Chapter 2

Minimal reducing subspaces of an operator weighted shift

2.1 Introduction

In this chapter, we consider a unilateral operator weighted shift S on $\ell^2_+(K)$ with a uniformly bounded sequence of weights $\{A_n\}_{n\in\mathbb{N}_0}$, and try to find its reducing and minimal reducing subspaces. The operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ are elements of the class $\mathcal{T}$. So, with reference to the definition of $\mathcal{T}$ given in the previous chapter, S is a unilateral operator weighted shift whose weights are not necessarily diagonalizable, and neither are these weights necessarily normal or self-adjoint.

2.2 Unitary equivalence

Let K be a separable complex Hilbert space with orthonormal basis $\{e_i\}_{i\in\mathbb{N}_0}$. Also for $i, j \in \mathbb{N}_0$, let $g_{i,j} := (0, \dots, e_i, 0, \dots)$ where e_i occurs at the j^{th} position. Then $\{g_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthonormal basis for $\ell^2_+(K)$.

We now consider the operator weighted sequence space $H^2(B)$, where B denotes a uniformly bounded sequence of operators $\{B_n\}_{n\in\mathbb{N}_0}$ on K. As $\|g_{i,j}\|_B = \|B_j e_i\|$, so if $f_{i,j} := \frac{g_{i,j}}{\|B_j e_i\|}$, then $\{f_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthonormal basis for the Hilbert space $H^2(B)$.

The unilateral shift U_+ on $H^2(B)$ is then defined as $U_+(f_0, f_1, \dots) = (0, f_0, f_1, \dots)$ and is bounded if and only if $\sup_{i,j} \frac{\|B_{j+1}e_i\|}{\|B_j e_i\|} < \infty$.

Theorem 2.2.1. *Let U_+ be the unilateral shift on $H^2(B)$, and for each $n \in \mathbb{N}_0$, we define the operator A_n on K as $A_n e_i = \left(\frac{\|B_{n+1}e_i\|}{\|B_n e_i\|}\right) e_i$. Then U_+ is unitarily equivalent to the unilateral operator weighted shift S on $\ell^2_+(K)$ with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$.*

Proof. Let $V : H^2(B) \to \ell^2_+(K)$ be defined as $V f_{i,j} = g_{i,j}$ for all $i, j \in \mathbb{N}_0$, and extend linearly. Then V is unitary and $V^* g_{i,j} = f_{i,j}$.
We claim: $U_+ = V^* S V$. To establish our claim choose $i, j \in \mathbb{N}_0$. Then,

$$\begin{aligned} U_+ f_{i,j} &= \frac{1}{\|B_j e_i\|} S g_{i,j} \\ &= \frac{g_{i,j+1}}{\|B_j e_i\|} \\ &= \frac{\|B_{j+1}e_i\|}{\|B_j e_i\|} f_{i,j+1}. \end{aligned}$$

Also, we have

$$\begin{aligned} V^* S V f_{i,j} &= V^* S(0, \dots, e_i, 0, \dots) \\ &= V^*(0, \dots, A_j e_i, 0, \dots), \end{aligned}$$

which implies $V^* S V f_{i,j} = \frac{\|B_{j+1}e_i\|}{\|B_j e_i\|} f_{i,j+1}$. Hence, $V^* S V = U_+$. □

For the converse, we consider a sequence $\{A_n\}_{n\in\mathbb{N}_0}$ of bounded linear operators on K such that $\sup_n \|A_n\| < \infty$. We first consider the case where A_n's are simultaneously diagonalizable with respect to $\{e_i\}_{i\in\mathbb{N}_0}$.

Theorem 2.2.2. *For $n \in \mathbb{N}_0$, let A_n be an invertible bounded linear operator on K such that the matrix of A_n with respect to $\{e_i\}_{i\in\mathbb{N}_0}$ is $diag(\delta_0^{(n)}, \delta_1^{(n)}, \delta_2^{(n)}, \dots)$. Also let $sup_n \|A_n\| < \infty$. If S is the unilateral operator weighted shift on $\ell^2_+(K)$ with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$, then S is unitarily equivalent to the unilateral shift U_+ on $H^2(B)$, where B denotes the sequence $\{B_n\}_{n\in\mathbb{N}_0}$ with $B_0 := I$ and $B_{n+1} := A_n A_{n-1} A_{n-2} \dots A_0$ for $n \in \mathbb{N}_0$.*

Proof. By Theorem 3.4 [34] we may assume that each A_n is positive. If $V : H^2(B) \to \ell^2_+(K)$ is defined linearly such that $Vf_{i,j} = g_{i,j}$ for all $i, j \in \mathbb{N}_0$, then V is unitary. Let $B_0 := I$ and $B_{n+1} := A_n A_{n-1} A_{n-2} \dots A_0$ for $n \in \mathbb{N}_0$. Then $\|B_{n+1}e_i\| = \delta_i^{(n)}\delta_i^{(n-1)} \dots \delta_i^{(0)}$ for all $i, n \in \mathbb{N}_0$ so that $\frac{\|B_{n+1}e_i\|}{\|B_n e_i\|} = \delta_i^{(n)}$. Then as in Theorem 2.2.1, it can be shown that $V^*SV = U_+$. □

Next, we consider the case where each A_n is in $\mathcal{T}$. Now, elements of $\mathcal{T}$ have a specific type of matrix representation with respect to $\{e_i\}_{i\in\mathbb{N}_0}$. Let $T \in \mathcal{T}$ and for $j \in \mathbb{N}_0$, let γ_j denote the non zero entry occurring in the j^{th} column of the matrix of T with respect to $\{e_i\}_{i\in\mathbb{N}_0}$. Then there exists a unique bijective map $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ such that γ_j occurs at the $\psi(j)^{th}$ row. Thus, if $[a_{i,j}]$ $(i, j \in \mathbb{N}_0)$ denotes the matrix of T with respect to $\{e_i\}_{i\in\mathbb{N}_0}$, then

$$a_{i,j} := \begin{cases} \gamma_j, & \text{if } i = \psi(j); \\ 0, & \text{otherwise.} \end{cases}$$

Thus for each $j \in \mathbb{N}_0$, $Te_j = \gamma_j e_{\psi(j)}$. Also $\|T\| = \sup_j |\gamma_j|$.

Since T is invertible in $\mathcal{B}(K)$, so $\gamma_j \neq 0$ for each $j \in \mathbb{N}_0$ and $T^{-1}e_{\psi(j)} = \frac{1}{\gamma_j}e_j$. Hence if $\varphi := \psi^{-1}$, then for each $i \in \mathbb{N}_0$,

$$\begin{aligned} T^{-1}e_i &= \frac{1}{\gamma_{\varphi(i)}} e_{\varphi(i)}, \text{ and} \\ \|T^{-1}\| &= \sup_i \frac{1}{|\gamma_{\varphi(i)}|} = \frac{1}{\inf_i |\gamma_{\varphi(i)}|} = \frac{1}{\inf_j |\gamma_j|}. \end{aligned}$$

If β_i denotes the non-zero entry in the i^{th} row of $[a_{i,j}]$, then for $x = \sum_{i\in\mathbb{N}_0} x_i e_i \in K$,

$$T(x_0, x_1, x_2, \dots) = (\beta_0 x_{\varphi(0)}, \beta_1 x_{\varphi(1)}, \dots).$$

Note that $K \cong \ell^2_+(\mathbb{C})$, so $x \cong (x_0, x_1, \dots)$. In [35], this operator T is called weighted pseudo shift and is denoted by $T_{b,\varphi}$, where $b = \{\beta_i\}_{i\in\mathbb{N}_0}$. We study this operator in Chapters 4 and 5.

Theorem 2.2.3. *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a sequence in $\mathcal{T}$ and $\sup_n \|A_n\| < \infty$. Then there exists a sequence $B = \{B_n\}_{n\in\mathbb{N}_0}$ of positive invertible diagonal bounded linear operators on K such that the unilateral operator weighted shift S on $\ell^2_+(K)$ with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$ is unitarily equivalent to the unilateral shift U_+ on $H^2(B)$.*

To prove the above theorem, we first prove the following lemmas.

Lemma 2.2.4. *Let $T \in \mathcal{T}$ and for $i \in \mathbb{N}_0$, let γ_i denote the only non zero entry in the matrix of T occurring in the ith column. If $T = UP$ is the polar decomposition of T, then P with respect to $\{e_i\}_{i\in\mathbb{N}_0}$ is diag$(|\gamma_0|, |\gamma_1|, |\gamma_2|, \dots)$ and U is unitary such that $U \in \mathcal{T}$ and $\frac{\gamma_i}{|\gamma_i|}$ is the only non-zero entry occurring in the ith column of the matrix of U with respect to the orthonormal basis $\{e_i\}_{i\in\mathbb{N}_0}$ of K.*

The proof being obvious is omitted.

Lemma 2.2.5. *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a sequence in $\mathcal{T}$ with $sup_n\|A_n\| < \infty$, and S be a unilateral operator weighted shift on $\ell^2_+(K)$ with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$. Then there exists a sequence $\{D_n\}_{n\in\mathbb{N}_0}$ of positive invertible diagonal operators on K such that S is unitarily equivalent to the operator weighted shift T on $\ell^2_+(K)$ with weight sequence $\{D_n\}_{n\in\mathbb{N}_0}$.*

Proof. For each $n \in \mathbb{N}_0$, there exists a bijective map $\psi_n : \mathbb{N}_0 \to \mathbb{N}_0$ such that $A_n e_i = \gamma_i^{(n)} e_{\psi_n(i)}$ for non-zero scalars $\gamma_i^{(n)}$ and $i \in \mathbb{N}_0$.

Let $A_n = U_n P_n$ be the polar decomposition of A_n. Then $P_n \geq 0$ is invertible diagonal and $P_n e_i = |\gamma_i^{(n)}| e_i$ for all $i \in \mathbb{N}_0$. Also U_n is unitary with $U_n e_i = \frac{\gamma_i^{(n)}}{|\gamma_i^{(n)}|} e_{\psi_n(i)}$ for all $i \in \mathbb{N}_0$. Define $P, M, U_+ : \ell^2_+(K) \to \ell^2_+(K)$ as follows:

$$P(x_0, x_1, \dots) = (P_0 x_0, P_1 x_1, \dots)$$

$$M(x_0, x_1, \dots) = (U_0 x_0, U_1 x_1, \dots)$$

$$U_+(x_0, x_1, \dots) = (0, x_0, x_1, \dots).$$

Then $S = (U_+M)P$, which is in fact the polar decomposition of S.

Let $V_0 = I$ and $V_{n+1} = U_n V_n$ for all $n \in \mathbb{N}_0$. Then each V_n is unitary on K. Let $V : \ell_+^2(K) \to \ell_+^2(K)$ be defined as $V(x_0, x_1, \dots) = (V_0 x_0, V_1 x_1, \dots)$. Then V is unitary and $U_+M = VU_+V^*$. Thus,

$$S = U_+MP = VU_+V^*P = V(U_+V^*PV)V^*.$$

As V is unitary, hence S is unitarily equivalent to U_+V^*PV.

Let $D_n := V_n^* P_n V_n$ for all $n \in \mathbb{N}_0$. For each $x \in K$,

$$\langle D_n x, x\rangle = \langle V_n^* P_n V_n x, x\rangle = \langle P_n V_n x, V_n x\rangle \geq 0.$$

This implies $D_n \geq 0$.

Also, as P_n is diagonal and V_n is unitary, so D_n is diagonal. If $T = U_+V^*PV$ then

$$T(x_0, x_1, \dots) = (0, D_0 x_0, D_1 x_1, \dots)$$

i.e, T is an operator weighted shift on $\ell_+^2(K)$ with weight sequence $\{D_n\}_{n\in\mathbb{N}_0}$ of positive invertible diagonal operators on K. □

Proof. Proof of Theorem 2.2.3.

By Lemma 2.2.5, there exists a sequence $\{D_n\}_{n\in\mathbb{N}_0}$ of positive invertible diagonal operators on K and an operator weighted shift T on $\ell_+^2(K)$ with weight sequence $\{D_n\}_{n\in\mathbb{N}_0}$ such that S is unitarily equivalent to T. By Theorem 2.2.2, T is unitarily equivalent to the unilateral shift U_+ on $H^2(B)$ with $B = \{B_n\}_{n\in\mathbb{N}_0}$ where $B_0 := I$ and $B_n := D_n D_{n-1} \dots D_0$ for $n \in \mathbb{N}_0$. Thus, S is also unitarily equivalent to U_+ on $H^2(B)$. □

Remark 2.2.6. Suppose we consider an operator $A \in \mathcal{T}$, whose matrix representation is $\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}$. The polar decomposition of A is then given by $A = VP$, where

$V = \begin{pmatrix} 0 & \frac{a}{|a|} & 0 \\ 0 & 0 & \frac{b}{|b|} \\ \frac{c}{|c|} & 0 & 0 \end{pmatrix}$ is the unitary matrix and $P = \begin{pmatrix} |c| & 0 & 0 \\ 0 & |a| & 0 \\ 0 & 0 & |b| \end{pmatrix}$ is the positive semi-definite matrix.

We can find $\{D_n\}_{n\in\mathbb{N}_0}$ as given in Lemma 2.2.5 in the following manner: Let $A_n \in \mathcal{T}$ and $A_n e_i = \gamma_i^{(n)} e_{\psi_n(i)}$, i.e, $\gamma_i^{(n)}$ occurs at the $\psi_n(i)$th row and ith column of the matrix representation of A_n. For each $n \in \mathbb{N}_0$, the polar decomposition of A_n is given by $A_n = U_n P_n$, where

$$P_n = diag(|\gamma_0^{(n)}|, |\gamma_1^{(n)}|, |\gamma_2^{(n)}|, \dots) \tag{2.2.1}$$

and if $[a_{i,j}]$ is the matrix representation of U_n with respect to $\{e_k\}_{k\in\mathbb{N}_0}$, then for each $j \in \mathbb{N}_0$, we must have

$$a_{i,j} := \begin{cases} \frac{\gamma_j^{(n)}}{|\gamma_j^{(n)}|}, & \text{if } i = \psi_n(j); \\ 0, & \text{elsewhere.} \end{cases} \tag{2.2.2}$$

From Lemma 2.2.5, we have for each $n \in \mathbb{N}_0$, $V_n : K \to K$ such that V_n is a unitary operator defined as

$$V_0 = I \,\text{and}\, V_{n+1} = U_n V_n \,\text{for all}\, n \in \mathbb{N}_0. \tag{2.2.3}$$

Also, $D_n := V_n^* P_n V_n$ for all $n \in \mathbb{N}_0$. Then 2.2.1 gives us $P_n e_i = |\gamma_i^{(n)}| e_i$ which clearly implies that $P_n^* e_i = |\gamma_i^{(n)}| e_i$. Again, 2.2.2 gives $U_n e_i = \frac{\gamma_i^{(n)}}{|\gamma_i^{(n)}|} e_{\psi_n(i)}$ so that $U_n^* e_i = \frac{|\gamma^{(n)}_{\psi_n^{-1}(i)}|}{\gamma_{\psi_n^{-1}(i)}} e_{\psi_n^{-1}(i)}$.

The recurrence relation 2.2.3 gives

$$V_n e_i = \frac{\gamma_i^{(0)} \gamma_{\psi_0(i)}^{(1)} \cdots \gamma_{\psi_{n-2}\psi_{n-3}\dots\psi_0(i)}^{(n-1)}}{\left|\gamma_i^{(0)} \gamma_{\psi_0(i)}^{(1)} \cdots \gamma_{\psi_{n-2}\psi_{n-3}\dots\psi_0(i)}^{(n-1)}\right|} e_{\psi_{n-1}\psi_{n-2}\dots\psi_0(i)}. \tag{2.2.4}$$

2.2.4 gives us the adjoint of V, i.e,

$$V_n^* e_i = \frac{\left|\gamma_{\psi_{n-1}^{-1}(i)}^{(n-1)} \gamma_{\psi_{n-2}^{-1}\psi_{n-1}^{-1}(i)}^{(n-2)} \cdots \gamma_{\psi_0^{-1}\psi_1^{-1}\dots\psi_{n-1}^{-1}(i)}^{(0)}\right|}{\gamma_{\psi_{n-1}^{-1}(i)}^{(n-1)} \gamma_{\psi_{n-2}^{-1}\psi_{n-1}^{-1}(i)}^{(n-2)} \cdots \gamma_{\psi_0^{-1}\psi_1^{-1}\dots\psi_{n-1}^{-1}(i)}^{(0)}} e_{\psi_0^{-1}\psi_1^{-1}\dots\psi_{n-1}^{-1}(i)}. \tag{2.2.5}$$

Hence, with the help of the relation $D_n := V_n^* P_n V_n$, we find D_n for each $n \in \mathbb{N}_0$ as

$$D_0 e_i = |\gamma_i^{(0)}| e_i$$
$$D_1 e_i = |\gamma_{\psi_0(i)}^{(1)}| e_i$$
$$D_2 e_i = |\gamma_{\psi_1\psi_0(i)}^{(2)}| e_i$$
$$D_3 e_i = |\gamma_{\psi_2\psi_1\psi_0(i)}^{(3)}| e_i, \ldots$$

Hence, for each $i \in \mathbb{N}_0$, $D_0 e_i = |\gamma_i^{(0)}| e_i$, and $D_n e_i = |\gamma_{\psi_{n-1}\psi_{n-2}\ldots\psi_0(i)}^{(n)}| e_i$ for $n > 0$. Thus, for each A_n, $n \in \mathbb{N}_0$ we get a positive invertible diagonal operator such that if A_n is given as $A_n = (\gamma_0^{(n)}, \gamma_1^{(n)}, \gamma_2^{(n)}, \ldots)$, where $\gamma_i^{(n)}$ occurs at the $\psi_n(i)$th row and ith column of the matrix representation of A_n, then the corresponding D_n is given as

$$D_0 = diag\big(|\gamma_0^{(0)}|, |\gamma_1^{(0)}|, |\gamma_2^{(0)}|, \ldots\big) \text{ for } n = 0,$$
$$D_n = diag\big(|\gamma_{\psi_{n-1}\psi_{n-2}\ldots\psi_0(0)}^{(n)}|, |\gamma_{\psi_{n-1}\psi_{n-2}\ldots\psi_0(1)}^{(n)}|, |\gamma_{\psi_{n-1}\psi_{n-2}\ldots\psi_0(2)}^{(n)}|, \ldots\big) \text{ for } n > 0.$$

The minimal reducing subspaces of U_+ on $H^2(B)$ is determined in [20], where it is assumed that B represents a uniformly bounded sequence of invertible diagonal operators on K. So in view of Theorem 2.2.3 and [20], we should be able to determine the minimal reducing subspaces of the unilateral operator weighted shift S on $\ell_+^2(K)$ with weights $\{A_n\}$ in $\mathcal{T}$. However, because of the complex transformations involved in the process, it is quite difficult to easily appreciate the end result. Hence in the present work, we adopt a different approach.

For unilateral operator weighted shift S with non diagonal operator weights, we first try and represent S as a direct sum of scalar weighted shift operators, as suggested in [44]. In this respect we have Theorem 3.9 [34] which we restate below for reference.

Theorem 2.2.7. *[34] The unilateral operator weighted shift S on $\ell_+^2(K)$ with operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ is a direct sum of scalar weighted shifts if and only if the* *weakly closed $*$ algebra generated by $\{I, A_0, A_1, \dots\}$ is diagonalizable.*

Note that an algebra $\mathcal{B}$ of operators is said to be diagonalizable if there is an orthonormal basis for the underlying space such that each operator in $\mathcal{B}$ is diagonal with respect to this basis.

We consider the unilateral operator weighted shift S on $\ell^2_+(K)$ with weights A_n in $\mathcal{T}$. In view of Lemma 2.2.5 and Theorem 2.2.7, it is possible to express S as a direct sum of scalar weighted shift operators. Based on these scalar weighted shifts, we then proceed to determine the minimal reducing subspaces of S.

2.3 Direct sum of scalar shifts

Since K is assumed to be a separable complex Hilbert space, so $K \cong \ell^2_+(\mathbb{C})$ where $\ell^2_+(\mathbb{C}) = \{x = (x_0, x_1, \dots) : x_i \in \mathbb{C}$ and $\sum_{i\in\mathbb{N}_0} |x_i|^2 < \infty\}$. Let $\{\xi_i\}_{i\in\mathbb{N}_0}$ denote the standard orthonormal basis for $\ell^2_+(\mathbb{C})$. If $\mu_{i,j} := (0, 0, \dots, \xi_j, 0, \dots)$ where ξ_j occurs at the i^{th} place, then $\{\mu_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthonormal basis for $\ell^2_+(\mathbb{C}) \oplus \ell^2_+(\mathbb{C}) \oplus \dots$.

Theorem 2.3.1. *Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$ where each A_n is positive invertible diagonal with respect to the orthonormal basis $\{e_i\}_{i\in\mathbb{N}_0}$ of K. Then there exists scalar weighted shift operators S_0, S_1,... on $\ell^2_+(\mathbb{C})$ such that S on $\ell^2_+(K)$ is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$ on $\ell^2_+(\mathbb{C}) \oplus \ell^2_+(\mathbb{C}) \oplus \dots$.*

Proof. For $n \in \mathbb{N}_0$, let A_n with respect to $\{e_i\}_{i\in\mathbb{N}_0}$ be the diagonal matrix given by $diag(\delta_0^{(n)}, \delta_1^{(n)}, \dots)$. Define S_n to be the scalar weighted shift on $\ell^2_+(\mathbb{C})$ with weight sequence $\{\delta_n^{(j)}\}_{j\in\mathbb{N}_0}$. Then $S_n\xi_j = \delta_n^{(j)}\xi_{j+1}$ for all $j \in \mathbb{N}_0$. Therefore,

$$(S_0 \oplus S_1 \oplus \dots)\mu_{i,j} = \delta_i^{(j)}\mu_{i,j+1}.$$

Also, $Sg_{i,j} = \delta_i^{(j)}g_{i,j+1}$. If $V : \ell^2(K) \to \ell^2_+(\mathbb{C}) \oplus \ell^2_+(\mathbb{C}) \oplus \dots$ is defined by $Vg_{i,j} = \mu_{i,j}$, then V is unitary and

$$\begin{aligned} VSV^*\mu_{i,j} &= VSg_{i,j} \\ &= \delta_i^{(j)} Vg_{i,j+1} \\ &= \delta_i^{(j)} \mu_{i,j+1} \\ &= (S_0 \oplus S_1 \oplus \dots)\mu_{i,j}. \end{aligned}$$

Thus, S on $\ell_+^2(K)$ is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$ on $\ell_+^2(\mathbb{C}) \oplus \ell_+^2(\mathbb{C}) \oplus \dots$. □

Remark 2.3.2. If $dim\, K < \infty$ then the above result can also be deduced using Lemma 2.1 [36]. A similar discussion can also be found in [6].

Theorem 2.3.3. *Let S be a unilateral operator weighted shift on $\ell_+^2(K)$ with uniformly bounded operator weights $\{A_n\}_{n\in\mathbb{N}_0}$ where each $A_n \in \mathcal{T}$. Then there exists scalar weighted shift operators S_0, S_1,... on $\ell_+^2(\mathbb{C})$ such that S on $\ell_+^2(K)$ is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$ on $\ell_+^2(\mathbb{C}) \oplus \ell_+^2(\mathbb{C}) \oplus \dots$.*

The proof follows immediately from Lemma 2.2.5 and Theorem 2.3.1. However, we include an independent proof so that the structure of S_n, which is often used in later sections, is explicitly given.

Proof. For each $A_n \in \mathcal{T}$, there exists a unique bijective map ψ_n on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ for all $j \in \mathbb{N}_0$. Let, $U : \ell_+^2(K) \to \ell_+^2(\mathbb{C}) \oplus \ell_+^2(\mathbb{C}) \oplus \dots$ be linearly defined such that

$$Ug_{i,j} := \begin{cases} \mu_{i,0}, & \text{if } j = 0; \\ \mu_{\psi_0^{-1}\psi_1^{-1}\dots\psi_{j-1}^{-1}(i),j}, & \text{if } j > 0. \end{cases}$$

Then U is unitary. For $n \in \mathbb{N}_0$, let S_n be scalar weighted shift on $\ell_+^2(\mathbb{C})$ with weight sequence $\{\gamma_n^{(0)}, \gamma_{\psi_0(n)}^{(1)}, \gamma_{\psi_1\psi_0(n)}^{(2)}, \dots\}$. i.e,

$$S_n\xi_j := \begin{cases} \gamma_n^{(0)}\xi_1, & \text{if } j = 0; \\ \gamma_{\psi_{j-1}\psi_{j-2}\dots\psi_0(n)}^{(j)}\xi_{j+1}, & \text{if } j > 0. \end{cases}$$

Therefore,

$$(S_0 \oplus S_1 \oplus \dots)\mu_{i,j} = \begin{cases} \gamma_i^{(0)}\mu_{i,1}, & \text{if } j = 0; \\ \gamma^{(j)}_{\psi_{j-1}\psi_{j-2}\dots\psi_0(i)}\mu_{i,j+1}, & \text{if } j > 0. \end{cases}$$

Hence for $j = 0$,

$$\begin{aligned} USU^*\mu_{i,0} &= USg_{i,0} \\ &= US(e_i, 0, 0, \dots) \\ &= U(0, A_0e_i, 0, \dots) \\ &= U(0, \gamma_i^{(0)}e_{\psi_0(i)}, 0, \dots) \\ &= \gamma_i^{(0)}Ug_{\psi_0(i),1} \\ &= \gamma_i^{(0)}\mu_{i,1} \\ &= (S_0 \oplus S_1 \oplus \dots)\mu_{i,0}. \end{aligned}$$

And for $j > 0$,

$$\begin{aligned} USU^*\mu_{i,j} &= USg_{\psi_{j-1}\psi_{j-2}\dots\psi_0(i),j} \\ &= US(0, \dots, e_{\psi_{j-1}\psi_{j-2}\dots\psi_0(i)}, 0, \dots) \\ &= U(0, \dots, 0, A_je_{\psi_{j-1}\psi_{j-2}\dots\psi_0(i)}, 0, \dots) \\ &= \gamma^{(j)}_{\psi_{j-1}\psi_{j-2}\dots\psi_0(i)}g_{\psi_j\psi_{j-1}\dots\psi_0(i),j+1} \\ &= \gamma^{(j)}_{\psi_{j-1}\psi_{j-2}\dots\psi_0(i)}\mu_{i,j+1} \\ &= (S_0 \oplus S_1 \oplus \dots)\mu_{i,j}. \end{aligned}$$

□

In view of Theorem 2.3.3, we now propose the following definitions.

Definition 2.3.4. Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. Let $S_0, S_1, \dots$ be scalar weighted shifts on $\ell^2_+(\mathbb{C})$ such that S is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$. For $n, m \in \mathbb{N}_0$, we say

'n is related to m with respect to S' denoted by $n \sim^S m$ if S_n and S_m are identical. Clearly $\sim^S$ is an equivalence relation on $\mathbb{N}_0$.

Definition 2.3.5. Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. Let $S_0, S_1, \dots$ be scalar weighted shifts on $\ell^2_+(\mathbb{C})$ such that S is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$. S is said to be of Type I if no two S_n's are identical. Otherwise, S is said to be of Type II. Thus, S is of Type II if and only if there exist distinct non negative integers n and m such that S_n and S_m are identical. An operator weighted shift S of Type II is said to be of Type III if $\sim^S$ partitions $\mathbb{N}_0$ into finite number of equivalence classes.

The above definition is motivated by similar definitions given in [50]. In fact for $dim\, K = N < \infty$ the two definitions refer to the same idea, as can be seen from the following discussion.

In [50], the minimal reducing subspaces of $M_z^N (N > 1)$ on the space $H^2(\beta) := \{f(z) = \sum_{n\in\mathbb{N}_0} f_n z^n : \|f\|^2_\beta = \sum_{n\in\mathbb{N}_0} |f_n|^2 \beta_n^2 < \infty\}$ is determined, where $\beta = \{\beta_0, \beta_1, \dots\}$ is a sequence of positive numbers.

If in the present study we consider $dim\, K = N$, and for each $n \in \mathbb{N}_0$, define

$$B_n := diag\big(\sqrt{\beta_{nN}}, \sqrt{\beta_{nN+1}}, \dots, \sqrt{\beta_{(n+1)N-1}}\big),$$

then M_z^N on $H^2(\beta)$ is unitarily equivalent to the unilateral shift U_+ on $H^2(B)$. Again if for each $n \in \mathbb{N}_0$, we define

$$A_n = diag\Big(\sqrt{\frac{\beta_{(n+1)N}}{w_{nN}}}, \sqrt{\frac{\beta_{(n+1)N+1}}{\beta_{nN+1}}}, \dots, \sqrt{\frac{\beta_{(n+2)N-1}}{\beta_{(n+1)N-1}}}\Big)$$

and consider S to be the unilateral operator weighted shift on $\ell^2_+(K)$ with weights $\{A_n\}_{n\in\mathbb{N}_0}$, then as in Theorem 2.2.1, U_+ is unitarily equivalent to S. Thus M_z^N on $H^2(\beta)$ is unitarily equivalent to the unilateral operator weighted shift S on $\ell^2_+(K)$

with weights $\{A_n\}_{n\in\mathbb{N}_0}$.

For $0 \le n \le N-1$, let S_n be the scalar weighted shift on $\ell^2_+(\mathbb{C})$ with weight sequence $\{\sqrt{\frac{\beta_{n+N}}{\beta_n}}, \sqrt{\frac{\beta_{n+2N}}{\beta_{n+N}}}, \sqrt{\frac{\beta_{n+3N}}{\beta_{n+2N}}}, \dots\}$. Then, as in Theorem 2.3.1, the unilateral operator weighted shift S on $\ell^2_+(K)$ with weights $\{A_n\}_{n\in\mathbb{N}_0}$ is unitarily equivalent to $S_0 \oplus \cdots \oplus S_{N-1}$ on $\ell^2_+(\mathbb{C}) \oplus \ell^2_+(\mathbb{C}) \oplus \cdots \oplus \ell^2_+(\mathbb{C})$ (N copies).

By Definition 2.3.5, S is of Type I if no two S_n's are identical. This means that for each $0 \le n \le N-1$ and $0 \le m \le N-1$ with $n \ne m$, there exists $l > 0$ such that $\sqrt{\frac{\beta_{n+lN}}{\beta_{n+(l-1)N}}} \ne \sqrt{\frac{\beta_{m+lN}}{\beta_{m+(l-1)N}}}$. If k is the smallest positive integer for which $\sqrt{\frac{\beta_{n+kN}}{\beta_{n+(k-1)N}}} \ne \sqrt{\frac{\beta_{m+kN}}{\beta_{m+(k-1)N}}}$, then $\frac{\beta_{n+kN}}{\beta_n} \ne \frac{\beta_{m+kN}}{\beta_m}$. So S is of Type I if for each $0 \le n \le N-1$ and $0 \le m \le N-1$ with $n \ne m$, there exists $k > 0$ such that $\frac{\beta_{n+kN}}{w_n} \ne \frac{\beta_{m+kN}}{w_m}$, and this according to [50] implies that the sequence β is of Type I.

2.4 Extremal functions of reducing subspaces

We begin the section by introducing a few definitions and notations which are to be used in subsequent results.

Definition 2.4.1. Let $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ be a non-zero vector in $\ell^2_+(K)$. The order of F, denoted as $o(F)$, is defined as the smallest non negative integer m such that $\alpha_m \ne 0$.

Definition 2.4.2. If $f = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$ is a non-zero vector in K, then order of f, denoted as $o(f)$, is defined to be the smallest non negative integer m such that $\alpha_m \ne 0$.

Definition 2.4.3. If $f = \sum_{i\in\mathbb{N}_0} \alpha_i e_i \in K$ then we define F_f in $\ell^2_+(K)$ as $F_f = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$. Clearly, for $f \ne 0$, $o(f) = o(F_f)$.

Definition 2.4.4. Let Y be a non-zero non-empty subset of K. Then order of Y, denoted as $o(Y)$, is defined to be the non negative integer m satisfying the following conditions:
(i) $o(f) \geq m$ for all $f \in Y$, and
(ii) there exists $\tilde{f} \in Y$ such that $o(\tilde{f}) = m$.

Definition 2.4.5. Let X be a subset of $\ell^2_+(K)$ and $\mathcal{L}_X := \{f_0 : (f_0, f_1, \dots) \in X\}$. If $\mathcal{L}_X$ is a non-zero subset of K, then order of X, denoted as $o(X)$, is defined as $o(\mathcal{L}_X)$.

Definition 2.4.6. Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. A linear expression $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ is said to be S-transparent if for every pair of non-zero scalars α_i and α_j, we have $i \sim^S j$.

Definition 2.4.7. Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$, and let $\mathcal{S}$ be the vector space of all finite linear combinations of finite products of S and S^*. For non-zero $F \in \ell^2_+(K)$, let $\mathcal{S}F := \{TF : T \in \mathcal{S}\}$. Then the closure of $\mathcal{S}F$ in $\ell^2_+(K)$ is a reducing subspace of S, denoted by X_F. Clearly X_F is the smallest reducing subspace of $\ell^2_+(K)$ containing F.

Lemma 2.4.8. *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$ and S be the unilateral operator weighted shift on $\ell^2_+(K)$ with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$. The following will hold :*

(i) *For each $n \in \mathbb{N}_0$, $A_n^* e_i = \gamma^{(n)}_{\psi_n^{-1}(i)} e_{\psi_n^{-1}(i)}$ for all $i \in \mathbb{N}_0$.*

(ii) $S^*(f_0, f_1, \dots) = (A_0^* f_1, A_1^* f_2, \dots)$ *for* $(f_0, f_1, \dots) \in \ell_+^2(K)$.

(iii) *For* $i, j \in \mathbb{N}_0$, $S g_{i,j} = \gamma_i^{(j)} g_{\psi_j(i), j+1}$ *and*

$$S^* g_{i,j} = \begin{cases} 0, & \text{if } j = 0; \\ \gamma^{(j-1)}_{\psi^{-1}_{j-1}(i)} g_{\psi^{-1}_{j-1}(i), j-1}, & \text{if } j > 0. \end{cases}$$

(iv) *For* $i, j \in \mathbb{N}_0$, $(S^*)^k S^k g_{i,j} = \begin{cases} [\gamma_i^{(j)}]^2 g_{i,j}, & \text{if } k = 1; \\ \left[\gamma_i^{(j)} \gamma_{\psi_j(i)}^{(j+1)} \dots \gamma^{(j+k-1)}_{\psi_{j+k-2} \dots \psi_j(i)}\right]^2 g_{i,j}, & \text{if } k > 1. \end{cases}$

(v) *For distinct non-negative integers* n *and* m, *if* $n \sim^S m$ *then* $\left\|(S^*)^k S^k g_{n,0}\right\| = \left\|(S^*)^k S^k g_{m,0}\right\|$ *for each* $k \in \mathbb{N}$.

Proof. (i) For $f = \sum_{j \in \mathbb{N}_0} \alpha_j e_j \in K$ and $n \in \mathbb{N}_0$,

$$\begin{aligned} \langle A_n f, e_i \rangle &= \sum_j \alpha_j \langle \gamma_j^{(n)} e_{\psi_n(j)}, e_i \rangle \\ &= \alpha_{\psi_n^{-1}(i)} \gamma^{(n)}_{\psi_n^{-1}(i)} \\ &= \langle f, \gamma^{(n)}_{\psi_n^{-1}(i)} e_{\psi_n^{-1}(i)} \rangle. \end{aligned}$$

Hence $A_n^* e_i = \gamma^{(n)}_{\psi_n^{-1}(i)} e_{\psi_n^{-1}(i)}$ for all $i \in \mathbb{N}_0$.

(ii) For $g = (g_0, g_1, \dots) \in \ell_+^2(K)$,

$$\begin{aligned} \langle Sg, f \rangle &= \sum_{i \in \mathbb{N}_0} \langle A_i g_i, f_{i+1} \rangle \\ &= \sum_{i \in \mathbb{N}_0} \langle g_i, A_i^* f_{i+1} \rangle \\ &= \langle g, (A_0^* f_1, A_1^* f_2, \dots) \rangle \end{aligned}$$

and so $S^*(f_0, f_1, \dots) = (A_0^* f_1, A_1^* f_2, \dots)$ for $f = (f_0, f_1, \dots) \in \ell_+^2(K)$.

(iii) follows from (i) and (ii), and (iv) follows from (iii).

(v) For $n \in \mathbb{N}_0$, let S_n be the scalar weighted shift on $\ell^2_+(\mathbb{C})$ with weight sequence $\{\gamma_n^{(0)}, \gamma_{\psi_0(n)}^{(1)}, \gamma_{\psi_1\psi_0(n)}^{(2)}, \dots\}$. Then by Theorem 2.3.3, S is unitarily equivalent to $S_0 \oplus S_1 \oplus \dots$. As $n \sim^S m$, so by Definition 2.3.4, S_n and S_m are identical. Therefore, $\gamma_n^{(0)} = \gamma_m^{(0)}$ and $\gamma_{\psi_k\psi_{k-1}\dots\psi_0(n)}^{(k+1)} = \gamma_{\psi_k\psi_{k-1}\dots\psi_0(m)}^{(k+1)}$ for all $k \geq 0$. The result now follows immediately from (iv). $\square$

Lemma 2.4.9. *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$ and S be the unilateral operator weighted shift on $\ell^2_+(K)$ with weight sequence $\{A_n\}_{n\in\mathbb{N}_0}$. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$. Let $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ be S-transparent in $\ell^2_+(K)$ with $o(F) = m$.*

If $\tilde{F}_k := \begin{cases} F, & \text{if } k = 0; \\ \sum_{i\in\mathbb{N}_0} \alpha_i g_{\psi_{k-1}\psi_{k-2}\dots\psi_0(i),k}, & \text{if } k > 1. \end{cases}$ *, then the following will hold :*

(i) $(S^*)^k S^k F = \begin{cases} \left[\gamma_m^{(0)}\right]^2 F, & \text{if } k = 1; \\ \left[\gamma_m^{(0)}\gamma_{\psi_0(m)}^{(1)} \dots \gamma_{\psi_{k-2}\dots\psi_0(m)}^{(k-1)}\right]^2 F, & \text{if } k > 1. \end{cases}$

(ii) $S\tilde{F}_k = \begin{cases} \gamma_m^{(0)} \tilde{F}_1, & \text{if } k = 0; \\ \gamma_{\psi_{k-1}\dots\psi_0(m)}^{(k)} \tilde{F}_{k+1}, & \text{if } k > 0. \end{cases}$

(iii) $S^*\tilde{F}_k = \begin{cases} 0, & \text{for } k = 0; \\ \gamma_m^{(0)} \tilde{F}_0, & \text{for } k = 1; \\ \gamma_{\psi_{k-2}\dots\psi_0(m)}^{(k-1)} \tilde{F}_{k-1}, & \text{for } k > 1, \end{cases}$

(iv) *X_F is the closed linear span of $\{\tilde{F}_k : k \in \mathbb{N}_0\}$.*

Proof. As $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ is S-transparent in $\ell^2_+(K)$ with $o(F) = m$, so the following must hold:

(a) $\alpha_m \neq 0$ and $\alpha_i = 0$ for $0 \leq i < m$.

(b) If $\alpha_i \neq 0$ and $\alpha_j \neq 0$, then $i \sim^S j$.

Thus we must have $i \sim^S m$ for all $i \in \mathbb{N}_0$ with $\alpha_i \neq 0$, and so,

$$\gamma_i^{(0)} = \gamma_m^{(0)} \text{ and } \gamma^{(k+1)}_{\psi_k\psi_{k-1}\dots\psi_0(i)} = \gamma^{(k+1)}_{\psi_k\psi_{k-1}\dots\psi_0(m)} \ \forall\ k \geq 0 \tag{2.4.1}$$

(i) Follows from 2.4.1 and Lemma 2.4.8(iv).

(ii) For $k = 0$, we get

$$S\tilde{F}_0 = SF = \sum_{i\in\mathbb{N}_0} \alpha_i S g_{i,0} = \sum_{i\in\mathbb{N}_0} \alpha_i \gamma_i^{(0)} g_{\psi_0(i),1} = \gamma_m^{(0)} \tilde{F}_1.$$

For $k > 0$,

$$\begin{aligned} S\tilde{F}_k &= \sum_{i\in\mathbb{N}_0} \alpha_i S g_{\psi_{k-1}\dots\psi_0(i),k} \\ &= \sum_{i\in\mathbb{N}_0} \alpha_i \gamma^{(k)}_{\psi_{k-1}\dots\psi_0(i)} g_{\psi_k\dots\psi_0(i),k+1} \\ &= \gamma^{(k)}_{\psi_{k-1}\dots\psi_0(m)} \tilde{F}_{k+1}. \end{aligned}$$

(iii) can be similarly shown using 2.4.1 and Lemma 2.4.8(iii).

(iv) By (ii) and (iii) each $\tilde{F}_k \in X_F$ and the *closed linear span*$\{\tilde{F}_k : k \in \mathbb{N}_0\}$ is a non-zero reducing subspace of S contained in X_F. Thus, by minimality of X_F, we have $X_F =$ *closed linear span*$\{\tilde{F}_k : k \in \mathbb{N}_0\}$. □

Definition 2.4.10. Let S be an operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. Let $\Omega_1, \Omega_2, \dots$ be the disjoint equivalence classes of $\mathbb{N}_0$ under the relation $\sim^S$. Consider $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0} \in l^2_+(K)$. For each k, let $q_k := \sum_{i\in\Omega_k} \alpha_i g_{i,0}$. Dropping those q_k which are zero, the remaining q_k's are arranged as $f_1, f_2, \dots$ in such a way that for $i < j$ we have $o(f_i) < o(f_j)$. The resulting decomposition $F = f_1 + f_2 + \dots$ is called the *canonical decomposition* of F with respect to S. Clearly each f_i is S-transparent in $\ell^2_+(K)$.

If there exists a finite positive integer n such that $F = f_1 + f_2 + \dots + f_n$, then F is said to have a *finite canonical decomposition*.

Lemma 2.4.11. *Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. Let X be a reducing subspace of S and $F = \sum_{i\in\mathbb{N}_0}\alpha_i g_{i,0}$ be in X. If F has a finite canonical decomposition $F = f_1+f_2+\cdots+f_n$, then each $f_i \in X_F$.*

Proof. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$. Let $o(f_i) = m_i$, so that $m_1 < m_2 < \cdots < m_n$. Clearly $m_i \nsim^S m_j$ for $i \neq j$.

Step I. As $m_1 \nsim^S m_n$ so either $\gamma_{m_1}^{(0)} \neq \gamma_{m_n}^{(0)}$, or there exists $k > 0$ such that $\gamma^{(k)}_{\psi_{k-1}\ldots\psi_0(m_1)} \neq \gamma^{(k)}_{\psi_{k-1}\ldots\psi_0(m_n)}$.

In case $\gamma_{m_1}^{(0)} = \gamma_{m_n}^{(0)}$, let k_1 be the smallest positive integer such that $\gamma^{(k_1)}_{\psi_{k_1-1}\ldots\psi_0(m_1)} \neq \gamma^{(k_1)}_{\psi_{k_1-1}\ldots\psi_0(m_n)}$. Let

$$Q_1 := \begin{cases} [(\gamma_{m_n}^{(0)})^2 - S^*S]F, & \text{if } \gamma_{m_1}^{(0)} \neq \gamma_{m_n}^{(0)}; \\ \left[(\gamma_{m_n}^{(0)}\gamma^{(1)}_{\psi_0(m_n)}\cdots\gamma^{(k_1)}_{\psi_{k_1-1}\ldots\psi_0(m_n)})^2 - (S^*)^{k_1+1}S^{k_1+1}\right]F, & \text{otherwise.}\end{cases}$$

For $1 \leq i \leq n-1$, let $\beta_i^{(1)} := (\gamma_{m_n}^{(0)})^2 - (\gamma_{m_i}^{(0)})^2$ if $\gamma_{m_1}^{(0)} \neq \gamma_{m_n}^{(0)}$; otherwise let

$$\beta_i^{(1)} := (\gamma_{m_n}^{(0)}\gamma^{(1)}_{\psi_0(m_n)}\cdots\gamma^{(k_1)}_{\psi_{k_1-1}\ldots\psi_0(m_n)})^2 - (\gamma_{m_i}^{(0)}\gamma^{(1)}_{\psi_0(m_i)}\cdots\gamma^{(k_1)}_{\psi_{k_1-1}\ldots\psi_0(m_i)})^2.$$

Then $\beta_1^{(1)} \neq 0$. Also since each f_i is S-transparent, so by applying Lemma 2.4.9(i), we get $Q_1 = \sum_{i=1}^{n-1}\beta_i^{(1)} f_i \in X_F$.

Step II. As $m_1 \nsim^S m_{n-1}$, so either $\gamma_{m_1}^{(0)} \neq \gamma_{m_n-1}^{(0)}$ or k_2 is the smallest positive integer such that $\gamma^{(k_2)}_{\psi_{k_2-1}\ldots\psi_0(m_1)} \neq \gamma^{(k_2)}_{\psi_{k_2-1}\ldots\psi_0(m_{n-1})}$. Let

$$Q_2 := \begin{cases} [(\gamma_{m_{n-1}}^{(0)})^2 - S^*S]Q_1, & \text{if } \gamma_{m_1}^{(0)} \neq \gamma_{m_{n-1}}^{(0)}; \\ \left[(\gamma_{m_{n-1}}^{(0)}\gamma^{(1)}_{\psi_0(m_{n-1})}\cdots\gamma^{(k_2)}_{\psi_{k_2-1}\ldots\psi_0(m_{n-1})})^2 - (S^*)^{k_2+1}S^{k_2+1}\right]Q_1, & \text{otherwise.}\end{cases}$$

For $1 \leq i \leq n-2$, let $\beta_i^{(2)} := (\gamma_{m_{n-1}}^{(0)})^2 - (\gamma_{m_i}^{(0)})^2$ if $\gamma_{m_1}^{(0)} \neq \gamma_{m_{n-1}}^{(0)}$; otherwise let

$$\beta_i^{(2)} := (\gamma_{m_{n-1}}^{(0)}\gamma^{(1)}_{\psi_0(m_{n-1})}\cdots\gamma^{(k_2)}_{\psi_{k_2-1}\ldots\psi_0(m_{n-1})})^2 - (\gamma_{m_i}^{(0)}\gamma^{(1)}_{\psi_0(m_i)}\cdots\gamma^{(k_2)}_{\psi_{k_2-1}\ldots\psi_0(m_i)})^2.$$

Then $\beta_1^{(2)} \neq 0$ and $Q_2 = \sum_{i=1}^{n-2} \beta_i^{(1)} \beta_i^{(2)} f_i \in X_F$.

Repeating the above argument $n-1$ times we get $Q_{n-1} = \beta_1^{(1)} \beta_1^{(2)} \dots \beta_1^{(n-1)} f_1 \in X_F$ with $\beta_1^{(i)} \neq 0$ for $1 \leq i \leq n-1$. This implies that $f_1 \in X_F$.
By a similar procedure it can be shown that $f_i \in X_F$ for $1 < i \leq n$. □

Lemma 2.4.12. *Let S be a unilateral operator weighted shift on $\ell_+^2(K)$ with uniformly bounded weights $\{A_n\}_{n \in \mathbb{N}_0}$ in $\mathcal{T}$. If X is a reducing subspace of S then $\mathcal{L}_X = 0$ if and only if $X = 0$.*

Proof. $X = 0 \Rightarrow \mathcal{L}_X = 0$.
Conversely, suppose $X \neq 0$, and let, if possible $\mathcal{L}_X = 0$. As $X \neq 0$ so we can choose $f = (0, f_1, f_2, \dots) \in X$ with $f_n \neq 0$. Then by Lemma 2.4.8(ii), $(S^*)^n f = (g_1, g_2, \dots)$ where $g_1 \neq 0$. As $(S^*)^n f \in X$, so $g_1 \in \mathcal{L}_X$, which is a contradiction. Thus, $X \neq 0 \Rightarrow \mathcal{L}_X \neq 0$. □

Theorem 2.4.13. *Let S be a unilateral operator weighted shift on $\ell_+^2(K)$ with uniformly bounded weights $\{A_n\}_{n \in \mathbb{N}_0}$ in $\mathcal{T}$. Let X be a non-zero reducing subspace of S with $o(X) = m$. Then the extremal problem*

$$\sup\{Re\ \alpha_m : F = (f_0, f_1, \dots) \in X,\ \|F\| \leq 1,\ f_0 = \sum_{i \in \mathbb{N}_0} \alpha_i e_i.\}$$

has a unique solution $G = \sum_{i \in \mathbb{N}_0} \alpha_i g_{i,0} \in X$ with $\|G\| = 1$ and $o(G) = m$.

Proof. Define $\varphi : X \to \mathbb{C}$ as $\varphi(F) = \alpha_m$, where $F = (f_0, f_1, \dots)$ and $f_0 = \sum_{i \in \mathbb{N}_0} \alpha_i e_i$. As $X \neq 0$, so by Lemma 2.4.12, $\mathcal{L}_X \neq 0$, and in view of Definition 2.4.5, $o(\mathcal{L}_X) = m = o(X)$. Therefore φ is a non-zero bounded linear functional on X. From [8] we know that there exists a unique $G \in X$ such that $\varphi(G) > 0$,

$\|G\| = 1$ and

$$\begin{aligned}\varphi(G) &= \sup\{Re\ \varphi(F) : F \in X,\ \|F\| \leq 1\} \\ &= \sup\{Re\ \alpha_m : F = (f_0, f_1, \dots) \in X,\ \|F\| \leq 1,\ f_0 = \sum_{i \in \mathbb{N}_0} \alpha_i e_i.\}\end{aligned}$$

We will show that $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ and $o(G) = m$. For this we consider $G = (g_0, g_1, \dots)$.

Claim I. If $F \in X$ and $\|F\| < 1$, then $Re\,\varphi(F) < \varphi(G)$.

Let, if possible, $Re\,\varphi(F) = \varphi(G)$. Let $H := \frac{F}{\|F\|}$. Then $H \in X$, $\|H\| = 1$ and $Re\,\varphi(H) > \varphi(G)$, contradicting the extremality of G. Hence, claim I is established.

Now for each $F \in X$, $Re\,\varphi(G + SF) = \varphi(G)$ and so by claim I, we must have $\|G + SF\| \geq 1$ which implies $G \perp SF$. In particular,

$$\begin{aligned}&\langle G, SS^*G\rangle = 0 \\ \Rightarrow &A_i^* g_{i+1} = 0\ \forall\ i \geq 0, \text{ by Lemma 2.4.8(ii)} \\ \Rightarrow &g_{i+1} = 0\ \forall\ i \geq 0.\end{aligned}$$

Thus, $G = (g_0, 0, 0, \dots)$. Let $g_0 = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$. Since, $o(\mathcal{L}_X) = m$, so $\alpha_i = 0$ for all $0 \leq i < m$. Also, $\varphi(G) > 0$ implies $\alpha_m \neq 0$.

Thus, $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ and $o(G) = m$. □

Remark 2.4.14. The function G in Theorem 2.4.13 is called the *extremal function* of the non-zero reducing subspace X of S.

Theorem 2.4.15. *Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. If the extremal function of a non-zero reducing subspace X of S has a finite canonical decomposition, then it must be S-transparent.*

Proof. Let X be a non-zero reducing subspace of order m and $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ be its extremal function. Also let $G = g_1 + g_2 + \cdots + g_n$ be the finite canonical decomposition of G.

Then, $g_1 = \sum_{i\in\mathbb{N}_0} \beta_i g_{i,0}$, such that $o(g_1) = m$ and $\beta_m = \alpha_m$. Also $\|g_1\| \le \|G\| = 1$. So by extremality of G, we must have $G = g_1$. As g_1, by definition, is S-transparent, so G is also S-transparent. □

2.5 Minimal Reducing subspaces

In this section we identify and study the minimal reducing subspaces of S in $\ell^2_+(K)$. It may be noted that in general there are many operators which have reducing subspaces that do not contain minimal reducing subspaces. One such operator is the operator of multiplication by z on the Bergman space $L^2(\mathbb{D}, dA)$, where $\mathbb{D}$ is the unit disc and dA is the area measure [26], [55].

Lemma 2.5.1. *Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. Let F be S-transparent and $o(F) = m$. If $G \in X_F$ is such that G is non zero and $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$, then $G = \lambda F$ for some non-zero scalar λ.*

Proof. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$.

As $G = (g, 0, 0, \dots)$ with $g \neq 0$ and $F = (f, 0, 0, \dots)$ with $f \neq 0$, so by Definition 2.4.7, $G = \sum_k \lambda_k (S^*)^k S^k F$ for scalars λ_k, not all zero. Let

$$\beta_k := \begin{cases} (\gamma_m^{(0)})^2, & \text{if } k = 1; \\ (\gamma_m^{(0)} \gamma_{\psi_0(m)}^{(1)} \dots \gamma_{\psi_{k-2}\cdots\psi_0(m)}^{(k-1)})^2, & \text{if } k > 1. \end{cases}$$

Then by Lemma 2.4.9(i), $(S^*)^k S^k F = \beta_k F$, where $\beta_k \neq 0$ for all k.

Therefore, $G = (\sum_k \lambda_k \beta_k) F = \lambda F$ for $\lambda = \sum_k \lambda_k \beta_k \neq 0$. □

Lemma 2.5.2. *Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. Let $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0}$ with $o(F) = m_1$. If $G \in X_F$ such that G is non zero and $G = \sum_{i\in\mathbb{N}_0} \beta_i g_{i,0}$, then $o(G) \ge m_1$.*

Proof. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$.

Let $F = f_1 + f_2 + \dots$ be the canonical decomposition of F with $o(f_i) = m_i$. If for each $i \in \mathbb{N}_0$,

$$\beta_k^{(i)} := \begin{cases} (\gamma_{m_i}^{(0)})^2, & \text{if } k = 1; \\ (\gamma_{m_i}^{(0)}\gamma_{\psi_0(m_i)}^{(1)} \dots \gamma_{\psi_{k-2}\dots\psi_0(m_i)}^{(k-1)})^2, & \text{if } k > 1. \end{cases}$$

then $(S^*)^k S^k f_i = \beta_k^{(i)} f_i$ for all $k \in \mathbb{N}_0$ and $i \in \mathbb{N}$. Now $G \in X_F$ implies

$$\begin{aligned} G &= \sum_{k\in\mathbb{N}_0} \lambda_k (S^*)^k S^k F \\ &= \sum_{k\in\mathbb{N}_0} \lambda_k (\sum_{i\in\mathbb{N}} \beta_k^{(i)} f_i) \\ &= \sum_{i\in\mathbb{N}} (\sum_{k\in\mathbb{N}_0} \lambda_k \beta_k^{(i)}) f_i. \end{aligned}$$

Therefore, $o(G) = o(f_1)$ if $\sum_{k\in\mathbb{N}_0} \lambda_k \beta_k^{(1)} \neq 0$, otherwise $o(G) > o(f_1)$. Hence $o(G) \geq m_1$. □

Theorem 2.5.3. *Let S be a unilateral operator weighted shift on $\ell_+^2(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$, and let X be a minimal reducing subspace of S . If $F = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0} \in X$, then F must be S-transparent.*

Proof. Let ψ_n denote the unique bijective map on $\mathbb{N}_0$ such that $A_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$ with $\gamma_j^{(n)} > 0$.

Let, if possible, F is not S-transparent. Then the canonical decomposition of $F = f_1 + f_2 + \dots$ will have at least two components f_1 and f_2.

Let $o(f_i) = n_i$. Then $n_1 \nsim^S n_2$ and so either $\gamma_{n_1}^{(0)} \neq \gamma_{n_2}^{(0)}$ or there exists a positive integer k such that $\gamma_{\psi_{k-1}\dots\psi_0(n_1)}^{(k)} \neq \gamma_{\psi_{k-1}\dots\psi_0(n_2)}^{(k)}$.

(i) If $\gamma_{n_1}^{(0)} \neq \gamma_{n_2}^{(0)}$, then define $G := S^*SF - (\gamma_{n_1}^{(0)})^2 F$ so that

$$G := [(\gamma_{n_2}^{(0)})^2 - (\gamma_{n_1}^{(0)})^2] f_2 + [(\gamma_{n_3}^{(0)})^2 - (\gamma_{n_1}^{(0)})^2] f_3 + \dots,$$

which implies $o(G) = o(f_2) = n_2$.

(ii) If $\gamma^{(0)}_{n_1} = \gamma^{(0)}_{n_2}$, then let k be the positive integer such that $\gamma^{(k)}_{\psi_{k-1}\dots\psi_0(n_1)} \neq \gamma^{(k)}_{\psi_{k-1}\dots\psi_0(n_2)}$ and $\gamma^{(i)}_{\psi_{i-1}\dots\psi_0(n_1)} = \gamma^{(i)}_{\psi_{i-1}\dots\psi_0(n_2)}$ for all $0 < i < k$. Then

$$\begin{aligned} G :&= (S^*)^{k+1}S^{k+1}F - (\gamma^{(0)}_{n_1}\gamma^{(1)}_{\psi_0(n_1)}\cdots\gamma^{(k)}_{\psi_{k-1}\dots\psi_0(n_1)})^2 F \\ &= [(\gamma^{(0)}_{n_2}\gamma^{(1)}_{\psi_0(n_2)}\cdots\gamma^{(k)}_{\psi_{k-1}\dots\psi_0(n_2)})^2 - (\gamma^{(0)}_{n_1}\gamma^{(1)}_{\psi_0(n_1)}\cdots\gamma^{(k)}_{\psi_{k-1}\dots\psi_0(n_1)})^2]f_2 + \dots \end{aligned}$$

which implies that $o(G) = o(f_2) = n_2$.

Thus, there exists $0 \neq G \in X$ such that $o(F) < o(G)$. Therefore X_G is a non-zero reducing subspace of S contained in X. By minimality of X, we must have $X_G = X$. But this implies $F \in X_G$ so that by Lemma 2.5.2, $o(F) \geq o(G)$ which is a contradiction. Thus, F must be S-transparent. ☐

Corollary 2.5.4. *Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. The extremal function of a minimal reducing subspace of S is always S-transparent.*

Theorem 2.5.5. *Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. Let X be a non-zero reducing subspace of S. Then X is minimal if and only if $X = X_F$ where $F \in X$ is S-transparent.*

Proof. If X is minimal then $X = X_G$ where G is the extremal function of X. Also by Corollary 2.5.4, G must be S-transparent.

Conversely, let $X = X_F$ where $F \in X$ is S-transparent. Then by Lemma 2.4.9, X_F is a reducing subspace of S. Thus, we only need to show that X_F is minimal reducing.

For this, let Y be a non-zero reducing subspace of S contained in X_F. If G is the extremal function of Y, then $G \in X_F$ and so by Lemma 2.5.1, $G = \lambda F$ for a non zero scalar λ. This implies that $F \in Y$.

Therefore $Y = X_F$, which shows that X_F is minimal. □

Corollary 2.5.6. *Let S be an operator weighted shift on $\ell^2_+(K)$ with weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. Every reducing subspace of S in $\ell^2_+(K)$, whose extremal function has a finite canonical decomposition must contain a minimal reducing subspace.*

The proof follows immediately from Lemma 2.4.11 and Theorem 2.5.5.

2.6 Conclusion

Theorem 2.6.1. *Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. If S is of Type I, then $X_{g_{n,0}}$ for $n \in \mathbb{N}_0$ are the only minimal reducing subspaces of S in $\ell^2_+(K)$.*

Proof. Let X be a minimal reducing subspace of S and G be the extremal function such that $X = X_G$. As S is of Type I, so the only S-transparent functions are $g_{n,0}$ and their scalar multiples. Hence, $X = X_{g_{n,0}}$ for $n \in \mathbb{N}_0$. □

Theorem 2.6.2. *Let S be a unilateral operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. If S is of Type II, then S has minimal reducing subspaces other than $X_{g_{n,0}}$ ($n \in \mathbb{N}_0$). In fact, for every S-transparent F, X_F is a minimal reducing subspace and hence S will have infinitely many minimal reducing subspaces in $\ell^2_+(K)$.*

Proof. Let Y be a non-zero reducing subspace of S such that $Y \subseteq X_F$. Let $Y = X_G$, where G is the extremal function. Then $G \in X_F$. So by Lemma 2.5.1, $G = \lambda F$, $\lambda \neq 0$, which implies $F \in Y$. Therefore $X_F = Y$. Hence, X_F is minimal. □

Theorem 2.6.3. *Let S be an operator weighted shift on $\ell^2_+(K)$ with uniformly bounded weights $\{A_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{T}$. If S is of Type III, then every reducing subspace of S must contain a minimal reducing subspace.*

Proof. Let X be a non-zero reducing subspace of S. If $X = X_F$ for some transparent function F, then X is minimal. Otherwise let $G = \sum_{i \in \mathbb{N}_0} \alpha_i g_{i,0} \in X$ and $G = f_1 + f_2 + \cdots + f_m$ be its canonical decomposition. Then by Lemma 2.4.11, each $f_i \in X$ and so X_{f_i} is a minimal reducing subspace in X. $\square$