Chapter 4

Generating functions and congruences for some partition functions using relations involving $R(q)$, $R(q^3)$ and $R(q^4)$

4.1 Introduction

Note that the generating functions in Chapters 2–3 have been found by using relations involving $R(q)$ and $R(q^2)$. In this chapter we find generating functions and congruences for some partition functions using relations involving $R(q)$, $R(q^3)$ and $R(q^4)$.

Recall from the introductory chapter that $p_3(n)$ denotes the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of 3 and the generating function for $p_3(n)$ is given by

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{1}{E_1E_3}.$$

In this chapter, we find the following exact generating function for $p_3(5n+1)$ by using relations involving $R(q)$ and $R(q^3)$.

Theorem 4.1.1. *For any nonnegative integer n, we have*

$$\sum_{n=0}^{\infty} p_3(5n+1)q^n = \frac{E_5^5}{E_1^6 E_{15}} + 10q\frac{E_5^{10}}{E_1^7 E_3^5} + q^2\frac{E_{15}^5}{E_3^6 E_5} + 45q^3\frac{E_5^5 E_{15}^5}{E_1^6 E_3^6} - 90q^5\frac{E_{15}^{10}}{E_1^5 E_3^7}. \tag{4.1.1}$$

We also deduce the following congruences, in which the first congruence is precisely (1.7.1).

Corollary 4.1.2. *We have*

$$p_3(25n+21) \equiv 0 \pmod 5 \tag{4.1.2}$$

and

$$\sum_{n=0}^{\infty} p_3(25n+21)q^n \equiv 10\left(\frac{E_{25}}{E_1^2 E_3} + q^2\frac{E_{75}}{E_1 E_3^2}\right) \pmod{25}. \tag{4.1.3}$$

Recall from Section 1.7 that $p_\beta(n)$ is defined

$$\sum_{n=0}^{\infty} \frac{q^{3n^2+3n+1}}{(q;q^3)_{n+1}(q^2;q^3)_{n+1}} = \sum_{n=0}^{\infty} p_\beta(n)q^n.$$

In this chapter, we present alternative proofs of the following congruences of Zhang and Shi [72] by employing the 5-dissections (1.4.1) and (1.4.2). In fact, we express the generating functions in terms of $R(q)$ and $R(q^3)$.

Theorem 4.1.3. *We have*

$$p_\beta(15n+7) \equiv 0 \pmod 5, \tag{4.1.4}$$

$$p_\beta(45n+23) \equiv 0 \pmod{15} \tag{4.1.5}$$

and

$$p_\beta(45n+41) \equiv 0 \pmod{15}. \tag{4.1.6}$$

We also find the following new generating function representations for $b_4(5n+3)$ and $a_4(5n)$, where $b_4(n)$ and $a_4(n)$ denote the number of 4-regular partitions and 4-core partitions, respectively, of n. In this case, we employ certain relations involving $R(q)$ and $R(q^4)$.

Theorem 4.1.4. *We have*

$$\sum_{n=0}^{\infty} b_4(5n+3)q^n = 3\frac{E_2^2E_{10}^6}{E_1^5E_4E_{20}^2} + q\frac{E_2^4E_5^5E_{20}^3}{E_1^6E_4^2E_{10}^4} + 4q^2\frac{E_2^3E_{10}E_{20}^3}{E_1^5E_4^2}. \tag{4.1.7}$$

Theorem 4.1.5. *We have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_4(5n)q^n \\
&= \frac{E_4^4E_{10}^{40}}{E_1^2E_2^8E_5^{15}E_{20}^{16}} - 3q\frac{E_4^2E_{10}^{15}}{E_1^5E_2^3E_{20}^6} + 4q\frac{E_4^3E_{10}^{30}}{E_1^3E_2^6E_5^{10}E_{20}^{11}} - 20q^2\frac{E_4E_5^5E_{10}^5}{E_1^6E_2E_{20}} \\
&\quad - 12q^2\frac{E_4^2E_{10}^{20}}{E_1^4E_2^4E_5^5E_{20}^6} + 24q^2\frac{E_4^3E_{10}^{35}}{E_1^2E_2^7E_5^{15}E_{20}^{11}} - 27q^3\frac{E_2E_5^{10}E_{20}^4}{E_1^7} \\
&\quad - 60q^3\frac{E_4E_{10}^{10}}{E_1^5E_2^2E_{20}} + 196q^3\frac{E_4^2E_{10}^{25}}{E_1^3E_2^5E_5^{10}E_{20}^6} - 83q^4\frac{E_5^5E_{20}^4}{E_1^6} \\
&\quad + 456q^4\frac{E_4E_{10}^{15}}{E_1^4E_2^3E_5^5E_{20}} + 296q^5\frac{E_{10}^5E_{20}^4}{E_1^5E_2} + 96q^5\frac{E_4E_{10}^{20}}{E_1^3E_2^4E_5^{10}E_{20}} \\
&\quad + 128q^6\frac{E_2E_5^5E_{20}^9}{E_1^6E_4E_{10}^5} + 592q^6\frac{E_{10}^{10}E_{20}^4}{E_1^4E_2^2E_5^5} + 512q^7\frac{E_{20}^9}{E_1^5E_4}.
\end{aligned} \tag{4.1.8}$$

We organize this chapter in the following way. In Section 4.2, we state some relations involving $R(q)$, $R(q^3)$ and $R(q^4)$ that will be used in the subsequent sections. In Section 4.3, we prove Theorem 4.1.1 and deduce the congruences in Corollary 4.1.2. In Section 4.4, we prove Theorem 4.1.3. In Section 4.5, we prove Theorem 4.1.4 whereas Theorem 4.1.5 is proved in the final section.

4.2 Preliminary lemmas

In the following lemma, we recall a few relations involving $R(q)$ and $R(q^3)$.

Lemma 4.2.1. (Ahmed, Baruah and Dastidar [1, Eqs. (3.11), (3.12), (3.13), (3.14)]) *If*

$$Z(q) = 5 - 18q^2\frac{E_1E_{15}^5}{E_3E_5^5} + 2\frac{E_3E_5^5}{q^2E_1E_{15}^5},$$

then

$$R(q)^4R(q^3)^2 + q^4\frac{R(q)^2}{R(q^3)^4} + \frac{R(q^3)^4}{R(q)^2} + \frac{q^4}{R(q)^4R(q^3)^2} = q^2Z(q),$$
$$q^2\frac{R(q)^3}{R(q^3)} - R(q)R(q^3)^3 - \frac{q^4}{R(q)R(q^3)^3} + q^2\frac{R(q^3)}{R(q)^3} = \frac{13q^2 - q^2Z(q)}{2},$$
$$R(q)^2R(q^3) + q^2\frac{R(q)}{R(q^3)^2} - \frac{R(q^3)^2}{R(q)} - \frac{q^2}{R(q)^2R(q^3)} = 3q$$

and

$$R(q)^3R(q^3)^4 - \frac{q^6}{R(q)^3R(q^3)^4} + q^2\frac{R(q^3)^3}{R(q)^4} - q^4\frac{R(q)^4}{R(q^3)^3}$$
$$= -\frac{51}{2}q^3 + \frac{3}{2}q^3Z(q) + q^2\frac{E_1^6}{qE_5^6} + \frac{E_3^6}{E_{15}^6} + 22q^3.$$

In the next lemma, we state a couple of relations involving $R(q)$ and $R(q^4)$.

Lemma 4.2.2. *We have*

$$R(q)R(q^4) + \frac{q^2}{R(q)R(q^4)} = 2q + \frac{E_1E_4E_{10}^{10}}{E_2^2E_5^2E_{20}^5} \tag{4.2.1}$$

and

$$\frac{R(q)^2R(q^2)}{R(q^4)} + \frac{R(q^4)}{R(q)^2R(q^2)} = 2 + 4q^2\frac{E_2E_{20}^5}{E_4E_{10}^5}. \tag{4.2.2}$$

Proof. From [43], we note that if $k_1 := k_1(q) = \dfrac{q}{R(q)R(q^4)}$, then

$$\frac{1 - 3k_1 + k_1^2}{k_1} = \frac{\psi^2(-q)}{q\psi^2(-q^5)}. \tag{4.2.3}$$

Now, recall from [19, Entry 9, p. 258] that

$$f(q,q^4)f(q^2,q^3) = \frac{\varphi(-q^5)E_5}{\chi(-q)}, \tag{4.2.4}$$

where

$$\chi(-q) := (q;q^2)_\infty = \frac{E_1}{E_2}. \tag{4.2.5}$$

Employing (4.2.4) in (2.2.9), and then using the resulting identity in (4.2.3), we arrive at (4.2.1).

Next, from [43], we also note that if $k_2 := k_2(q) = \dfrac{R(q^2)}{R(q^{1/2})^2R(q)}$, then

$$\frac{(1-k_2)^2}{4k_2} = q\frac{\chi(-q)}{\chi(-q^5)^5}, \tag{4.2.6}$$

Replacing q by q^2 in (4.2.6) and then employing (4.2.5), we arrive at (4.2.2). □

4.3 Proofs of Theorem 4.1.1 and Corollary 4.1.2

Proof of Theorem 4.1.1. We have,

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{1}{E_1E_3}.$$

Using (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+1}, dividing both sides of the resulting identity by q and then replacing q^5 by q, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_3(5n+1)q^n \\
&= \frac{E_5^5E_{15}^5}{E_1^6E_3^6}\Bigg(\Big(R(q)^3R(q^3)^4 - \frac{q^6}{R(q)^3R(q^3)^4}\Big) + q\Big(2\Big(R(q)^4R(q^3)^2 + \frac{q^4}{R(q)^4R(q^3)^2}\Big) \\
&\quad + 3\Big(R(q)R(q^3)^3 + \frac{q^4}{R(q)R(q^3)^3}\Big) + 2\Big(\frac{R(q^3)^4}{R(q)^2} + q^4\frac{R(q)^2}{R(q^3)^4}\Big)\Big) \\
&\quad + q^2\Big(6\Big(R(q)^2R(q^3) - \frac{q^2}{R(q)^2R(q^3)}\Big) - 6\Big(\frac{R(q^3)^2}{R(q)} - q^2\frac{R(q)}{R(q^3)^2}\Big) \\
&\quad + \Big(\frac{R(q^3)^3}{R(q)^4} - q^2\frac{R(q)^4}{R(q^3)^3}\Big)\Big) + q^3\Big(25 - 3\Big(\frac{R(q)^3}{R(q^3)} + \frac{R(q^3)}{R(q)^3}\Big)\Big)\Bigg) \\
&= \frac{E_5^5E_{15}^5}{E_1^6E_3^6}\Big(25q^3 + 6q^2\Big(R(q)^2R(q^3) + q^2\frac{R(q)}{R(q^3)^2} - \frac{R(q^3)^2}{R(q)} - \frac{q^2}{R(q)^2R(q^3)}\Big) \\
&\quad + 2q\Big(R(q)^4R(q^3)^2 + q^4\frac{R(q)^2}{R(q^3)^4} + \frac{R(q^3)^4}{R(q)^2} + \frac{q^4}{R(q)^4R(q^3)^2}\Big)
\end{aligned}$$

$$- 3q\Big(q^2\frac{R(q)^3}{R(q^3)} - R(q)R(q^3)^3 - \frac{q^4}{R(q)R(q^3)^3} + q^2\frac{R(q^3)}{R(q)^3}\Big)$$
$$+ \Big(R(q)^3R(q^3)^4 - \frac{q^6}{R(q)^3R(q^3)^4}\frac{R(q^3)^3}{R(q)^4} - q^4\frac{R(q)^4}{R(q^3)^3}\Big)\Big)$$

Now, using Lemma 4.2.1 in the above and then simplifying, we arrive at (4.1.1) to finish the proof. □

Proof of Corollary 4.1.2. Clearly, from (4.1.1), we have

$$p_3(5n+1) \equiv \frac{E_5^5}{E_1^6E_{15}} + q^2\frac{E_{15}^5}{E_3^6E_5} \pmod 5,$$

which by (2.4.2) gives

$$p_3(5n+1) \equiv \frac{E_5^4}{E_1E_{15}} + q^2\frac{E_{15}^4}{E_3E_5} \pmod 5. \tag{4.3.1}$$

Now, using (1.4.2) and then extracting the terms involving q^{5n+4}, we arrive at

$$p_3(5(5n+4)+1) = p_3(25n+21) \equiv 0 \pmod 5,$$

which is (4.1.2).

We now prove (4.1.3).

Employing (2.4.2) in (4.1.1), we have

$$\sum_{n=0}^{\infty} p_3(5n+1)q^n \equiv \frac{E_5^5}{E_1^6E_{15}} + 10q\frac{E_5^9}{E_1^2E_{15}} + q^2\frac{E_{15}^5}{E_3^6E_5} + 45q^3\frac{E_5^4E_{15}^4}{E_1E_3}$$
$$- 90q^5\frac{E_{15}^9}{E_3^2E_5} \pmod{25}. \tag{4.3.2}$$

Let $[q^{5n+r}]\{F(q)\}$, $r = 0, 1, \ldots, 4$ denote the terms after extracting the terms involving q^{5n+r}, dividing by q^r and then replacing q^5 by q.

Now, from (1.1.4), (1.1.7) and (2.4.2), we have

$$[q^{5n+4}]\left\{\frac{E_5^5}{E_1^6}\frac{1}{E_{15}}\right\} = \frac{5}{E_3}\Big(63 \times \frac{E_5^6}{E_1^7} + 52 \times 5^3q\frac{E_5^{12}}{E_1^{13}} + 63 \times 5^5q^2\frac{E_5^{18}}{E_1^{19}}$$
$$+ 6 \times 5^8q^3\frac{E_5^{24}}{E_1^{25}} + 5^{10}q^4\frac{E_5^{30}}{E_1^{31}}\Big)$$
$$\equiv 5 \times 63\frac{E_5^6}{E_1^7E_3}$$

$$\equiv 5\times 63\frac{E_5^5}{E_1^2E_3}$$
$$\equiv 5\times 63\frac{E_{25}}{E_1^2E_3} \pmod{25}. \tag{4.3.3}$$

Next, by employing (1.4.2) and (3.2.2), we see that

$$\begin{aligned}
\left[q^{5n+4}\right]\left\{10q\frac{E_5^9}{E_1^2E_{15}}\right\} &= \left[q^{5n+3}\right]\left\{10\frac{E_5^9}{E_1^2E_{15}}\right\}\\
&= 10\frac{E_1^9}{E_3}\left(15q+10\left(R(q)^5-\frac{q^2}{R(q)^5}\right)\right)\frac{E_5^{10}}{E_1^{12}}\\
&= 10\frac{E_5^{10}}{E_1^3E_3}\left(15q+10\left(11q+\frac{E_1^6}{E_5^6}\right)\right)\\
&= 10\frac{E_5^{10}}{E_1^3E_3}\left(125q+10\frac{E_1^6}{E_5^6}\right)\\
&= 25\left(10q\frac{E_5^{10}}{E_1^3E_3}+4\frac{E_1^3E_5^4}{E_3}\right)\\
&\equiv 0 \pmod{25}.
\end{aligned} \tag{4.3.4}$$

Again, by (1.1.7) and (2.4.2), we have

$$\begin{aligned}
\left[q^{5n+4}\right]\left\{q^2\frac{E_{15}^5}{E_3^6E_5}\right\} &= \frac{5}{E_1}\Big(63\times\frac{E_{15}^6}{E_3^7}+52\times 5^3q\frac{E_{15}^{12}}{E_3^{13}}+63\times 5^5q^2\frac{E_{15}^{18}}{E_3^{19}}\\
&\quad +6\times 5^8q^3\frac{E_{15}^{24}}{E_3^{25}}+5^{10}q^4\frac{E_{15}^{30}}{E_3^{31}}\Big)\\
&\equiv 5\times 63q^2\frac{E_{15}^6}{E_1E_3^7}\\
&\equiv 5\times 63q^2\frac{E_{15}^5}{E_1E_3^2}\\
&\equiv 5\times 63q^2\frac{E_{75}}{E_1E_3^2} \pmod{25}.
\end{aligned} \tag{4.3.5}$$

Similarly, by (4.3.1) and (2.4.2), we find that

$$\begin{aligned}
\left[q^{5n+4}\right]\left\{45q^3\frac{E_5^4E_{15}^4}{E_1E_3}\right\} &= \left[q^{5n+1}\right]\left\{45\frac{E_5^4E_{15}^4}{E_1E_3}\right\}\\
&\equiv 45E_1^4E_3^4\left(\frac{E_5^4}{E_1E_{15}}+q^2\frac{E_{15}^4}{E_3E_5}\right)\\
&\equiv 45\left(\frac{E_1^3E_5^4E_3^4}{E_{15}}+q^2\frac{E_1^4E_3^3E_{15}^4}{E_5}\right)\\
&\equiv 45\left(\frac{E_{25}}{E_1^2E_3}+q^2\frac{E_{75}}{E_1E_3^2}\right) \pmod{25}.
\end{aligned} \tag{4.3.6}$$

Finally, employing (1.4.2) and (3.2.2), we have

$$\begin{aligned}[q^{5n+4}]\left\{90q^5\frac{E_{15}^9}{E_3^2E_5}\right\} &= 90\frac{E_{25}^{10}}{E_1E_3^3}\left(15q^5+10q^2\left(R(q^3)^5-\frac{q^6}{R(q^3)^5}\right)\right)\\ &= 90\frac{E_{25}^{10}}{E_1E_3^3}\left(15q^5+10q^2\left(11q^3+\frac{E_3^6}{E_{15}^6}\right)\right)\\ &= 11250q^5\frac{E_{25}^{10}}{E_1E_3}+900\frac{E_3^3E_{25}^{10}}{E_1E_{15}^6}\\ &\equiv 0 \pmod{25}. \end{aligned} \tag{4.3.7}$$

Employing (4.3.3)-(4.3.7) in (4.3.2), we readily arrive at (4.1.3) to complete the proof. □

4.4 Proof of Theorem 4.1.3

Recall from Section 1.7 that

$$\sum_{n=0}^{\infty}p_\beta(3n+1)q^n=\frac{E_3^3}{E_1^2}.$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+2}, dividing both sides of the resulting identity by q^2, and then replacing q^5 by q, we find that

$$\begin{aligned}&\sum_{n=0}^{\infty}p_\beta(15n+7)q^n\\ &=5\frac{E_5^{10}E_{15}^3}{E_1^{12}}\Bigg(\Big(R(q)^6R(q^3)^3-\frac{q^6}{R(q)^6R(q^3)^3}\Big)+q\Big(-12\Big(R(q)^4R(q^3)^2+\frac{q^4}{R(q)^4R(q^3)^2}\Big)\\ &\quad+4\Big(R(q)R(q^3)^3+\frac{q^4}{R(q)R(q^3)^3}\Big)\Big)+q^2\Big(10\Big(R(q)^5-\frac{q^2}{R(q)^5}\Big)\\ &\quad+12\Big(\frac{R(q^3)^2}{R(q)}-q^2\frac{R(q)}{R(q^3)^2}\Big)+4\Big(\frac{R(q^3)^3}{R(q)^4}-q^2\frac{R(q)^4}{R(q^3)^3}\Big)\Big)\\ &\quad+q^3\Big(15-3\Big(\frac{R(q)^6}{R(q^3)^2}+\frac{R(q^3)^2}{R(q)^6}\Big)\Big)\Bigg).\end{aligned}$$

Congruence (4.1.4) follows immediately from the above. Note that if the terms $\left(R(q)^6R(q^3)^3-\frac{q^6}{R(q)^6R(q^3)^3}\right)$, $\left(R(q)^4R(q^3)^2+\frac{q^4}{R(q)^4R(q^3)^2}\right)$, etc. appearing in the above can be written in terms of E_1, E_3, E_5 and E_{15}, analogous to those in Lemma 2.2.1, then one may be able to find congruences modulo higher powers of 5, including the following congruences conjectured by Zhang and Shi [72, Conjecture 6]:

$$p_\beta(3\cdot 5^2 n+22)\equiv p_\beta(3\cdot 5^2 n+52)\equiv p_\beta(3\cdot 5^2 n+67)\equiv 0 \pmod{5^2},$$
$$p_\beta(3\cdot 5^4 n+547)\equiv p_\beta(3\cdot 5^4 n+1297)\equiv p_\beta(3\cdot 5^4 n+1672)\equiv 0 \pmod{5^3}.$$

Now we prove the remaining two congruences of Theorem 4.1.3.

From Section 1.7, we also recall that

$$\sum_{n=0}^{\infty} p_\beta(9n+5)q^n = 3\frac{E_3^6}{E_1^5}.$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+2}, dividing both sides of the resulting identity by q^2, and then replacing q^5 by q, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_\beta(45n+23)q^n \\
&= 15\frac{E_5^{25}E_{15}^6}{E_1^{30}}\Bigg(4x^{18}z^6 + q\Big(9x^{19}z^4 - 228x^{16}z^5 + 406x^{13}z^6\Big) + q^2\Big(-6x^{20}z^2 + 130x^{17}z^3 \\
&\quad + 1836x^{14}z^4 - 7182x^{11}z^5 + 3136x^8z^6\Big) + q^3\Big(-24x^{18}z - 2736x^{15}z^2 + 7420x^{12}z^3 \\
&\quad + 22401x^9z^4 - 22632x^6z^5 + 1870x^3z^6\Big) + q^4\Big(1558x^{16} + 6\frac{x^{19}}{z} - 2436x^{13}z \\
&\quad - 55332x^{10}z^2 + 34370x^7z^3 + 26370x^4z^4 - 1740xz^5 + 2240\frac{z^6}{x^2}\Big) + q^5\Big(49077x^{11} \\
&\quad - 2\frac{x^{20}}{z^3} - 390\frac{x^{17}}{z^2} + 1224\frac{x^{14}}{z} - 18816x^8z - 92598x^5z^2 + 22400x^2z^3 - 2610\frac{z^4}{x} \\
&\quad - 17580\frac{z^5}{x^4} - 3437\frac{z^6}{x^7}\Big) + q^6\Big(154652x^6 + 36\frac{x^{18}}{z^4} - 912\frac{x^{15}}{z^3} - 22260\frac{x^{12}}{z^2} + 14934\frac{x^9}{z} \\
&\quad - 11220x^3z - 71670z^2 - 18700\frac{z^3}{x^3} + 33948\frac{z^4}{x^6} + 14934\frac{z^5}{x^9} + 742\frac{z^6}{x^{12}}\Big) + q^7\Big(11890x \\
&\quad + \frac{x^{19}}{z^6} + 228\frac{x^{16}}{z^5} + 3654\frac{x^{13}}{z^4} - 18444\frac{x^{10}}{z^3} - 103110\frac{x^7}{z^2} + 17580\frac{x^4}{z} - 13440\frac{z}{x^2} \\
&\quad + 92598\frac{z^2}{x^5} + 31360\frac{z^3}{x^8} - 10773\frac{z^4}{x^{11}} - 1224\frac{z^5}{x^{14}} - 13\frac{z^6}{x^{17}}\Big) + q^8\Big(\frac{120130}{x^4} \\
&\quad + 204\frac{x^{14}}{z^6} + 7182\frac{x^{11}}{z^5} + 28224\frac{x^8}{z^4} - 30866\frac{x^5}{z^3} - 67200\frac{x^2}{z^2} - \frac{1740}{xz} + 20622\frac{z}{x^7}
\end{aligned}$$

$$- 55332\frac{z^2}{x^{10}} - 4060\frac{z^3}{x^{13}} + 342\frac{z^4}{x^{16}} + 6\frac{z^5}{x^{19}}\Big) + q^9\Big(- \frac{102049}{x^9} + 2489\frac{x^9}{z^6}$$
$$+ 22632\frac{x^6}{z^5} + 16830\frac{x^3}{z^4} - \frac{23890}{z^3} + \frac{56100}{x^3z^2} + \frac{22632}{x^6z} - 4452\frac{z}{x^{12}} + 2736\frac{z^2}{x^{15}}$$
$$+ 40\frac{z^3}{x^{18}}\Big) + q^{10}\Big(\frac{8364}{x^{14}} + 2930\frac{x^4}{z^6} + 1740\frac{x}{z^5} + \frac{20160}{x^2z^4} + \frac{30866}{x^5z^3} - \frac{94080}{x^8z^2} - \frac{7182}{x^{11}z}$$
$$+ 78\frac{z}{x^{17}} - 6\frac{z^2}{x^{20}}\Big) + q^{11}\Big(- \frac{41}{x^{19}} - \frac{290}{xz^6} + \frac{17580}{x^4z^5} - \frac{30933}{x^7z^4} - \frac{18444}{x^{10}z^3} + \frac{12180}{x^{13}z^2}$$
$$+ \frac{228}{x^{16}z}\Big) + q^{12}\Big(\frac{3772}{x^6z^6} - \frac{14934}{x^9z^5} + \frac{6678}{x^{12}z^4} + \frac{912}{x^{15}z^3} - \frac{120}{x^{18}z^2}\Big) + q^{13}\Big(- \frac{1197}{x^{11}z^6}$$
$$+ \frac{1224}{x^{14}z^5} - \frac{117}{x^{17}z^4} - \frac{2}{x^{20}z^3}\Big) + q^{14}\Big(\frac{38}{x^{16}z^6} - \frac{6}{x^{19}z^5}\Big)\Bigg),$$

where $x = R(q)$ and $z = R(q^3)$. Now (4.1.5) follows immediately from the above. In a similar way, extracting the terms involving q^{5n+4} , dividing both sides of the resulting identity by q^4, and then replacing q^5 by q, we can arrive at

$$p_\beta(45n + 41) \equiv 0 \pmod{15},$$

which is (4.1.6). Thus, we complete the proof.

4.5 Proof of Theorem 4.1.4

We have

$$\sum_{n=0}^{\infty} b_4(n)q^n = \frac{E_4}{E_1}.$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+3}, dividing both sides of the resulting identity by q^3, and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} b_4(5n+3)q^n = \frac{E_5^5E_{20}}{E_1^6}\Big(3\Big(R(q)R(q^4) + \frac{q^2}{R(q)R(q^4)}\Big) + q\Big(- 5 - \Big(\frac{R(q)^4}{R(q^4)} - \frac{R(q^4)}{R(q)^4}\Big)\Big)\Big). \tag{4.5.1}$$

Now, in Lemma 4.2.2, if we set

$$c_1 := R(q)R(q^4) + \frac{q^2}{R(q)R(q^4)} = 2q + \frac{E_1E_4E_{10}^{10}}{E_2^2E_5^2E_{20}^5} \tag{4.5.2}$$

and

$$c_2 := \frac{R(q)^2R(q^2)}{R(q^4)} + \frac{R(q^4)}{R(q)^2R(q^2)} = 2 + 4q^2\frac{E_2E_{20}^5}{E_4E_{10}^5},$$

then

$$c_3 := \frac{R(q)^4}{R(q^4)} - \frac{R(q^4)}{R(q)^4} = c_2a_2 + b_2, \tag{4.5.3}$$

where a_2 is given by (2.3.5), and b_2 is the expression for a_2 when q is replaced by q^2.

Using (4.5.2) and (4.5.3) in (4.5.1), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} b_4(5n+3)q^n \\
&= 3\frac{E_4E_{10}^{10}}{E_1^5E_2^2E_{20}^4} + q\frac{E_5^5E_{20}}{E_1^6} - 8q^2\frac{E_{10}^5E_{20}}{E_1^5E_2} - 4q^3\frac{E_2E_5^5E_{20}^6}{E_1^6E_4E_{10}^5} - 16q^4\frac{E_{20}^6}{E_1^5E_4} \\
&= \left(\frac{E_{10}^5}{E_2^4E_{20}^3} - 4q^2\frac{E_{20}^2}{E_2^3E_4}\right)\left(3\frac{E_2^2E_4E_{10}^5}{E_1^5E_{20}} + q\frac{E_2^4E_5^5E_{20}^4}{E_1^6E_{10}^5} + 4q^2\frac{E_2^3E_{20}^4}{E_1^6}\right).
\end{aligned}$$

Now, replacing q by q^2 in (2.2.5) and using the resulting identity in the above, we arrive at (4.1.7).

4.6 Proof of Theorem 4.1.5

We have

$$\sum_{n=0}^{\infty} a_4(n)q^n = \frac{E_4^4}{E_1}.$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n}, and then replacing q^5 by q, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_4(5n)q^n \\
&= \frac{E_5^5E_{20}^4}{E_1^6}\Bigg(\Big(R(q)^4R(q^4)^4 + \frac{q^8}{R(q)^4R(q^4)^4}\Big) + q\Big(-4\Big(R(q)^3R(q^4)^3 + \frac{q^6}{R(q)^3R(q^4)^3}\Big) \\
&\quad - 3\Big(\frac{R(q^4)^4}{R(q)} - q^6\frac{R(q)}{R(q^4)^4}\Big)\Big) + q^2\Big(4\Big(R(q)^2R(q^4)^2 + \frac{q^4}{R(q)^2R(q^4)^2}\Big) \\
&\quad - 8\Big(\frac{R(q^4)^3}{R(q)^2} - \frac{R(q)^2}{R(q^4)^3}\Big)\Big) + q^3\Big(24\Big(R(q)R(q^4) + \frac{q^2}{R(q)R(q^4)}\Big)
\end{aligned}$$

$$- 2\Big(\frac{R(q^4)^2}{R(q)^3} - q^2\frac{R(q)^3}{R(q^4)^2}\Big)\Big) + q^4\Big(- 25 - 8\Big(\frac{R(q)^4}{R(q^4)} - \frac{R(q^4)}{R(q)^4}\Big)\Big)\Big) \tag{4.6.1}$$

Now, we can easily see that

$$c_4 := \frac{R(q^4)^2}{R(q)^3} - q^2\frac{R(q)^3}{R(q^4)^2} = -c_1c_3 + a_5, \tag{4.6.2}$$

$$c_5 := \frac{R(q^4)^3}{R(q)^2} - q^4\frac{R(q)^2}{R(q^4)^3} = c_1c_4 + q^2c_3, \tag{4.6.3}$$

$$c_6 := \frac{R(q^4)^4}{R(q)} - q^6\frac{R(q)}{R(q^4)^4} = c_1c_5 - q^2c_4, \tag{4.6.4}$$

$$c_7 := R(q)^2R(q^4)^2 + \frac{q^4}{R(q)^2R(q^4)^2} = c_1^2 - 2q^2, \tag{4.6.5}$$

$$c_8 := R(q)^3R(q^4)^3 + \frac{q^6}{R(q)^3R(q^4)^3} = c_1^3 - 3q^2c_1 \tag{4.6.6}$$

and

$$c_9 := R(q)^4R(q^4)^4 + \frac{q^8}{R(q)^4R(q^4)^4} = c_7^2 - 2q^4, \tag{4.6.7}$$

where a_5 is given by (2.3.8).

Employing (4.5.2), (4.5.3), (4.6.2)–(4.6.7) in (4.6.1), we arrive at (4.1.8).