Chapter 3

Minimal reducing subspaces of the unilateral shift U_+ on $H^2(B)$

3.1 Introduction

Our aim in this chapter is to investigate the minimal reducing subspaces of a unilateral shift U_+ on an operator weighted sequence space $H^2(B)$. We consider the operator weights $B = \{B_n\}_{n\in\mathbb{N}_0}$ on the separable Hilbert space K as a sequence of uniformly bounded invertible linear operators in the class $\mathcal{T}$. We recall that the unilateral shift U_+ is defined on $H^2(B)$ as

$$U_+(f_0, f_1, \dots) = (0, f_0, f_1, \dots)$$

for $(f_0, f_1, \dots)$ in $H^2(B)$. Clearly, U_+ is bounded if and only if $\sup_{i,j} \frac{\|B_{j+1}e_i\|}{\|B_j e_i\|} < \infty$.

In Corollary 2 of Theorem 3 [48], Shields has shown that U_+ on $H^2(\beta)$ is irreducible. Here, β denotes a sequence of positive numbers $\{\beta_n\}_{n\in\mathbb{N}_0}$ with $\beta_0 = 1$. In the case of operator shifts, the reducing subspaces of U_+ on $H^2(B)$ has been determined under specific assumptions on the weight sequence $\{B_n\}$. In [17], the weights $\{B_n\}$ are assumed to be commuting normal operators; in [50] it is assumed that $dim\, K = N < \infty$ and the weights $\{B_n\}$ are positive diagonal with respect to a fixed basis for K; in [20] $dim\, K = \aleph_0$ and $\{B_n\}$ are positive diagonals on K. In all these results we observe that B_n's are always assumed to be mutually commuting. Hence, in this chapter, we try to drop this assumption and consider the weight sequence $B = \{B_n\}_{n\in\mathbb{N}_0}$ to be in the more general class of operators $\mathcal{T}$.

As we are considering the operator weighted sequence space $H^2(B)$, where the uniformly bounded weight sequence $B = \{B_n\}_{n\in\mathbb{N}_0}$ is in $\mathcal{T}$, so for each $n \in \mathbb{N}_0$ there exists a unique bijective map ψ_n on $\mathbb{N}_0$ such that $B_n e_j = \gamma_j^{(n)} e_{\psi_n(j)}$, where $\gamma_j^{(n)}$ denotes the unique non zero entry occurring in the j^{th} column of the matrix of B_n.

Theorem 3.1.1. *For $i, j \in \mathbb{N}_0$, let $f_{i,j}$ (or $x^i y^j$) $\in H^2(B)$ be the vector that has e_i as the jth entry and zero as all other entries. Then, $\{f_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthogonal basis for $H^2(B)$.*

Proof. For $i, j \in \mathbb{N}_0$, we get $\| f_{i,j} \|_B^2 = \| B_j e_i \|^2 = |\gamma_i^{(j)}|^2$.

$$\langle f_{i,j}, f_{p,q}\rangle_B = \begin{cases} \langle B_j e_i, B_q e_p\rangle, & \text{if } j = q; \\ 0, & \text{if } j \neq q. \end{cases}$$

Since ψ_n is a bijective function for each $n \in \mathbb{N}_0$, so we get

$$\langle f_{i,j}, f_{p,q}\rangle_B = \begin{cases} \gamma_i^{(j)} \bar{\gamma}_p^{(q)}, & \text{if } j = q, i = p; \\ 0, & \text{otherwise.} \end{cases}$$

i.e,

$$\langle f_{i,j}, f_{p,q}\rangle_B := \begin{cases} |\gamma_i^{(j)}|^2, & \text{if } j = q,\ i = p; \\ 0, & \textit{otherwise}. \end{cases}$$

Let $X = (X_0, X_1, \dots) \in H^2(B)$ such that $\langle X, f_{i,j}\rangle_B = 0$ for all $i, j \in \mathbb{N}_0$. Also,

$X_0 = \sum_{p\in\mathbb{N}_0} \lambda_p e_p$, where $\{\lambda_p\}_{p\in\mathbb{N}_0}$ is a sequence of scalars. Thus for each $i \in \mathbb{N}_0$,

$$\begin{aligned}
&\langle X, f_{i,0}\rangle_B = 0 \\
\Rightarrow &\langle B_0 X_0, B_0 e_i\rangle = 0 \\
\Rightarrow &\sum_{p\in\mathbb{N}_0} \lambda_p \langle B_0 e_p, B_0 e_i\rangle = 0 \\
\Rightarrow &\lambda_i \|B_0 e_i\|^2 = 0 \\
\Rightarrow &\lambda_i = 0.
\end{aligned}$$

Therefore, $X_0 = 0$. Similarly, $X_1 = 0$, $X_2 = 0$ and so on. Thus for all $i, j \in \mathbb{N}_0$, $\langle X, f_{i,j}\rangle_B = 0$ implies that $X = 0$. This implies that $\{f_{i,j}\}_{i,j\in\mathbb{N}_0}$ is an orthogonal basis for $H^2(B)$. □

On the orthogonal basis $\{f_{i,j}\}_{i,j\in\mathbb{N}_0}$, the unilateral shift U_+ acts as $U_+ f_{i,j} = f_{i,j+1}$, or equivalently $U_+(x^i y^j) = x^i y^{j+1}$ for each $i, j \in \mathbb{N}_0$. Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the jth column of the matrix of B_n. On the basis of these scalars $\gamma_j^{(n)}$, we classify the weights into three classes: types I, II and III.

Definition 3.1.2. The weight sequence $\{B_n\}$ is said to be of *type* I if for each pair of distinct non negative integers m and n there exist some positive integer k such that

$$\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} \neq \frac{\gamma_n^{(k)}}{\gamma_n^{(0)}}.$$

Otherwise, it is said to be of *type* II. Thus $\{B_n\}$ is of *type* II if there exist distinct non negative integers m and n such that

$$\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} = \frac{\gamma_n^{(k)}}{\gamma_n^{(0)}}$$

for every positive integer k.

Definition 3.1.3. Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the jth column of the matrix of B_n. Two non negative integers m and n are said to be B-related (denoted by $m \sim^B n$) if for every positive integer k, we have

$$\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} = \frac{\gamma_n^{(k)}}{\gamma_n^{(0)}}.$$

Clearly, $\sim^B$ is an equivalence relation on the set $\mathbb{N}_0$.

Definition 3.1.4. Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$. A weight sequence $\{B_n\}$ of *type* II is said to be of *type* III if $\sim^B$ partitions $\mathbb{N}_0$ into a finite number of equivalence classes.

Remark 3.1.5. The above definitions are motivated by similar definitions given in [50]. In fact for $dim\, K = N < \infty$ the two definitions refer to the same idea. In [50] the minimal reducing subspaces of $M_z^N (N > 1)$ on the space $H^2(\beta)$ is determined, where $\beta = \{\beta_0, \beta_1, \dots\}$ is a sequence of positive numbers. If in the present study, we consider $dim\, K = N$, and for each $n \in \mathbb{N}_0$ if we define

$$B_n = diag\big(\sqrt{\beta_{nN}}, \sqrt{\beta_{nN+1}}, \dots, \sqrt{\beta_{(n+1)N-1}}\big),$$

then M_z^N on $H^2(\beta)$ is unitarily equivalent to the unilateral shift U_+ on $H^2(B)$.

Definition 3.1.6. Let $F = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ be a non-zero vector in $H^2(B)$. The order of F, denoted as $o(F)$, is defined as the smallest non negative integer m such that $\alpha_m \neq 0$.

Definition 3.1.7. If $f = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$ is a non-zero vector in K, then order of f, denoted as $o(f)$, is defined to be the smallest non negative integer m such that $\alpha_m \neq 0$.

Definition 3.1.8. Let Y be a non-zero non-empty subset of K. Then order of Y, denoted as $o(Y)$, is defined to be the non negative integer m satisfying the following conditions:

(i) $o(f) \geq m$ for all $f \in Y$, and

(ii) there exists $\tilde{f} \in Y$ such that $o(\tilde{f}) = m$.

Definition 3.1.9. Let X be a subset of $H^2(B)$ and $\mathcal{L}_X := \{f_0 : (f_0, f_1, \dots) \in X\}$. If $\mathcal{L}_X$ is a non-zero subset of K, then order of X, denoted as $o(X)$, is defined as $o(\mathcal{L}_X)$.

Definition 3.1.10. Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$. A linear expression $F = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ in $H^2(B)$ is said to be B-transparent if for every pair of non-zero scalars α_i and α_j, we have $i \sim^B j$.

Definition 3.1.11. Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and U_+ be the unilateral shift on $H^2(B)$. Let $\mathcal{S}$ be the vector space of all finite linear combinations of finite products of U_+ and U_+^*. For non-zero $F \in H^2(B)$, let $\mathcal{S}F := \{TF : T \in \mathcal{S}\}$. Then the closure of $\mathcal{S}F$ in $H^2(B)$ is a reducing subspace of U_+, denoted by X_F. Clearly X_F is the smallest reducing subspace of $H^2(B)$ containing F.

Lemma 3.1.12. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. If U_+ is the unilateral shift on $H^2(B)$, then for $i, j \in \mathbb{N}_0$, the following will hold:*

(i) $U_+^* f_{i,j} = \begin{cases} 0 & \text{if } j = 0, \\ \left|\frac{\gamma_i^{(j)}}{\gamma_i^{(j-1)}}\right| f_{i,j-1} & \text{if } j > 0. \end{cases}$

(ii) *For any non negative integer k, $(U_+^k)^* U_+^k f_{i,j} = \left|\frac{\gamma_i^{(j+k)}}{\gamma_i^{(j)}}\right|^2 f_{i,j}$.*

Proof. (i) For $i \in \mathbb{N}_0$, we have $\langle U_+X, f_{i,0}\rangle = 0$ for all $X \in H^2(B)$. This implies $U_+^* f_{i,0} = 0$.

Next we consider $X = (x_0, x_1, \dots)$ in $H^2(B)$, where $x_j = \sum_{t\in\mathbb{N}_0} \alpha_t^{(j)} e_t$ for each $j \in \mathbb{N}_0$. Then for $j > 0$, we have

$$\langle U_+X, f_{i,j}\rangle = \frac{1}{|\gamma_i^{(j)}|}\langle B_j x_{j-1}, B_j e_i\rangle = \alpha_i^{(j-1)}|\gamma_i^{(j)}|.$$

Choosing $\lambda_{i,j} = \left|\frac{\gamma_i^{(j)}}{\gamma_i^{(j-1)}}\right|$, we get

$$\begin{aligned}\langle X, \lambda_{i,j} f_{i,j-1}\rangle &= \frac{\lambda_{i,j}}{|\gamma_i^{(j-1)}|}\langle B_{j-1}x_{j-1}, B_{j-1}e_i\rangle \\ &= \frac{\lambda_{i,j}}{|\gamma_i^{(j-1)}|}\alpha_i^{(j-1)}\|B_{j-1}e_i\|^2 \\ &= \alpha_i^{(j-1)}|\gamma_i^{(j)}|.\end{aligned}$$

Therefore, $\langle U_+X, f_{i,j}\rangle = \langle X, \lambda_{i,j} f_{i,j-1}\rangle$, and so $U_+^* f_{i,j} = \lambda_{i,j} f_{i,j-1}$ for $j > 0$.

(ii) As $U_+^*U_+ f_{i,j} = \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| U_+^* f_{i,j+1} = \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right|^2 f_{i,j}$, so the result holds for $k = 1$. Suppose, $(U_+^*)^n U_+^n f_{i,j} = \left|\frac{\gamma_i^{(j+n)}}{\gamma_i^{(j)}}\right|^2 f_{i,j}$ holds for $n = k$. We will show that it also holds for $n = k + 1$.

$$\begin{aligned}(U_+^*)^{k+1}U_+^{k+1} f_{i,j} =& \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| U_+^*({U_+^*}^k U_+^k) f_{i,j+1} \\ =& \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| \left|\frac{\gamma_i^{(j+1+k)}}{\gamma_i^{(j+1)}}\right|^2 U_+^* f_{i,j+1} \\ =& \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| \left|\frac{\gamma_i^{(j+1+k)}}{\gamma_i^{(j+1)}}\right|^2 \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right| f_{i,j} \\ =& \left|\frac{\gamma_i^{(j+1+k)}}{\gamma_i^{(j)}}\right|^2 f_{i,j}.\end{aligned}$$

Thus, the results holds for all $k \in \mathbb{N}_0$ by induction. □

Lemma 3.1.13. *If U_+ is the unilateral shift on $H^2(B)$, then for any non negative integer k, $(U_+^k)^*U_+^k(x^iy^j) = \left|\frac{\gamma_i^{(j+k)}}{\gamma_i^{(j)}}\right|^2 x^iy^j$.*

Proof. As $U_+^*U_+(x^iy^j) = U_+^*(x^iy^{j+1}) = \left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right|^2 x^iy^j$, so the result holds for $k = 1$.

Suppose, $(U_+^*)^n U_+^n(x^iy^j) = \left|\frac{\gamma_i^{(j+n)}}{\gamma_i^{(j)}}\right|^2 x^iy^j$ holds for all $n = k$. We will show that it also holds for $n = k+1$.

$$\begin{aligned}(U_+^*)^{k+1}U_+^{k+1}(x^iy^j) =& U_+^*({U_+^*}^k U_+^k)(x^iy^{j+1})\\ =& U_+^*\left|\frac{\gamma_i^{(j+1+k)}}{\gamma_i^{(j+1)}}\right|^2 x^iy^{j+1}\\ =& \left|\frac{\gamma_i^{(j+1+k)}}{\gamma_i^{(j+1)}}\right|^2\left|\frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}}\right|^2 x^iy^j\\ =& \left|\frac{\gamma_i^{(j+1+k)}}{\gamma_i^{(j)}}\right|^2 x^iy^j.\end{aligned}$$

Thus, the results holds for all $k \in \mathbb{N}_0$ by induction. □

Lemma 3.1.14. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. Let $F = \sum_{i\in\mathbb{N}_0}\alpha_i f_{i,0}$ be B-transparent in $H^2(B)$ with $o(F) = m$. If for each $k \in \mathbb{N}_0$, $\tilde{F}_k := \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right| \sum_{i\in\mathbb{N}_0}\alpha_i\, f_{i,k}$, then the following will hold:*

(i) $(U_+^k)^*U_+^kF = \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right|^2 F$.

(ii) $U_+\tilde{F}_k = \tilde{F}_{k+1}$ *and* $U_+^*\tilde{F}_k = \begin{cases} 0, & \text{if } k = 0;\\ \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(k-1)}}\right|^2 \tilde{F}_{k-1}, & \text{if } k > 0.\end{cases}$

(iii) X_F *is the closed linear span of* $\{\tilde{F}_k : k \in \mathbb{N}_0\}$.

Proof. Since $o(F) = m$, so $\alpha_i = 0$ for all $i < m$. Let, $\Lambda = \{i \geq m : \alpha_i \neq 0\}$. Then $m \in \Lambda$, and for $i \in \Lambda$, $\frac{\gamma_i^{(k)}}{\gamma_i^{(0)}} = \frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}$ for each positive integer k.

(i) For $i \in \Lambda$ and positive integer k, by Lemma 3.1.12 (ii) we have

$$(U_+^k)^*U_+^kf_{i,0} = \left|\frac{\gamma_i^{(k)}}{\gamma_i^{(0)}}\right|^2 f_{i,0} = \left|\frac{\gamma_m^{(k)}}{\gamma_m^{(0)}}\right|^2 f_{i,0}.$$

Thus, we have

$$
\begin{aligned}
(U_+^k)^* U_+^k F &= (U_+^k)^* U_+^k (\sum_{i\in\Lambda} \alpha_i f_{i,0}) \\
&= \sum_{i\in\Lambda} \alpha_i \Big| \frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} \Big|^2 f_{i,0} \\
&= \Big| \frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} \Big|^2 F.
\end{aligned}
$$

(ii) For $i, j \in \mathbb{N}_0$, $U_+ f_{i,j} = \Big| \frac{\gamma_i^{(j+1)}}{\gamma_i^{(j)}} \Big| f_{i,j+1}$, and so

$$
\begin{aligned}
U_+ \tilde{F}_k &= \Big| \frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} \Big| \sum_{i\in\Lambda} \alpha_i \, U_+ f_{i,k} \\
&= \sum_{i\in\Lambda} \alpha_i \Big| \frac{\gamma_i^{(k)}}{\gamma_i^{(0)}} \Big| \Big| \frac{\gamma_i^{(k+1)}}{\gamma_i^{(k)}} \Big| f_{i,k+1} \\
&= \Big| \frac{\gamma_m^{(k+1)}}{\gamma_m^{(0)}} \Big| \sum_{i\in\Lambda} \alpha_i f_{i,k+1} \\
&= \tilde{F}_{k+1}.
\end{aligned}
$$

As $U_+^* f_{i,0} = 0$, so we have $U_+^* \tilde{F}_0 = 0$.

For $k > 0$,

$$
\begin{aligned}
U_+^* \tilde{F}_k &= \Big| \frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} \Big| \sum_{i\in\mathbb{N}_0} \alpha_i \, U_+^* f_{i,k} \\
&= \Big| \frac{\gamma_m^{(k)}}{\gamma_m^{(0)}} \Big| \sum_{i\in\mathbb{N}_0} \alpha_i \Big| \frac{\gamma_i^{(k)}}{\gamma_i^{(k-1)}} \Big| f_{i,k-1} \\
&= \Big| \frac{\gamma_m^{(k)}}{\gamma_m^{(k-1)}} \Big|^2 \tilde{F}_{k-1}.
\end{aligned}
$$

(iii) By (ii), each $\tilde{F}_k \in X_F$ and so the *closed linear span*$\{\tilde{F}_k : k \in \mathbb{N}_0\}$ is a non-zero reducing subspace of U_+ contained in X_F. Thus, by minimality of X_F, we have $X_F =$ *closed linear span*$\{\tilde{F}_k : k \in \mathbb{N}_0\}$. □

Lemma 3.1.15. *If $F = f(x)$ in $H^2(B)$ is transparent, then $X_F = Span\{Fy^k : k \in \mathbb{N}_0\}$.*

Proof. Let $X = Span\{Fy^k : k \in \mathbb{N}_0\}$ and let $F = \sum_{i\in\mathbb{N}_0} \alpha_i x_i$. Then

$$U_+^k F = U_+^k(\sum_{i\in\mathbb{N}_0} \alpha_i x_i) = \sum_{i\in\mathbb{N}_0} \alpha_i S^k x^i = \sum_{i\in\mathbb{N}_0} \alpha_i x^i y^k = Fy^k.$$

Hence, $Fy^k = S^k F \in X_F$ for all $k \in \mathbb{N}_0$. So $F \in X \subseteq X_F$. We claim that X is reducing for S.

For any $G \in X$, $SG = Gy$ and $X \subseteq H^2(B)$. So X is invariant under U_+. Also $U_+^*(x^i) = 0$ for all $i \geq 0$ and $F = f(x)$. So $U_+^*(F) = 0$. For any positive integer k, $U_+^*(Fy^k) = U_+^* U_+(Fy^c)$ where $c = k-1 \geq 0$. If the order of zero of F at the origin is m, then since F is transparent, so by Lemma 3.1.14 we have

$$\begin{aligned} U_+^*(Fy^k) =& U_+^* U_+(Fy^c) \\ =& U_+^* U_+(\sum_{i\in\mathbb{N}_0} \alpha_i x^i y^c) \\ =& \sum_{i\in\mathbb{N}_0} \alpha_i U_+^* U_+ x^i y^c \\ =& \sum_{i\in\mathbb{N}_0} \alpha_i |\frac{\alpha_i^{(c+1)}}{\alpha_i^{(c)}}|^2 x^i y^c \\ =& \sum_{i\in\mathbb{N}_0} \alpha_i |\frac{\alpha_i^{(k)}}{\alpha_i^{(c)}}|^2 x^i y^c \in X. \end{aligned}$$

Thus for any $G \in X$, $U_+^* G \in X$. Therefore, X is reducing under U_+. Since X_F is the smallest reducing subspace of U_+ containing F, so we must have $X = X_F$. □

Definition 3.1.16. Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and U_+ be the unilateral shift on $H^2(B)$. Let $\Omega_1, \Omega_2, \ldots$ be the disjoint equivalence classes of $\mathbb{N}_0$ under the relation $\sim^B$. Consider $F = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ in $H^2(B)$. For each $k = 1, 2, \ldots$, let $q_k := \sum_{i\in\Omega_k} \alpha_i g_{i,0}$. Dropping those q_k which are zero, the remaining q_k's are arranged as $f_1, f_2, \ldots$ in such a way that for $i < j$ we have $o(f_i) < o(f_j)$. The resulting decomposition $F = f_1 + f_2 + \ldots$ is called the *canonical decomposition* of F. Clearly each f_i is B-transparent in $H^2(B)$.

If there exists a finite positive integer n such that $F = f_1 + f_2 + \cdots + f_n$, then F in the above case is said to have a *finite canonical decomposition.*

Lemma 3.1.17. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. Let U_+ be the unilateral shift on $H^2(B)$, and X be a reducing subspace of U_+ in $H^2(B)$. If $F = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ in X has a finite canonical decomposition $F = f_1 + f_2 + \cdots + f_n$, then each f_i is in X_F.*

Proof. Let $o(f_i) = m_i$ so that $m_1 < m_2 < \cdots < m_n$ and no two of them are B-related. For $2 \le i \le n$, as $m_1 \not\sim^B m_i$, and so there exists a positive integer k_i such that $\frac{\gamma_{m_1}^{(k_i)}}{\gamma_{m_1}^{(0)}} \neq \frac{\gamma_{m_i}^{(k_i)}}{\gamma_{m_i}^{(0)}}$. Let k_i be the smallest positive integer having this property.
Let $q_1 := F$ and for $2 \le i \le n$, $q_i := \left[\left|\frac{\gamma_{m_i}^{(k_i)}}{\gamma_{m_i}^{(0)}}\right|^2 - (U_+^{k_i})^* U_+^{k_i}\right] q_{i-1}$. Then $q_i \in X_F$ for all $1 \le i \le n$. Also $q_n = (\beta_2 \ldots \beta_n)\, f_1$, where $\beta_i = \left|\frac{\gamma_{m_i}^{(k_i)}}{\gamma_{m_i}^{(0)}}\right|^2 - \left|\frac{\gamma_{m_1}^{(k_i)}}{\gamma_{m_1}^{(0)}}\right|^2$ for $2 \le i \le n$. As each $\beta_i \neq 0$, so $q_n \in X_F$ implies that $f_1 \in X_F$.
In a similar way it can be shown that $f_2, \ldots, f_n$ are also in X_F. □

3.2 An Extremal Problem

Theorem 3.2.1. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and U_+ be the unilateral shift on $H^2(B)$. Let X be a non zero reducing subspace of U_+ in $H^2(B)$ with $o(X) = m$. Then the extremal problem*

$$\sup\{Re\ \alpha_m : F = (f_0, f_1, \ldots) \in X,\ \|F\| \le 1,\ f_0 = \sum_{i\in\mathbb{N}_0} \alpha_i e_i.\}$$

has a unique solution $G = \sum_{i\in\mathbb{N}_0} \alpha_i g_{i,0} \in X$ with $\|G\| = 1$ and $o(G) = m$.

Proof. For $F = (f_0, f_1, \ldots) \in X$, we define $\varphi : X \to \mathbb{C}$ as $\varphi(F) = \alpha_m$ where $f_0 = \sum_{i\in\mathbb{N}_0} \alpha_i e_i$. Since $o(X) = m$, so φ is a non zero bounded linear functional on X. From [8], it follows that the extremal problem has a unique solution G in X

such that $\|G\| = 1$, $\varphi(G) > 0$ and

$$\begin{aligned}\varphi(G) &= \sup\{Re\ \varphi(F) : F \in X,\ \|F\| \leq 1\} \\ &= \sup\{Re\ \alpha_m : F = (f_0, f_1, \dots) \in X,\ \|F\| \leq 1,\ f_0 = \sum_{i\in\mathbb{N}_0} \alpha_i e_i.\}\end{aligned}$$

We claim that G has the form $G = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ with $o(G) = m$.

If $F \in X$ and $\|F\| < 1$, then by the maximality of G we must have $Re\varphi(F) < \varphi(G)$. Now as $Re\varphi(G + SF) = \varphi(G)$ for all $F \in X$, so we must have $\|G + SF\| \geq 1$. This implies that $G \perp SF$ for all $F \in X$. In particular $\langle G, U_+U_+^*G\rangle = 0$ which implies that $U_+^*G = 0$. Thus G is of the form $G = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$. Also $\varphi(G) > 0$ and $o(X) = m$ together imply $o(G) = m$. □

Note: The function G in Theorem 3.2.1 will be called the *extremal function* of X.

Theorem 3.2.2. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and U_+ be the unilateral shift on $H^2(B)$. If the extremal function of a non-zero reducing subspace of U_+ in $H^2(B)$ has a finite canonical decomposition, then it must be B-transparent.*

Proof. Let X be a non-zero reducing subspace of U_+ in $H^2(B)$ and $G = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ be its extremal function with $o(G) = m$. Let $G = g_1 + g_2 + \cdots + g_n$ be the finite canonical decomposition of G. Each g_i is B-transparent and also by Lemma 3.1.17, each of them is in X_G. Clearly $o(g_1) = m$ and $\|g_1\| \leq \|G\| = 1$. So by the extremality of G, we must have $G = g_1$. Thus G is B-transparent. □

3.3 Minimal reducing subspaces

Theorem 3.3.1. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and for each $n \in \mathbb{N}_0$ let $\gamma_j^{(n)}$ denote the unique non zero entry occurring in the j^{th} column of the matrix of B_n. Also let U_+ be the unilateral shift on $H^2(B)$. If X*

is a minimal reducing subspace of U_+ *in* $H^2(B)$ *and* $F = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ *is in* X*, then* F *is* B*-transparent.*

Proof. Let $o(F) = m$, and if possible, F is not B-transparent. So we must have a positive integer $k > m$ such that $\alpha_k \neq 0$ and $k \nsim^B m$. This means that there exists a positive integer l such that $\frac{\gamma_k^{(l)}}{\gamma_k^{(0)}} \neq \frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}$.

We define $G := (U_+^l)^* U_+^l F - \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2 F$. Clearly, G is in X, and we get

$$\begin{aligned}
G =&(U_+^l)^* U_+^l F - \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2 F \\
=&(U_+^l)^* U_+^l (\sum_{i=m}^{\infty} \alpha_i f_{i,0}) - \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2 (\sum_{i=m}^{\infty} \alpha_i f_{i,0}) \\
=&\sum_{i=m}^{\infty} \alpha_i \left|\frac{\gamma_i^{(l)}}{\gamma_i^{(0)}}\right|^2 f_{i,0} - \sum_{i=m}^{\infty} \alpha_i \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2 f_{i,0} \\
=&\sum_{i=m+1}^{\infty} \alpha_i \Big[\left|\frac{\gamma_i^{(l)}}{\gamma_i^{(0)}}\right|^2 - \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2\Big] f_{i,0}.
\end{aligned}$$

Thus, $G = \sum_{i=m+1}^{\infty} \gamma_i f_{i,0}$, where $\gamma_i = \alpha_i \Big[\left|\frac{\gamma_i^{(l)}}{\gamma_i^{(0)}}\right|^2 - \left|\frac{\gamma_m^{(l)}}{\gamma_m^{(0)}}\right|^2\Big]$. Also since $\gamma_k \neq 0$, so $G \neq 0$. Moreover, $o(F) < o(G)$ implies $F \notin X_G$. Hence X_G is a non-zero reducing subspace properly contained in X which contradicts the minimality of X. Hence F must be B-transparent. □

As an immediate corollary of the above result we have the following :

Corollary 3.3.2. *The extremal function of a minimal reducing subspace of* U_+ *in* $H^2(B)$ *is always* B*-transparent.*

Theorem 3.3.3. *Let* $B = \{B_n\}_{n\in\mathbb{N}_0}$ *be a uniformly bounded sequence of operators in* $\mathcal{T}$*, and* U_+ *be the unilateral shift on* $H^2(B)$*. Let* X *be a reducing subspace of* U_+ *in* $H^2(B)$*. Then* X *is minimal if and only if* $X = X_F$ *where* F *is* B*-transparent.*

Proof. If X is a minimal reducing subspace and G is the associated extremal function, then the reducing subspace $X_G \subseteq X$. The minimality of X gives $X = X_G$.

Note that by Corollary 3.3.2, G is B-transparent.

Conversely, let $X = X_F$, where F is B-transparent. Clearly X_F is a reducing subspace. We claim that X_F is minimal. Let Y be a non zero reducing subspace of U_+ contained in X_F and H be its extremal function, which is transparent. Then $H \in X_F$ and so by Lemma 3.1.14 (i), H is a scalar multiple of F. In particular, $F \in Y$. Thus, $Y = X_F$ which means that X_F must be minimal. □

Corollary 3.3.4. *Every reducing subspace of U_+ in $H^2(B)$, whose extremal function has a finite canonical decomposition, contains a minimal reducing subspace.*

Proof. Let X be a reducing subspace of U_+ in $H^2(B)$ whose associated extremal function G has a finite canonical decomposition. By Theorem 3.2.2, G is B-transparent and so X_G is a minimal reducing subspace of U_+ which is contained in X. Hence, the result. □

3.4 Conclusion

Theorem 3.4.1. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and U_+ be the unilateral shift on $H^2(B)$. If the weight sequence $\{B_n\}_{n\in\mathbb{N}_0}$ is of type I, then $X_{f_{n,0}}$ for $n \in \mathbb{N}_0$ are the only minimal reducing subspaces of U_+ in $H^2(B)$.*

Proof. Let X be a minimal reducing subspace of U_+ and G be its extremal function so that $X = X_G$. Since the weight sequence $\{B_n\}_{n\in\mathbb{N}_0}$ is of *type* I, so the only transparent functions are $f_{n,0}$ for $n \in \mathbb{N}_0$ and their scalar multiples. The result now follows from Theorem 3.3.3. □

Theorem 3.4.2. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and U_+ be the unilateral shift on $H^2(B)$. If $\{B_n\}_{n\in\mathbb{N}_0}$ is of type II, then U_+ has minimal reducing subspaces other than $X_{f_{n,0}}$, $n \in \mathbb{N}_0$.*

Proof. Since the weight sequence $\{B_n\}_{n\in\mathbb{N}_0}$ is of *type* II, so we can form a transparent function $F = \sum_{i\in\mathbb{N}_0} \alpha_i f_{i,0}$ where more than one α_i's are non zero. Clearly, X_F is a minimal reducing subspace of U_+ in $H^2(B)$ such that $X_F \neq X_{f_{n,0}}$ for any $n \in \mathbb{N}_0$. □

Theorem 3.4.3. *Let $B = \{B_n\}_{n\in\mathbb{N}_0}$ be a uniformly bounded sequence of operators in $\mathcal{T}$, and U_+ be the unilateral shift on $H^2(B)$. If $\{B_n\}_{n\in\mathbb{N}_0}$ is of type III, then every reducing subspace of U_+ in $H^2(B)$ must contain a minimal reducing subspace.*

Proof. Let X be a reducing subspace of U_+ and G be its extremal function. Since the weight sequence $\{B_n\}_{n\in\mathbb{N}_0}$ is of *type* III, so G must have a finite canonical decomposition, say $g_1 + g_2 + \cdots + g_n$. By Lemma 3.1.17, for each $1 \leq i \leq n$, $g_i \in X$ and so each X_{g_i} is a minimal reducing subspace of U_+ in $H^2(B)$ contained in X. □