Chapter 1

Introduction

1.1 Introduction

In the branch of operator theory, shift operators are a class of very widely and extensively studied linear operators on Hilbert spaces. These operators are of fundamental importance in many parts of operator theory. Among the many adequate and comprehensive references are [10], [41], [45] and [48]. A recent approach is the weighted shifts on trees [28]. These operators have many algebraic and analytic properties and very often it is the technique applied in proving these properties which are very valuable even though the properties may not have immediate visible application.

In our work, we have studied weighted unilateral shifts of higher multiplicity with operator weights. We determine the reducing subspaces of a class of operator weighted shifts where the operator weights are not necessarily normal or self adjoint. It may be recalled that a reducing subspace of an operator T on a Hilbert space H is a closed subspace M of H such that T and T^* both maps M into M. A reducing subspace M is called minimal if the only reducing subspace contained in M are M and $\{0\}$. Also, an operator T on H is irreducible if the only reducing subspaces of T are H and $\{0\}$.

It is well known that the unweighted unilateral shift of multiplicity one is irreducible. The structure of the reducing subspace lattice for unweighted unilateral shifts of higher multiplicities was described in [18] and [42]. The reducing subspaces of more general weighted shifts are discussed in [50]. Similar work on reducing subspaces of analytic Toeplitz operators can be found in [1, 7, 15, 26, 38, 46]. Both scalar and operator shifts have time and again proved to be a fertile ground for providing examples and counter examples in various branches of operator theory.

We begin with an introduction of shift operators along with some of their basic properties. It may be mentioned that different authors often tend to use different notations. For our purpose, we follow the notation given in [10].

Let K be a Hilbert space, and let $\ell^2_+(K) = K \oplus K \oplus \dots$ be the Hilbert space of all sequences $x = \{x_n\}_{n=0}^{\infty}$ of vectors $x_n \in K$ such that $\|x\|^2 = \sum_{n=0}^{\infty} \|x_n\|^2 < \infty$. The unilateral shift U_+ on $\ell^2_+(K)$ is defined as

$$U_+(x_0, x_1, \dots) = (0, x_0, x_1, \dots).$$

The multiplicity of U_+ is the cardinal number $n = \dim K$. It follows immediately that the adjoint of U_+ is given by

$$U_+^*(x_0, x_1, \dots) = (x_1, x_2, \dots)$$

and U_+^* is called the backward shift. Two unilateral shift operators are unitarily equivalent if and only if they have the same multiplicity.

Again, let $\ell^2(K) = \sum_{-\infty}^{\infty} \oplus K$ be the Hilbert space of two-way sequences $x = (\dots, x_{-1}, [x_0], x_1, \dots)$ of vectors from K with $\|x\|^2 = \sum_{n=-\infty}^{\infty} \|x_n\|^2 < \infty$, and let the bilateral shift U on $\ell^2(K)$ be defined as

$$U(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, x_{-2}, [x_{-1}], x_0, \dots).$$

Here, $[\cdot]$ denotes the central 0th entry of $x = (\dots, x_{-1}, [x_0], x_1, \dots)$.
The multiplicity of U is $\dim K$. Just like unilateral shifts, bilateral shifts are also unitarily equivalent if and only if they have the same multiplicity.

1.2 Review of literature

Let $\mathbb{C}$ denote the complex plane, and $\mathbb{T}$ and $\mathbb{D}$ denote the unit circle and open unit disc in $\mathbb{C}$ respectively. Also, $\mathbb{Z}$ and $\mathbb{N}$ denotes the set of integers and the set of natural numbers respectively. The set of non negative integers is denoted by $\mathbb{N}_0$.

Let L^2 be the space of all square integrable functions with respect to the normalized Lebesgue measure μ on the unit circle $\mathbb{T}$. For $f, g \in L^2$,

$$\begin{aligned}\langle f, g\rangle &:= \int_{\mathbb{T}} f(z)\overline{g(z)}d\mu(z)\\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}d\theta \text{ (writing } f(\theta) \text{ for } f(e^{i\theta}) \text{ and } g(\theta) \text{ for } g(e^{i\theta})).\end{aligned}$$

For $n \in \mathbb{Z}$, $e_n : \mathbb{T} \to \mathbb{C}$ is defined as $e_n(e^{i\theta}) = e^{in\theta}$, or equivalently as $e_n(z) = z^n$ for $z = e^{i\theta}$. We usually write $e_n(\theta)$ instead of $e_n(e^{i\theta})$ for simplicity whenever there is no confusion. The functions $e_n(\theta) = e^{in\theta}$, $n \in \mathbb{Z}$ form a complete orthonormal set in L^2.

Let H^2 be the closed linear span of the $\{e_n\}_{n\in\mathbb{N}_0}$. Thus

$$H^2 = \{f \in L^2 : \langle f, e_n\rangle = 0 \text{ for } n < 0\}.$$

Let $e(\theta) := e_1(\theta) = e^{i\theta}$ and $M_e : L^2 \to L^2$ be defined as

$$M_e(f) = ef \text{ for all } f \in L^2.$$

Therefore, $(M_e f)(\theta) = e^{i\theta} f(\theta)$ for all $f \in L^2$.
Then M_e is unitarily equivalent to the bilateral shift of multiplicity one. Also, $M_e|_{H^2}$ is unitarily equivalent to the unilateral shift of multipicity one.

Let $L^2(K)$ consist of all measurable functions f from $\mathbb{T}$ to K such that $\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \|f(\theta)\|^2 d\theta < \infty$. Two functions f and g in $L^2(K)$ are considered equal if they differ only on a set of measure zero. $L^2(K)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} \langle f(\theta), g(\theta)\rangle d\theta.$$

Functions in $L^2(K)$ admit two kinds of orthogonal expansions, as shown below:

(1) Let $\{b_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal basis for K. Let $f \in L^2(K)$. For each $\alpha \in \Lambda$, let $f_\alpha : \mathbb{T} \to \mathbb{C}$ be defined as $f_\alpha(\theta) = \langle f(\theta), b_\alpha\rangle$. Here, f_α's are called the co-ordinate functions of f. Then $f(\theta) = \sum_{\alpha\in\Lambda} f_\alpha(\theta) b_\alpha$, where convergence is in the norm of K. Here, each $f_\alpha \in L^2$ and $\|f\|^2 = \sum_{\alpha\in\Lambda} \|f_\alpha\|^2$. Thus the map $f \mapsto \sum_{\alpha\in\Lambda} \oplus f_\alpha$ is an isometry from $L^2(K)$ onto $\sum_{\alpha\in\Lambda} \oplus L^2$. This also shows that $L^2(K)$ is complete.

(2) The second kind of expansion is as a Fourier series with vector coefficients. It can be shown that each $f \in L^2(K)$ admits a unique expansion $f(\theta) = \sum_{k=-\infty}^{\infty} x_k e^{ik\theta}$, with $x_k \in K$ and $\|f\|^2 = \sum_{k=-\infty}^{\infty} \|x_k\|^2$. This is to be understood in the weak sense: for each $x \in K$, the Fourier expansion of $\langle f(\theta), x\rangle$ is $\sum_{k=-\infty}^{\infty} \langle x_k, x\rangle e^{ik\theta}$. The map $f \mapsto \{x_k\}$ is an isometry from $L^2(K)$ onto $\ell^2(K)$.

Multiplication by e, i.e, M_e on $L^2(K)$ is the bilateral shift U of multiplicity $\dim K$.

The subspace $H^2(K) := \{f \in L^2(K) : f(\theta) = \sum_{k=0}^{\infty} x_k e^{ik\theta}\}$ is invariant under U, and $U|_{H^2(K)}$ is the unilateral shift U_+ of multiplicity $\dim K$.

The invariant and reducing subspaces of these general shift operators are well understood. A comprehensive reference for the same is [10]. Of particular interest is the case when $\dim K = 1$, for which a lucid treatment can be found in [39].

Motivated by the theory of unilateral and bilateral shifts, another class of operators namely weighted shift operators was defined on $\ell^2(K)$. Here, we refer to the definition given by Shields in [48], where he considers weighted shifts on $\ell^2(K)$ for $\dim K = 1$. For a bounded sequence of scalars $\{\beta_n\}_{n\in\mathbb{Z}}$, he defines S on $\ell^2_+(K)$ as

$$S(x_0, x_1, \dots) = (0, \beta_0 x_0, \beta_1 x_1, \dots);$$

and W on $\ell^2(K)$ as

$$W(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, \beta_{-2}x_{-2}, [\beta_{-1}x_{-1}], \beta_0 x_0, \dots)$$

and call them the unilateral weighted shift and bilateral weighted shift respectively.

In the same paper, Shields also introduced the definition of a weighted sequence space. Let β denote a sequence of positive numbers $\{\beta_n\}_{n\in\mathbb{N}_0}$ with $\beta_0 = 1$. Then $H^2(\beta)$ is defined to be the space of all formal power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$ such that $\sum_{n=0}^{\infty} |f_n|^2 \beta_n^2 < \infty$. Similarly, $L^2(\beta)$ is defined as the space of formal Laurent series $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$ such that $\sum_{n\in\mathbb{Z}} |f_n|^2 \beta_n^2 < \infty$. For $f, g \in L^2(\beta)$, $\langle f, g\rangle := \sum_{n\in\mathbb{Z}} f_n \overline{g}_n \beta_n^2$. Thus,

$$H^2(\beta) = \{(x_0, x_1, \dots) : x_i \in \mathbb{C}, \sum_{n=0}^{\infty} |x_n|^2 \beta_n^2 < \infty\}, \text{ and}$$

$$L^2(\beta) = \{(\dots, x_{-1}, [x_0], x_1, \dots) : x_i \in \mathbb{C}, \sum_{n\in\mathbb{Z}} |x_n|^2\beta_n^2 < \infty\}.$$

The following results were established in [48]:

Theorem 1.2.1. *The unilateral shift U_+ on $H^2(\beta)$ is unitarily equivalent to the unilateral weighted shift S on $\ell^2_+(\mathbb{C})$ with weights $\{w_n\}_{n\in\mathbb{N}_0}$ given by $w_n = \frac{\beta_{n+1}}{\beta_n}$ for all $n \in \mathbb{N}_0$. Also, U_+ on $H^2(\beta)$ is bounded if and only if $\sup\{\frac{\beta_{n+1}}{\beta_n} : n \geq 0\} < \infty$.*

Theorem 1.2.2. *If S is a unilateral weighted shift on $\ell^2_+(\mathbb{C})$ with non-zero weights $\{w_n\}_{n\in\mathbb{N}_0}$ then S is unitarily equivalent to U_+ on $H^2(\beta)$ where $\beta_0 := 1$ and $\beta_n := w_0w_1\dots w_{n-1}$ for $n > 0$.*

Similar results also hold for the bilateral shift U on $L^2(\beta)$. In Corollary 2 of Theorem 3 [48], Shields has shown that U_+ on $H^2(\beta)$ is irreducible. In view of Theorem 1.2.2, this implies that S on $\ell^2_+(\mathbb{C})$ is also irreducible. In 1967, N. K. Nikolskii [40] introduced operator weighted shifts as a generalization of scalar weighted shifts. Invariant subspaces of the weighted shifts was first studied by Nikolskii [40]. For this, he considered a sequence of uniformly bounded operators $\{A_n\}_{n\in\mathbb{N}_0}$ on K. The operator S on $\ell^2_+(K)$ is defined as

$$S(x_0, x_1, \dots) = (0, A_0x_0, A_1x_1, \dots)$$

and is called the unilateral operator weighted shift with weights $\{A_n\}_{n\in\mathbb{N}_0}$. A bilateral operator weighted shift is similarly defined on $\ell^2(K)$.

The unilateral operator weighted shift S is bounded and $\|S\| = \sup_n \|A_n\|$. If each A_n is invertible, then S is an *invertibly weighted operator shift* [34]. Operator weighted shifts are a generalization of the scalar weighted shifts. However, this generalization is not just formal. For example, by means of an operator weighted shift, Pearcy and Petrovic [43] proved that an n-normal operator is power bounded if and only if it is similar to a contraction. Since its introduction, operator weighted shifts have been widely studied. For a general understanding of its various properties we refer the following: [3, 11, 25, 27, 34, 36, 40].

Our interest is to determine the minimal reducing subspaces of an invertibly weighted operator shift S on $\ell^2_+(K)$.

Definition 1.2.3. A subspace M of $\ell^2_+(K)$ is *invariant* under S if $S(M) \subseteq M$. If M is invariant under both S and S^* then M is said to be *reducing* for S. A reducing subspace M is said to be *minimal reducing* if it does not contain any proper non zero reducing subspace.

The invariant and reducing subspaces of specific types of invertibly weighted operator shifts are known from [13, 14, 16, 17, 20, 30, 34, 40, 44, 50, 55]. However, we observe that in all these cases S is an operator weighted shift with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$, where it is either assumed that the A_n's are commuting normal operators, or that each A_n is positive diagonal.

In [34], Lambert considered the unilateral operator weighted shift S on $\ell^2_+(K)$ with uniformly bounded operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ such that each A_n is invertible, though the A_n's need not be mutually commuting. For each $n \in \mathbb{N}_0$, he constructed operators S_n on K such that $S_0 := I$, and for $n \geq 1$, $S_n := A_{n-1}A_{n-2}\ldots A_0$ and considered $\mathcal{T}(S)$ to be the weakly closed $*$-algebra generated by $\{S_n^*S_n\}_{n\in\mathbb{N}_0}$. Lambert proved that S is irreducible if and only if $\mathcal{T}(S) = \mathcal{B}(K)$, where $\mathcal{B}(K)$ denotes the space of all bounded linear operators on K. As against this result, we know that the unweighted shift U_+ on $\ell^2_+(K)$ is always irreducible. For the unilateral operator weighted shift S on $\ell^2_+(K)$, Lambert proved the following two significant results in [34]:

Theorem 1.2.4. *If M is a subspace of $\ell^2_+(K)$ which reduces S, then we have $M = \sum_{n\in\mathbb{N}_0} \oplus S_n M_0$ for some subspace M_0 of K.*

Theorem 1.2.5. *If $M = \sum_{n\in\mathbb{N}_0} \oplus S_n M_0$ is a subspace of $\ell^2_+(K)$, then the following are equivalent:*
(i) *M reduces S.*
(ii) *$S_n M_0$ is invariant for $A_n^* A_n$, $n \in \mathbb{N}_0$.*
(iii) *$(S_n M_0)^\perp = S_n (M_0)^\perp$, $n \in \mathbb{N}_0$.*
(iv) *$S_n^* S_n M_0 = M_0$, $n \in \mathbb{N}_0$.*

Following this work, there has been a continuous attempt to identify the reducing subspaces of unilateral and bilateral operator weighted shifts S and W respectively with the operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ satisfying different sets of conditions.

In [44], Pilidi deduced a reducibility condition for W on $\ell^2(K)$ with invertible weights $\{A_n\}_{n=-\infty}^{\infty}$. Let $\mathcal{B}(K)$ denote the space of all bounded linear operators on K. For any integer p, the sequence $\{T^{(n,p)}\}_{n\in\mathbb{Z}}$ in $\mathcal{B}(K)$ is defined as follows:

$$T^{(0,p)} = 1, \text{ and for } n \geq 1,$$
$$T^{(n,p)} = |A_{n+p-1} A_{n+p-2} \dots A_p|,$$
$$T^{(-n,p)} = |A^{-1}_{-n+p} A^{-1}_{-n+p+1} \dots A^{-1}_{p-1}|,$$

where $|B| = (B^* B)^{\frac{1}{2}}$ for $B \in \mathcal{B}(K)$. $\mathcal{U}_T^{(p)}$ denotes the w^*-algebra generated by $\{T^{(n,p)}\}_{n\in\mathbb{Z}}$. In [44], Pilidi established the following results for the bilateral operator weighted shift W on $\ell^2(K)$:

Theorem 1.2.6. *The bilateral operator weighted shift W on $\ell^2(K)$ with positive weights $\{A_n\}_{n\in\mathbb{Z}}$ is reducible if and only if one of the following conditions hold:*
(a) $\mathcal{U}_T^{(0)} \neq \mathcal{B}(K)$.
(b) *The sequence $\{A_n\}_{n\in\mathbb{Z}}$ is unitarily periodic.*

Note that a sequence $\{A_n\}_{n\in\mathbb{Z}}$ in $\mathcal{B}(K)$ is said to be unitarily periodic if for some integer $p \neq 0$ the sequences $\{A_n\}_{n\in\mathbb{Z}}$ and $\{A_{n-p}\}_{n\in\mathbb{Z}}$ are unitarily equivalent. That is, if there exists a unitary operator $U \in \mathcal{B}(K)$ such that $A_n = U^* A_{n-p} U$ for all $n \in \mathbb{Z}$.

Theorem 1.2.7. *If W on $\ell^2(K)$ has positive operator weights $\{A_n\}_{n\in\mathbb{Z}}$ such that the sequence $\{A_n\}_{n\in\mathbb{Z}}$ is completely non-periodic, then $M \subset \ell^2(K)$ reduces W if and only if $M = \ell^2(N)$, where $N \subset K$ is a common reducing subspace of the operators A_n.*

Note that a sequence $\{A_n\}_{n\in\mathbb{Z}}$ in $\mathcal{B}(K)$ is said to be completely non-periodic if the conditions $p \in \mathbb{Z}$, $p \neq 0$, $A \in \mathcal{B}(K)$ and $A_{n+p}A = AA_n$ for all $n \in \mathbb{Z}$ together imply $A = 0$.

In [17], Guyker extended these results to the case of normal weights with dense range. He proved the following:

Theorem 1.2.8. *Let $\{A_n\}_{n\in\mathbb{Z}}$ be a commuting family of compact, normal operators with dense range. Then W on $\ell^2(K)$ is unitarily equivalent to the countable direct sum $\sum \oplus W_n$ of bilateral weighted shifts W_n on $\ell^2(\mathbb{C})$ with non-zero scalar weights. Moreover, a subspace M reduces $\sum \oplus W_n$ if and only if $M = \sum \oplus M_n$, where M_n reduces W_n for every n.*

In [50], Stessin and Zhu considers U_+^N on $H^2(\beta)$ for $N > 1$, and gives a complete description of its reducing subspaces. In [20], Hazarika and Arora considers U_+ on the operator weighted sequence space $H^2(B)$. The space $H^2(B)$ is defined as follows: Let B denote a uniformly bounded sequence $\{B_n\}_{n\in\mathbb{N}_0}$ of positive invertible operators on K, and

$$H^2(B) := \{(x_0, x_1, \dots) : x_i \in K, \sum_{i\in\mathbb{N}_0} \|B_i x_i\|^2 < \infty\}.$$

For $f = (f_i)$ and $g = (g_i)$ in $H^2(B)$, $\langle f, g\rangle := \sum_{i\in\mathbb{N}_0}\langle B_i f_i, B_i g_i\rangle$. If $\dim K = N < \infty$, then U_+^N on $H^2(\beta)$ can be considered as U_+ on $H^2(B)$ where each B_n is an invertible diagonal operator on K defined as $B_n e_i = \beta_{Nn+i} e_i$ for $i = 0, 1, \ldots, N-1$. Here $\{e_i\}_{i=0}^{N-1}$ is an orthonormal basis for K.

In [20], the results of [50] are extended to the case where $\dim K = \aleph_0$. In both these papers, the minimal reducing subspaces of the shift are also discussed.

1.3 Chapter-wise brief summary of the thesis

The thesis comprises of five chapters. The first chapter is introductory in nature. It includes a brief background leading to the problem in hand. The operators and spaces referred to in the sequel are also defined in this chapter.

In our work, we consider K to be a separable complex Hilbert space with $\dim K = \aleph_0$, and orthonormal basis $\{e_i\}_{i\in\mathbb{N}_0}$. Though the literature on scalar weighted shifts is extensive, the same is not true for operator weighted shifts. There are many aspects of the class of operator weighted shifts which are yet to be fully understood. One of these is regarding their reducing and minimal reducing subspaces. As regards the available literature, we find that most of the work imposes specific restrictions on the operator weights. One such set of conditions is to assume that the weights are self adjoint and invertible. In another situation, it is assumed that the operator weights are simultaneously diagonalizable i.e, they are mutually commuting. In our study, we try to go beyond these assumptions and consider a more general class of operator weights. We begin with a brief introduction to these weights:

Let $\mathcal{B}(K)$ denote the set of all bounded linear operators on the separable complex

Hilbert space K having an orthonormal basis $\{e_n\}_{n\in\mathbb{N}_0}$, and $\mathcal{T}$ be the subset of $\mathcal{B}(K)$ defined as follows:

$\mathcal{T} := \{T \in \mathcal{B}(K)\big|$ T is invertible in $\mathcal{B}(K)$ and the matrix of T with respect to $\{e_n\}_{n\in\mathbb{N}_0}$ has exactly one non zero entry in each row and each column.$\}$

We observe the following:

(i) If $T_1, T_2 \in \mathcal{T}$, then $T_1T_2 \in \mathcal{T}$. However, T_1 and T_2 need not commute and hence elements of $\mathcal{T}$ are not simultaneously diagonalizable with respect to $\{e_n\}_{n\in\mathbb{N}_0}$.

(ii) If $T \in \mathcal{T}$ then its Hilbert adjoint T^* and inverse T^{-1} are also in $\mathcal{T}$.

(iii) Elements of $\mathcal{T}$ need not be self adjoint or normal.

In Chapter 2, we consider the operator weighted shift S on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$ and determine its reducing and minimal reducing subspaces.

In Chapter 3, we consider the unilateral shift U_+ on the operator weighted sequence space $H^2(B)$, where B denotes a uniformly bounded sequence of operators $\{B_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. We determine the reducing and minimal reducing subspaces of U_+ on $H^2(B)$.

In Chapters 4 and 5, we extend our study to a more general class of operators called operator pseudo shifts. The motivation to define operator pseudo shift comes from that of scalar weighted pseudo shift [12, 35]. We consider an injective map φ on $\mathbb{N}_0$ and for a uniformly bounded sequence of operators $\{A_n\}_{n\in\mathbb{N}_0}$ in K, we define T_φ on

$\ell^2_+(K)$ as

$$T_\varphi(f_0, f_1, \dots) = (A_0 f_{\varphi(0)}, A_1 f_{\varphi(1)}, \dots).$$

T_φ, also denoted as $T_{\varphi,\{A_n\}}$ is called the operator pseudo shift induced by φ, with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$.

For a given injective map φ, we define the following sets:

(i) $\mathcal{M}_1 = \{n \in \mathbb{N}_0 : n \notin R(\varphi)\}$, where $R(\varphi)$ denotes the range of φ.

(ii) $\mathcal{M}_2 = \{n \in \mathbb{N}_0 : n = \varphi^k(m)$ for some $m \in \mathcal{M}_1, k > 0\}$.

(iii) $\mathcal{M}_3 = \{n \in \mathbb{N}_0 : n = \varphi^k(n)$ for some $k > 0\}$.

(iv) $\mathcal{M}_4 = \mathbb{N}_0 - (\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3)$.

Based on these sets, we classify an operator pseudo shift T_φ into three types, as mentioned below:

(i) T_φ is said to be of Type I if $\mathcal{M}_3 = \phi$ and $\mathcal{M}_4 = \phi$.

(ii) T_φ is said to be of Type II if $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are empty and $\mathcal{M}_4 \neq \phi$.

(iii) T_φ is said to be of Type III if $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_4$ are empty and $\mathcal{M}_3 \neq \phi$.

In Chapter 4, we consider T_φ to be of type I and determine its reducing and minimal reducing subspaces.

In Chapter 5, we discuss about the reducing and minimal reducing subspaces of T_φ when it is of type II and type III.