Chapter 5

Proofs of some conjectures of S. H. Chan on Appell-Lerch sums

5.1 Introduction

In this chapter we prove some conjectural congruences of S. H. Chan [26] as well as find some new congruences for the Appell-Lerch sums. Recall from Section 1.8 that $a(n)$ is defined by

$$\sum_{n=1}^{\infty} a(n)q^n := \phi(q),$$

where

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{n+1}}{(q;q^2)_{n+1}^2}.$$

We find the following representation of the generating function of $a(10n+9)$.

Theorem 5.1.1. *For any nonnegative integer n,*

$$\sum_{n=0}^{\infty} a(10n+9)q^n = 5\Big(46\frac{E_5E_{10}^2}{E_2^2} + 460q\frac{E_{10}^5}{E_1^3E_2} + 1125q^2\frac{E_{10}^8}{E_1^6E_5} + 1875q\frac{E_2^8E_5^9}{E_1^{16}} + 15625q^2\frac{E_2^8E_5^{15}}{E_1^{22}}\Big). \tag{5.1.1}$$

S. H. Chan's congruence (1.8.1) follows immediately from the above. In this chapter, we also prove (1.8.2) from the above representation of the generating function of $a(10n+9)$. Furthermore, we find the following new congruences:

For any nonnegative integer n, we have

$$a(1250n + 250r + 219) \equiv 0 \pmod{125}, \quad \text{for } r = 1, 3, 4. \tag{5.1.2}$$

We also prove the conjectural congruences (1.8.7) and (1.8.8) of S. H. Chan [26].

In the next section, we state some preliminary results. In Section 5.3, we prove Theorem 5.1.1, (1.8.2) and (5.1.2). In the final section, we prove (1.8.7) and (1.8.8).

The contents of this chapter have been accepted for publication in *The Ramanujan Journal* [17].

5.2 Preliminary results

Let $f(a, b)$ denotes Ramanujan's general theta function as defined in Section 1.2. We recall the following identities from [19].

Lemma 5.2.1. (Berndt [19, p. 45, Entry 29])*If $ab = cd$, then*

$$f(a, b)f(c, d) + f(-a, -b)f(-c, -d) = 2f(ac, bd)f(ad, bc) \tag{5.2.1}$$

and

$$f(a, b)f(c, d) - f(-a, -b)f(-c, -d) = 2af\left(\frac{b}{c}, ac^2d\right) f\left(\frac{b}{d}, acd^2\right). \tag{5.2.2}$$

In the next lemma, we state an identity analogous to Jacobi's identity (2.2.10).

Lemma 5.2.2. (Berndt [20, Corollary 1.3.22]) *We have*

$$\frac{E_1^2 E_4^2}{E_2} = \sum_{n=-\infty}^{\infty} (3n + 1)q^{3n^2+2n}. \tag{5.2.3}$$

5.3 Proofs of Theorem 5.1.1 and (5.1.2)

Proof of Theorem 5.1.1. From [26, Eq. (5.1)], we have

$$\sum_{n=0}^{\infty} a(2n + 1)q^n = \frac{E_2^8}{E_1^7},$$

which, with the aid of (2.2.6) and (2.2.7), may be simplified as

$$\begin{aligned}\sum_{n=0}^{\infty} a(2n+1)q^n &= \frac{E_2^3E_{10}}{E_1^2E_5} + 5q\frac{E_2^4E_{10}^4}{E_1^5E_5^2} \\ &= \left(\frac{E_5^2}{E_1} + q\frac{E_{10}^5}{E_2E_5^3}\right) + 5q\left(\frac{E_{10}^5}{E_2E_5^3} + 5q\frac{E_{10}^8}{E_1^3E_5^4}\right) \\ &= \frac{E_5^2}{E_1} + 6q\frac{E_{10}^5}{E_2E_5^3} + 25q^2\frac{E_{10}^8}{E_1^3E_5^4}.\end{aligned}$$

Employing (1.4.2) in the above, extracting the terms involving q^{5n+4}, dividing both sides of the resulting identity by q^4, and then replacing q^5 by q, we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} a(10n+9)q^n \\ &= 5\frac{E_5^5}{E_1^4} + 30q\frac{E_{10}^5}{E_1^3E_2} + 1775q^2\frac{E_2^8E_5^{15}}{E_1^{22}} + 4425\frac{E_2^8E_5^{15}}{E_1^{22}}\left(R(q)^5 - \frac{q^2}{R(q)^5}\right) \\ &\quad + 225\frac{E_2^8E_5^{15}}{E_1^{22}}\left(R(q)^{10} + \frac{q^4}{R(q)^{10}}\right). \qquad (5.3.1)\end{aligned}$$

By (3.2.2), the above can be simplified to

$$\sum_{n=0}^{\infty} a(10n+9)q^n = 5\Big(\frac{E_5^5}{E_1^4} + 6q\frac{E_{10}^5}{E_1^3E_2} + 45\frac{E_2^8E_5^3}{E_1^{10}} + 1875q\frac{E_2^8E_5^9}{E_1^{16}} + 15625q^2\frac{E_2^8E_5^{15}}{E_1^{22}}\Big).$$

Employing (2.2.6) in the above, we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} a(10n+9)q^n = 5\Big(&\frac{E_2^3E_5^2E_{10}}{E_1^5} + 5q\frac{E_{10}^5}{E_1^3E_2} + 45\frac{E_2^8E_5^3}{E_1^{10}} + 1875q\frac{E_2^8E_5^9}{E_1^{16}} \\ &+ 15625q^2\frac{E_2^8E_5^{15}}{E_1^{22}}\Big).\end{aligned}$$

With the aid of (2.2.7), the above can be rewritten in the form

$$\begin{aligned}\sum_{n=0}^{\infty} a(10n+9)q^n &= 5\Big(45\left(\frac{E_2^3E_5^2E_{10}}{E_1^5} + 5q\frac{E_2^4E_5E_{10}^4}{E_1^8}\right) + \frac{E_2^3E_5^2E_{10}}{E_1^5} + 5q\frac{E_{10}^5}{E_1^3E_2} \\ &\quad + 1875q\frac{E_2^8E_5^9}{E_1^{16}} + 15625q^2\frac{E_2^8E_5^{15}}{E_1^{22}}\Big) \\ &= 5\Big(46\Big(\frac{E_5E_{10}^2}{E_2^2} + 5q\frac{E_{10}^5}{E_1^3E_2}\Big) + 5q\frac{E_{10}^5}{E_1^3E_2} + 225q\Big(\frac{E_{10}^5}{E_1^3E_2} \\ &\quad + 5q\frac{E_{10}^8}{E_1^6E_5}\Big) + 1875q\frac{E_2^8E_5^9}{E_1^{16}} + 15625q^2\frac{E_2^8E_5^{15}}{E_1^{22}}\Big),\end{aligned}$$

which is equivalent to (5.1.1).

As corollaries to the above theorem, we now deduce the congruences in (1.8.2) originally conjectured by S. H. Chan in [26] and the new congruences in (5.1.2).

Corollary 5.3.1. *The congruences in* (1.8.2) *hold true.*

Proof. Taking congruences modulo 5 in (5.1.1) and using (2.4.2), we see that

$$\sum_{n=0}^{\infty} a(10n+9)q^n \equiv 5\times 46\ E_5E_{10}E_2^3 \pmod{25},$$

which can be rewritten with the aid of (2.2.10) as

$$\sum_{n=0}^{\infty} a(10n+9)q^n \equiv 5\times 46\ E_5E_{10}\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)} \pmod{25}.$$

As $k(k+1)\equiv 0,\ 1,$ or $2 \pmod 5$, equating the coefficients of q^{5n+r}, $r=3,4$ from both sides of the above, we easily arrive at the last two congruences of (1.8.2). Furthermore, we note that $k(k+1)\equiv 1 \pmod 5$ only when $k\equiv 2 \pmod 5$, that is, only when $2k+1\equiv 0 \pmod 5$. Therefore, equating the coefficients of q^{5n+1} from both sides of the above we arrive at the other congruence of (1.8.2), to complete the proof. □

Corollary 5.3.2. *The congruences in* (5.1.2) *hold true.*

Proof. From (5.1.1), we have

$$\sum_{n=0}^{\infty} a(10n+9)q^n \equiv 5\times 46\ \left(\frac{E_5E_{10}^2}{E_2^2}+10q\frac{E_{10}^5}{E_1^3E_2}\right) \pmod{125}. \tag{5.3.2}$$

Now, let $[q^{5n+r}]\left\{F(q)\right\}$, $r=0,1,\ldots,4$ denotes the terms after extracting the terms involving q^{5n+r}, dividing by q^r and then replacing q^5 by q.

With the aid of (1.4.2), we have

$$[q^{5n+1}]\left\{\frac{E_5E_{10}^2}{E_2^2}\right\}=\frac{E_1E_{10}^{10}}{E_2^{10}}\ \left(15q^3+10q\left(R(q^2)^5-\frac{q^4}{R(q^2)^5}\right)\right),$$

which, by (3.2.2), implies

$$\left[q^{5n+1}\right]\left\{\frac{E_5E_{10}^2}{E_2^2}\right\}=10q\frac{E_1E_{10}^4}{E_2^4}+125q^3\frac{E_1E_{10}^{10}}{E_2^{10}}. \tag{5.3.3}$$

Again, by (2.2.5) and (1.4.2), we obtain

$$\begin{aligned}&\left[q^{5n+1}\right]\left\{10q\frac{E_{10}^5}{E_1^3E_2}\right\}\\&=\left[q^{5n+1}\right]\left\{\frac{5}{2}\left(\frac{E_5^5}{E_1^4}-\frac{E_5E_{10}^2}{E_2^2}\right)\right\}\\&=5\Big(2\frac{E_5^{20}}{E_1^{19}}\left(R(q)^{15}-\frac{q^6}{R(q)^{15}}\right)+209q\frac{E_5^{20}}{E_1^{19}}\left(R(q)^{10}+\frac{q^4}{R(q)^{10}}\right)\\&\quad+5q\frac{E_1E_{10}^{10}}{E_2^{10}}\left(R(q^2)^5-\frac{q^4}{R(q^2)^5}\right)+920q^2\frac{E_5^{20}}{E_1^{19}}\left(R(q)^5-\frac{q^2}{R(q)^5}\right)\\&\quad+\frac{q^3}{2}\left(1015\frac{E_5^{20}}{E_1^{19}}-15\frac{E_1E_{10}^{10}}{E_2^{10}}\right)\Big).\end{aligned}$$

Employing (3.2.2), and then simplifying by using the identities in Lemma 2.2.2, we find that

$$\begin{aligned}&\left[q^{5n+1}\right]\left\{10q\frac{E_{10}^5}{E_1^3E_2}\right\}\\&=10\Big(\frac{E_1^{14}E_{10}^3}{E_2^{15}E_5}+150q\frac{E_1^{11}E_{10}^6}{E_2^{14}E_5^2}+5650q^2\frac{E_1^8E_{10}^9}{E_2^{13}E_5^3}+101825q^3\frac{E_1^5E_{10}^{12}}{E_2^{12}E_5^4}\\&\quad+1068125q^4\frac{E_1^2E_{10}^{15}}{E_2^{11}E_5^5}+7042500q^5\frac{E_{10}^{18}}{E_1E_2^{10}E_5^6}\\&\quad+29800000q^6\frac{E_{10}^{21}}{E_1^4E_2^9E_5^7}+79000000q^7\frac{E_{10}^{24}}{E_1^7E_2^8E_5^8}\\&\quad+120000000q^8\frac{E_{10}^{27}}{E_1^{10}E_2^7E_5^9}+80000000q^9\frac{E_{10}^{30}}{E_1^{13}E_2^6E_5^{10}}\Big).\end{aligned} \tag{5.3.4}$$

Invoking (5.3.3) and (5.3.4) in (5.3.2), we obtain

$$\sum_{n=0}^{\infty}a(50n+19)q^n\equiv 5^2\times 92\ \left(\frac{E_1^{14}E_{10}^3}{E_2^{15}E_5}+q\frac{E_1E_{10}^4}{E_2^4}\right)\pmod{125},$$

which, by (2.4.2) reduces to

$$\sum_{n=0}^{\infty}a(50n+19)q^n\equiv 5^2\times 92\ \left(\frac{E_5^2}{E_1}+qE_1E_2E_{10}^3\right)\pmod{125}.$$

Employing once again (1.4.2) in the above, extracting the terms involving q^{5n+4}, dividing both sides by q^4, and then replacing q^5 by q, we arrive at

$$\sum_{n=0}^{\infty} a(250n+219)q^n \equiv 5^2 \times 92\ E_5E_{10}E_2^3 \pmod{125}.$$

Employing (2.2.10) in the above and then proceeding as in the proof of the previous corollary, we conclude that

$$a(250(5n+r)+219) \equiv 0 \pmod{125}, \quad \text{for } r = 1, 3, 4.$$

Thus, we complete the proof of (5.1.2). □

Remark 5.3.3. *Proceeding as in the proof of Theorem 5.1.1, we may obtain the exact generating function of* $a(50n+19)$*, but the calculations and expressions are too lengthy and tedious even if we use* ***Mathematica****. Therefore, we decided not to include that lengthy generating function of* $a(50n+19)$*.*

5.4 Proofs of (1.8.7) and (1.8.8)

In this section, we prove the congruences (1.8.7) and (1.8.8) originally conjectured by S. H. Chan [26].

At first, setting $k = 5$ and $j = 1$ and 3 in [60, Eq. (2.7)], we have

$$4\sum_{n=0}^{\infty} a_{1,10}(n)q^n = \frac{(-q;q^{10})_\infty^2(-q^9;q^{10})_\infty^2E_{10}^5}{(q;q^{10})_\infty^2(q^9;q^{10})_\infty^2E_{20}^4} - 2\frac{E_{10}}{E_{20}^2}\sum_{n=-\infty}^{\infty}\frac{q^{5n(n+1)}}{1+q^{10n}}$$

and

$$4\sum_{n=0}^{\infty} a_{3,10}(n)q^n = \frac{(-q^3;q^{10})_\infty^2(-q^7;q^{10})_\infty^2E_{10}^5}{(q^3;q^{10})_\infty^2(q^7;q^{10})_\infty^2E_{20}^4} - 2\frac{E_{10}}{E_{20}^2}\sum_{n=-\infty}^{\infty}\frac{q^{5n(n+1)}}{1+q^{10n}},$$

respectively.

In view of (1.2.3), we can rewrite the above identities as

$$4\sum_{n=0}^{\infty} a_{1,10}(n)q^n = \frac{f^2(q,q^9)E_{10}^5}{f^2(-q,-q^9)E_{20}^4} - 2A(q^2) \tag{5.4.1}$$

and

$$4\sum_{n=0}^{\infty} a_{3,10}(n)q^n = \frac{f^2(q^3,q^7)E_{10}^5}{f^2(-q^3,-q^7)E_{20}^4} - 2A(q^2), \tag{5.4.2}$$

where

$$A(q) := \frac{E_5}{E_{10}^2}\sum_{n=-\infty}^{\infty}\frac{q^{5n(n+1)/2}}{1+q^{5n}}.$$

Proof of (1.8.7). To prove (1.8.7), first we aim to find a generating function of $a_{1,10}(2n+1)$. For that purpose, we need to find a 2-dissection of the first term of the right side of (5.4.1). To that end, we recast (5.4.1) as

$$4\sum_{n=0}^{\infty} a_{1,10}(n)q^n = \frac{f^2(q,q^9)f^2(-q^3,-q^7)E_{10}^5}{f^2(-q,-q^9)f^2(-q^3,-q^7)E_{20}^4} - 2A(q^2). \tag{5.4.3}$$

By Jacobi triple product identity, (1.2.3), we have

$$f(-q,-q^9)f(-q^3,-q^7) = \frac{(q;q^2)_\infty E_{10}^2}{(q^5;q^{10})_\infty} = \frac{E_1E_{10}^3}{E_2E_5}, \tag{5.4.4}$$

and hence, (5.4.3) reduces to

$$\begin{aligned}4\sum_{n=0}^{\infty} a_{1,10}(n)q^n &= \frac{E_2^2E_5^2}{E_1^2E_{10}E_{20}^4}f^2(q,q^9)f^2(-q^3,-q^7) - 2A(q^2)\\ &= \frac{E_2}{E_{20}^4}\cdot\frac{\varphi(-q^5)}{\varphi(-q)}\left(f^2(q,q^9)f^2(-q^3,-q^7)\right) - 2A(q^2),\end{aligned} \tag{5.4.5}$$

where (1.2.1) is used to arrive at the last equality.

Now, setting $a=q$, $b=q^9$, $c=-q^3$, and $d=-q^7$ in (5.2.1) and (5.2.2), and then adding, we find that

$$f(q,q^9)f(-q^3,-q^7) = f(-q^4,-q^{16})f(-q^8,-q^{12}) + qf(-q^6,-q^{14})f(-q^2,-q^{18}). \tag{5.4.6}$$

But, by Jacobi triple product identity, (1.2.3), we have

$$f(-q,-q^4)f(-q^2,-q^3) = E_1E_5.$$

Using the above identity and (5.4.4) in (5.4.6), we see that

$$f(q,q^9)f(-q^3,-q^7) = E_4E_{20} + q\frac{E_2E_{20}^3}{E_4E_{10}}.$$

Therefore, (5.4.5) can be rewritten as

$$4\sum_{n=0}^{\infty} a_{1,10}(n)q^n = \frac{E_2}{E_{20}^4}\cdot\frac{\varphi(-q^5)}{\varphi(-q)}\left(E_4^2E_{20}^2 + q^2\frac{E_2^2E_{20}^6}{E_4^2E_{10}^2} + 2q\frac{E_2E_{20}^4}{E_{10}}\right) - 2A(q^2). \tag{5.4.7}$$

Replacing q by $-q$ in the above, and then subtracting the resulting identity from (5.4.7), we find that

$$\begin{aligned}
&4\sum_{n=0}^{\infty} a_{1,10}(n)q^n - 4\sum_{n=0}^{\infty} a_{1,10}(n)(-q)^n \\
&= \frac{E_2}{E_{20}^4}\left(E_4^2E_{20}^2 + q^2\frac{E_2^2E_{20}^6}{E_4^2E_{10}^2}\right)\left(\frac{\varphi(-q^5)}{\varphi(-q)} - \frac{\varphi(q^5)}{\varphi(q)}\right) + 2q\frac{E_2^2}{E_{10}}\left(\frac{\varphi(-q^5)}{\varphi(-q)} + \frac{\varphi(q^5)}{\varphi(q)}\right).
\end{aligned}$$

With the aid of the trivial identity $\varphi(q)\varphi(-q) = \varphi^2(-q^2) = E_2^4/E_4^2$, we can rewrite the above as

$$\begin{aligned}
&4\sum_{n=0}^{\infty} a_{1,10}(n)q^n - 4\sum_{n=0}^{\infty} a_{1,10}(n)(-q)^n \\
&= \frac{E_4^2}{E_2^3E_{20}^4}\left(E_4^2E_{20}^2 + q^2\frac{E_2^2E_{20}^6}{E_4^2E_{10}^2}\right)\left(\varphi(q)\varphi(-q^5) - \varphi(-q)\varphi(q^5)\right) \\
&\quad + 2q\frac{E_4^2}{E_2^2E_{10}}\left(\varphi(q)\varphi(-q^5) + \varphi(-q)\varphi(q^5)\right).
\end{aligned} \tag{5.4.8}$$

Now, recall from [19, p. 278] that

$$\varphi(q)\varphi(-q^5) - \varphi(-q)\varphi(q^5) = 4qE_4E_{20}.$$

Furthermore, from Entries 25(i) and 25(ii) of [19, p. 40], it is easy to show that

$$\varphi(q)\varphi(-q^5) + \varphi(-q)\varphi(q^5) = 2\varphi(q^4)\varphi(q^{20}) - 8q^6\psi(q^8)\psi(q^{40}).$$

Therefore, (5.4.8) becomes

$$\begin{aligned}
&4\sum_{n=0}^{\infty} a_{1,10}(n)q^n - 4\sum_{n=0}^{\infty} a_{1,10}(n)(-q)^n \\
&= 4q\frac{E_4^3}{E_2^3E_{20}^3}\left(E_4^2E_{20}^2 + q^2\frac{E_2^2E_{20}^6}{E_4^2E_{10}^2}\right) + 2q\frac{E_4^2}{E_2^2E_{10}}\left(2\varphi(q^4)\varphi(q^{20}) - 8q^6\psi(q^8)\psi(q^{40})\right).
\end{aligned}$$

Extracting the terms involving q^{2n+1} from both sides of the above, dividing by q, and then replacing q^2 by q, we find that

$$2\sum_{n=0}^{\infty} a_{1,10}(2n+1)q^n$$

$$= \frac{E_2^3}{E_1^3E_{10}^3}\left(E_2^2E_{10}^2 + q\frac{E_1^2E_{10}^6}{E_2^2E_5^2}\right) + \frac{E_2^2}{E_1^2E_5}\left(\varphi(q^2)\varphi(q^{10}) - 4q^3\psi(q^4)\psi(q^{20})\right)$$
$$= \frac{E_1^2}{E_5} + 6q\frac{E_2E_{10}^3}{E_1E_5^2} + \frac{E_2^2}{E_1^2E_5}\left(\varphi(q^2)\varphi(q^{10}) - 4q^3\psi(q^4)\psi(q^{20})\right), \tag{5.4.9}$$

where we have used (2.2.7) to arrive at the last equality.

Now, to prove (1.8.7), we see from the above that it is enough to show that the coefficients of q^{5n+2} of the terms on the right side of the above are multiples of 5. We accomplish this in the remaining part of the proof.

With the aid of (1.4.1) and (1.4.2), we find that

$$\left[q^{5n+2}\right]\left\{\frac{E_1^2}{E_5} + 6q\frac{E_2E_{10}^3}{E_1E_5^2}\right\}$$
$$= -\frac{E_5^2}{E_1} + 6\frac{E_2^3E_5^5E_{10}}{E_1^8}\left(\left(R(q)^3R(q^2) + \frac{q^2}{R(q)^3R(q^2)}\right) - 2q\left(\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2}\right) - 5q\right).$$

Employing (2.2.2) and (2.2.4) in the above, and then simplifying by using (2.2.5), we obtain

$$\left[q^{5n+2}\right]\left\{\frac{E_1^2}{E_5} + 6q\frac{E_2E_{10}^3}{E_1E_5^2}\right\} = 5\frac{E_5^2}{E_1} + 30q\frac{E_2E_5E_{10}^3}{E_1^4}. \tag{5.4.10}$$

Next, by (2.4.2), we have

$$\frac{E_2^2}{E_1^2E_5}\ \varphi(q^2)\varphi(q^{10}) = \varphi(q^{10})\frac{E_4^5}{E_1^2E_5E_8^2}$$
$$\equiv \frac{E_{20}\varphi(q^{10})}{E_5^2E_{40}}\ E_1^3E_8^3 \pmod 5,$$

which, with the aid of Jacobi's identity, (2.2.10), can be written as

$$\frac{E_2^2}{E_1^2E_5}\ \varphi(q^2)\varphi(q^{10})$$
$$\equiv \frac{E_{20}\varphi(q^{10})}{E_5^2E_{40}}\ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}(2j+1)(2k+1)q^{j(j+1)/2+4k(k+1)} \pmod 5. \tag{5.4.11}$$

We now observe that $j(j+1)/2 + 4k(k+1) \equiv 2 \pmod 5$ only when $j \equiv 2 \pmod 5$ and $k \equiv 2 \pmod 5$; i.e., only when both $2j+1$ and $2k+1$ are multiples of 5. Therefore, from (5.4.11), we find that

$$\left[q^{5n+2}\right]\left\{\frac{E_2^2}{E_1^2E_5}\ \varphi(q^2)\varphi(q^{10})\right\} \equiv 0 \pmod 5. \tag{5.4.12}$$

Finally, by (1.2.2) and (2.4.2), we have

$$\begin{aligned} q^3\frac{E_2^2}{E_1^2E_5}\ \psi(q^4)\psi(q^{20}) &= q^3\frac{E_2^2E_8^2}{E_1^2E_4E_5}\ \psi(q^{20}) \\ &\equiv \frac{\psi(q^{20})}{E_5^2}\ E_1^3\cdot\frac{E_2^2E_8^2}{E_4}\ (\text{mod } 5), \end{aligned}$$

which can be rewritten, with the help of (2.2.10) and (5.2.3), as

$$\begin{aligned} &q^3\frac{E_2^2}{E_1^2E_5}\ \psi(q^4)\psi(q^{20}) \\ &\equiv \frac{\psi(q^{20})}{E_5^2}\ \sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty}(-1)^j(2j+1)(3k+1)q^{j(j+1)/2+2k(3k+2)+3}\ (\text{mod } 5). \end{aligned} \tag{5.4.13}$$

We observe that $j(j+1)/2+2k(3k+1)+3 \equiv 2\ (\text{mod } 5)$ only when $j \equiv 2\ (\text{mod } 5)$ and $k \equiv 3\ (\text{mod } 5)$; i.e., only when both $2j+1$ and $3k+1$ are multiples of 5. Therefore, from (5.4.13), we arrive at

$$\left[q^{5n+2}\right]\left\{q^3\frac{E_2^2}{E_1^2E_5}\ \psi(q^4)\psi(q^{20})\right\} \equiv 0\ (\text{mod } 5). \tag{5.4.14}$$

With the aid of (5.4.10), (5.4.12) and (5.4.14), we conclude from (5.4.9) that

$$a_{1,10}(10n+5) \equiv 0\ (\text{mod } 5).$$

Thus, we complete the proof of (1.8.7).

Proof of (1.8.8). We can recast (5.4.2) as

$$4\sum_{n=0}^{\infty}a_{3,10}(n)q^n = \frac{f^2(q^3,q^7)f^2(-q,-q^9)E_{10}^5}{f^2(-q^3,-q^7)f^2(-q,-q^9)E_{20}^4} - 2A(q^2),$$

Proceeding exactly in the same way as in the proof of (1.8.7), we find that

$$2\sum_{n=0}^{\infty}a_{3,10}(2n+1)q^n = \frac{E_1^2}{E_5} + 6q\frac{E_2E_{10}^3}{E_1E_5^2} - \frac{E_2^2}{E_1^2E_5}\left(\varphi(q^2)\varphi(q^{10}) - 4q^3\psi(q^4)\psi(q^{20})\right).$$

We notice that the right side of the above is almost the same as that of (5.4.9) except a negative sign. Hence, (1.8.8) can be deduced as in the previous case.