i^* C_i x = a(\alpha_i^{(\varphi^2(\lambda))})^2 g_{i,\lambda} + b(\alpha_i^{(\lambda)})^2 g_{i,\varphi(\lambda)} + c(\alpha_i^{(\varphi(\lambda))})^2 g_{i,\varphi^2(\lambda)}. \tag{5.5.14}$$

Case I. Suppose, $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))}$ but $\alpha_i^{(\lambda)} \neq \alpha_i^{(\varphi^2(\lambda))}$. Then

$$\begin{aligned}&[(\alpha_i^{(\lambda)})^2 - (\alpha_i^{(\varphi^2(\lambda))})^2]ag_{i,\lambda} + [(\alpha_i^{(\lambda)})^2 - (\alpha_i^{(\varphi(\lambda))})^2]cg_{i,\varphi^2(\lambda)} \\ =&(\alpha_i^{(\lambda)})^2 x - C_i^* C_i x \in X_x. \end{aligned} \tag{5.5.15}$$

As $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))}$ and $\alpha_i^{(\lambda)} \neq \alpha_i^{(\varphi^2(\lambda))}$, so we get $g_{i,\lambda} \in X_x$, and so $X_x = K_\lambda^{(i)}$, showing that X_x cannot be a proper subspace for C_i.

Case II. Suppose, $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi^2(\lambda))}$ but $\alpha_i^{(\lambda)} \neq \alpha_i^{(\varphi(\lambda))}$. Again by (5.5.15), we see that $g_{i,\varphi^2(\lambda)} \in X_x$, which implies $X_x = K_\lambda^{(i)}$. So, X_x cannot be a proper subspace for C_i.

Case III. Suppose, $\alpha_i^{(\varphi(\lambda))} = \alpha_i^{(\varphi^2(\lambda))}$ but $\alpha_i^{(\lambda)} \neq \alpha_i^{(\varphi(\lambda))}$. Then

$$\begin{aligned}&[(\alpha_i^{(\varphi^2(\lambda))})^2 - (\alpha_i^{(\lambda)})^2]bg_{i,\varphi(\lambda)} + [(\alpha_i^{(\varphi^2(\lambda))})^2 - (\alpha_i^{(\varphi(\lambda))})^2]cg_{i,\varphi^2(\lambda)} \\ =&(\alpha_i^{(\varphi^2(\lambda))})^2 x - C_i^* C_i x \in X_x. \end{aligned} \tag{5.5.16}$$

As $\alpha_i^{(\lambda)} \neq \alpha_i^{(\varphi^2(\lambda))}$ and $\alpha_i^{(\varphi(\lambda))} = \alpha_i^{(\varphi^2(\lambda))}$. So, we get $g_{i,\varphi(\lambda)} \in X_x$, and so $X_x = K_\lambda^{(i)}$, showing that X_x cannot be a proper subspace for C_i. □

Theorem 5.5.13. *Let $\lambda \in \mathcal{M}_3$ with $o(\lambda) = 3$. For $i \in \mathbb{N}_0$, let $x = ag_{i,\lambda} + bg_{i,\varphi(\lambda)} + cg_{i,\varphi^2(\lambda)}$ with $a, b, c \neq 0$. Suppose X_x is a proper reducing subspace for C_i in $K_\lambda^{(i)}$. If $\alpha_i^{(\lambda)}, \alpha_i^{(\varphi(\lambda))}, \alpha_i^{(\varphi^2(\lambda))}$ are all distinct, then*

$$\frac{(\alpha_i^{(\lambda)})^2 - (\alpha_i^{(\varphi(\lambda))})^2}{(\alpha_i^{(\varphi(\lambda))})^2 - (\alpha_i^{(\varphi^2(\lambda))})^2} = \frac{(\alpha_i^{(\varphi(\lambda))})^2 - (\alpha_i^{(\varphi^2(\lambda))})^2}{(\alpha_i^{(\varphi^2(\lambda))})^2 - (\alpha_i^{(\lambda)})^2} = \frac{(\alpha_i^{(\varphi^2(\lambda))})^2 - (\alpha_i^{(\lambda)})^2}{(\alpha_i^{(\lambda)})^2 - (\alpha_i^{(\varphi(\lambda))})^2}.$$

Proof. From (5.5.12) and (5.5.13), we have

$$(\alpha_i^{(\lambda)})^2 x - C_i C_i^* x = b\big[(\alpha_i^{(\lambda)})^2 - (\alpha_i^{(\varphi(\lambda))})^2\big]g_{i,\varphi(\lambda)} + c\big[(\alpha_i^{(\lambda)})^2 - (\alpha_i^{(\varphi^2(\lambda))})^2\big]g_{i,\varphi^2(\lambda)} \in X_x. \tag{5.5.17}$$

Then from (5.5.16) and (5.5.17), we get

$$b\big[[(\alpha_i^{(\lambda)})^2 - (\alpha_i^{(\varphi(\lambda))})^2][(\alpha_i^{(\varphi^2(\lambda))})^2 - (\alpha_i^{(\varphi(\lambda))})^2] - [(\alpha_i^{(\varphi^2(\lambda))})^2 - (\alpha_i^{(\lambda)})^2]^2\big]g_{i,\varphi(\lambda)} \in X_x. \tag{5.5.18}$$

Let, if possible

$$\frac{(\alpha_i^{(\varphi(\lambda))})^2-(\alpha_i^{(\varphi^2(\lambda))})^2}{(\alpha_i^{(\varphi^2(\lambda))})^2-(\alpha_i^{(\lambda)})^2}\neq\frac{(\alpha_i^{(\varphi^2(\lambda))})^2-(\alpha_i^{(\lambda)})^2}{(\alpha_i^{(\lambda)})^2-(\alpha_i^{(\varphi(\lambda))})^2}.$$

Then, from (5.5.18), $g_{i,\varphi(\lambda)} \in X_x$. This implies $X_x = K_\lambda^{(i)}$, a contradiction. Hence, we must have

$$\frac{(\alpha_i^{(\varphi(\lambda))})^2-(\alpha_i^{(\varphi^2(\lambda))})^2}{(\alpha_i^{(\varphi^2(\lambda))})^2-(\alpha_i^{(\lambda)})^2}=\frac{(\alpha_i^{(\varphi^2(\lambda))})^2-(\alpha_i^{(\lambda)})^2}{(\alpha_i^{(\lambda)})^2-(\alpha_i^{(\varphi(\lambda))})^2}.$$

A similar situation occurs for

$$\frac{(\alpha_i^{(\lambda)})^2-(\alpha_i^{(\varphi(\lambda))})^2}{(\alpha_i^{(\varphi(\lambda))})^2-(\alpha_i^{(\varphi^2(\lambda))})^2}\neq\frac{(\alpha_i^{(\varphi(\lambda))})^2-(\alpha_i^{(\varphi^2(\lambda))})^2}{(\alpha_i^{(\varphi^2(\lambda))})^2-(\alpha_i^{(\lambda)})^2}, \text{ or}$$

$$\frac{(\alpha_i^{(\lambda)})^2-(\alpha_i^{(\varphi(\lambda))})^2}{(\alpha_i^{(\varphi(\lambda))})^2-(\alpha_i^{(\varphi^2(\lambda))})^2}\neq\frac{(\alpha_i^{(\varphi^2(\lambda))})^2-(\alpha_i^{(\lambda)})^2}{(\alpha_i^{(\lambda)})^2-(\alpha_i^{(\varphi(\lambda))})^2}.$$

□

Theorem 5.5.14. *Sufficiency condition for minimality:*

Let $\lambda \in \mathcal{M}_3$ with $o(\lambda) = 3$. Let Λ be a subset of $\mathbb{N}_0$, and a, b, c be non zero scalars such that $f = \sum_{i\in\Lambda}[ag_{i,\lambda} + bg_{i,\varphi(\lambda)} + cg_{i,\varphi^2(\lambda)}] \in K_\lambda$. Then X_f is a minimal reducing subspace of $T_\varphi|_{K_\lambda}$ if

(i) $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$, and

(ii) there exists $\mu > 0$ such that $\alpha_i^{(\lambda)} = \alpha_i^{(\varphi(\lambda))} = a_i^{\varphi^2(\lambda)} = \mu$ for all $i \in \Lambda$.

Proof. For $i \in \Lambda$, let $\delta_i = ag_{i,\lambda} + bg_{i,\varphi(\lambda)} + cg_{i,\varphi^2(\lambda)}$. Then $f = \sum_{i\in\Lambda}\delta_i$ and $T_\varphi|_{K_\lambda} = \sum_{i\in\Lambda} C_i$. By Theorem 5.5.11, for each $i \in \Lambda$, we have $C_i\delta_i = (\frac{b\mu}{a})\delta_i$ and $C_i^*\delta_i = (\frac{c\mu}{a})\delta_i$. Therefore, $T_\varphi f = (\frac{b\mu}{a})f$ and $T_\varphi^* f = (\frac{c\mu}{a})f$. Hence, $X_f = span\{f\}$, and so X_f is a minimal reducing subspace of T_φ on K_λ. □

5.5.4 For $\lambda \in \mathcal{M}_3$, $o(\lambda) = r$

From Theorem 5.5.4 and Theorem 5.5.14, we can propose a sufficiency condition for minimality where $\lambda \in \mathcal{M}_3$, $o(\lambda) = r$, $r \geq 2$.

Theorem 5.5.15. *Let $\lambda \in \mathcal{M}_3$ and $o(\lambda) = r$ with $r \geq 2$. Let Λ be a subset of $\mathbb{N}_0$, and $a_0, a_1, \ldots, a_{r-1}$ be non zero scalars such that $f = \sum_{i\in\Lambda}[a_0 g_{i,\lambda} + a_1 g_{i,\varphi(\lambda)} + \cdots + a_{r-1} g_{i,\varphi^{r-1}(\lambda)}] \in K_\lambda$. Then X_f is a minimal reducing subspace of $T_\varphi|_{K_\lambda}$ if*
(i) $\frac{a_0}{a_1} = \frac{a_1}{a_2} = \cdots = \frac{a_{r-2}}{a_{r-1}} = \frac{a_{r-1}}{a_0}$, *and*
(ii) there exists $\mu > 0$ such that $\alpha_i^{(\varphi^t(\lambda))} = \mu$ for all $0 \leq t < r$ and $i \in \Lambda$.

The proof of Theorem 5.5.15 is similar to that of Theorem 5.5.14, and hence it is omitted.