Chapter 1

Introduction

In this thesis, we find several new exact generating functions and congruences for various partition functions by using dissections of q-products, Ramanujan's theta function identities and some identities for the Rogers-Ramanujan continued fraction. Several of these functions are related to Ramanujan/Watson mock theta functions. The thesis comprises of five chapters including this introductory chapter. In this chapter, we present some background material and a brief outline of the work in the subsequent chapters of the thesis.

1.1 The partition function and Ramanujan's partition congruences

A partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ of a nonnegative integer n is a finite sequence of non-increasing positive integers (called *parts*) $\pi_1, \pi_2, \ldots, \pi_k$ such that $\pi_1 + \pi_2 + \cdots + \pi_k = n$.

The partition function $p(n)$ is defined as the number of partitions of n. For example, $p(5)$=7, since there are seven partitions of 5, namely,

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1,1), \text{ and } (1,1,1,1,1).$$

By convention, $p(0) = 1$. The generating function for $p(n)$, due to Euler, is given by

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}, \tag{1.1.1}$$

where, for any complex number a and q, with $|q| < 1$, we define

$$(a;q)_0 := 1,$$
$$(a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k), \quad n \geq 1$$

and

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n.$$

In the sequel, for any positive integer j, we use

$$E_j := (q^j;q^j)_\infty$$

and

$$(a_1, a_2, \ldots, a_k;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty \cdots (a_k;q)_\infty.$$

Ramanujan [61], found nice congruence properties for $p(n)$ modulo 5, 7 and 11, namely, for any nonnegative integer n,

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.1.2}$$
$$p(7n+5) \equiv 0 \pmod 7 \tag{1.1.3}$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

He also found the exact generating functions of $p(5n+4)$ and $p(7n+5)$ as given below:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\,\frac{E_5^5}{E_1^6} \tag{1.1.4}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\,\frac{E_7^3}{E_1^4} + 49q\frac{E_7^7}{E_1^8}, \tag{1.1.5}$$

which immediately imply (1.1.2) and (1.1.3), respectively. It can also be shown from the above generating functions that

$$p(25n+24) \equiv 0 \pmod{25} \tag{1.1.6}$$

and

$$p(49n+47) \equiv 0 \pmod{49},$$

which were known to Ramanujan [62, p. 139 and p. 144] (See also [21]).

In 1939, Zuckerman [73] found the generating functions of $p(25n+24)$, $p(49n+47)$ and $p(13n+6)$ analogous to (1.1.4) and (1.1.5). In particular, he showed that

$$\begin{aligned}\sum_{n=0}^{\infty} p(25n+24)q^n =& 63 \times 5^2 \frac{E_5^6}{E_1^7} + 52 \times 5^5 q \frac{E_5^{12}}{E_1^{13}} + 63 \times 5^7 q^2 \frac{E_5^{18}}{E_1^{19}} + 6 \times 5^{10} q^3 \frac{E_5^{24}}{E_1^{25}} \\ &+ 5^{12} q^4 \frac{E_5^{30}}{E_1^{31}},\end{aligned} \tag{1.1.7}$$

which readily shows (1.1.6).

In [61], Ramanujan also offered a more general conjecture for congruences of $p(n)$ modulo arbitrary powers of 5, 7 and 11. In particular, if $\alpha \geq 1$ and if δ_α is the reciprocal modulo 5^α of 24, then

$$p(5^\alpha n + \delta_\alpha) \equiv 0 \pmod{5^\alpha}.$$

In his unpublished manuscript [62, pp. 240 – 241] (See also [21]), Ramanujan gave a proof of the above. Hirschhorn and Hunt [47] gave an elementary proof of the above by finding the generating function of $p(5^\alpha n + \delta_\alpha)$.

Since our proofs of the results mainly rely on various properties of Ramanujan's theta functions, some identities for the Rogers-Ramanujan continued fraction and dissections of certain q-products, we now introduce these topics.

1.2 Ramanujan's theta functions

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1.$$

We have the following two useful cases:

$$\varphi(-q) := f(-q,-q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} = (q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{E_1^2}{E_2} \tag{1.2.1}$$

and

$$\psi(q) := f(q,q^3) = \sum_{j=0}^{\infty} q^{j(j+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{E_2^2}{E_1}, \tag{1.2.2}$$

where the product representations arise from Jacobi's famous triple product identity [19, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.2.3}$$

We refer to Berndt's book [19] for various properties satisfied by $f(a,b)$.

1.3 The Rogers-Ramanujan continued fraction

For $|q| < 1$, the famous Rogers-Ramanujan continued fraction $\mathcal{R}(q)$ is defined by

$$\mathcal{R}(q) := \frac{q^{1/5}}{1} \underset{+}{} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \cdots.$$

In [63], Rogers proved that this continued fraction has the q-product representation

$$\mathcal{R}(q) = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

We refer to Andrews and Berndt's book [4] for many diversified results on $\mathcal{R}(q)$.

1.4 t-dissection

If $P(q)$ denotes a power series in q, then a t-dissection of $P(q)$ is given by

$$[P(q)]_{t-\text{dissection}} = \sum_{k=0}^{t-1} q^k P_k(q^t),$$

where P_k's are power series in q^t. For example, the 5-dissections of E_1, $1/E_1$ and $\varphi(-q)$ are given by

$$E_1 = E_{25}\left(R(q^5) - q - \frac{q^2}{R(q^5)}\right), \tag{1.4.1}$$

$$\frac{1}{E_1} = \frac{E_{25}^5}{E_5^6}\left(R(q^5)^4 + qR(q^5)^3 + 2q^2R(q^5)^2 + 3q^3R(q^5) + 5q^4 - \frac{3q^5}{R(q^5)} + \frac{2q^6}{R(q^5)^2} - \frac{q^7}{R(q^5)^3} + \frac{q^8}{R(q^5)^4}\right) \tag{1.4.2}$$

and

$$\varphi(-q) = \frac{E_1^2}{E_2} = \frac{E_{25}^2}{E_{50}} - 2q(q^{15}, q^{35}, q^{50}; q^{50})_\infty + 2q^4(q^5, q^{45}, q^{50}; q^{50})_\infty, \tag{1.4.3}$$

where $R(q) = q^{1/5}/\mathcal{R}(q)$. For a proof of the above, we refer to Berndt's books [19, p. 40] and [20, p. 165].

In the remaining part of this chapter, we give a brief outline of the work done in this thesis.

1.5 Partitions into distinct parts

Let $Q(n)$ denote the number of partitions of n into distinct parts. For example, $Q(5) = 3$ since there are three partitions of 5 into distinct parts, namely, $(5), (4,1)$ and $(3,2)$. One of Euler's famous results on partitions is that the number of partitions of n into distinct parts is equinumerous to the number of partitions of n into odd parts. Note that there are also three partitions of 5 into odd parts, namely, $(5), (3,1,1)$ and $(1,1,1,1,1)$. The generating function for $Q(n)$ is given by

$$\sum_{n=0}^{\infty} Q(n)q^n = (-q;q)_\infty.$$

Equivalently, by Euler's result,

$$\sum_{n=0}^{\infty} Q(n)q^n = \frac{1}{(q;q^2)_\infty}. \tag{1.5.1}$$

In [64], Rödseth found the following infinite family of congruences modulo powers of 5 for $Q(n)$:

If $\gamma_j = \dfrac{25^{[(j+1)/2]} - 1}{24}$, then for any nonnegative integer n,

$$Q\left(5^{2j+1}n + \gamma_{2j+1}\right) \equiv 0 \pmod{5^j}. \tag{1.5.2}$$

By using the theory of modular forms and Hecke operators, Lovejoy [55, 56] found some more infinite families of congruences modulo powers of 5 for $Q(n)$. In particular, we recall from [56, Theorem 4] that, for $r = 1, 3, 4$ and for all nonnegative integers n,

$$Q\left(5^{2j+1}n + \gamma_{2j} + r5^{2j}\right) \equiv 0 \pmod{5^j}. \tag{1.5.3}$$

This is (1.5.2) when $r = 1$.

When $j = 1$ in (1.5.2), we have

$$Q\left(125n + 26\right) \equiv 0 \pmod{5}. \tag{1.5.4}$$

In the second chapter of our thesis, we find the exact generating functions of $Q(5n + 1)$, $Q(25n + 1)$ and $Q(125n + 26)$ that are analogous to (1.1.4) and (1.1.5). Our generating function representation for $Q(125n+26)$ immediately implies (1.5.4). In our proofs, we employ Ramanujan's simple theta function identities and some identities involving $R(q)$ and $R(q^2)$, where $R(q)$ is as defined in the previous section.

We also deduce the cases $j = 1$ and $j = 2$ of (1.5.2) and some other congruences.

1.6 Partition functions related to mock theta functions

In his last letter to Hardy [22, pp. 220–223] Ramanujan defines 17 functions and calls them as mock theta functions. The discovery of Ramanujan's lost notebook in 1976 by G.E. Andrews brought to light that Ramanujan recorded many more results on these functions. Since then, these functions have been studied quite intensively. For details, we refer to Andrews and Berndt's recent book [5] and the references therein.

Recently, partition-theoretic interpretations of mock theta functions have been the subject of prominent study. Garthwaite [38] showed the existence of infinitely many congruences for the third order mock theta function

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} a_\omega(n) q^n. \tag{1.6.1}$$

However, the first explicit congruences were given by Waldherr [66]:

$$a_\omega(40n+27) \equiv a_\omega(40n+35) \equiv 0 \pmod{5}. \tag{1.6.2}$$

Recently, Andrews, Dixit and Yee [8] introduced partition functions associated with $\omega(q)$ and $\nu(q)$, where the latter one is a third-order mock theta function,

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}.$$

It is worthwhile to note that $\omega(q)$ and $\nu(q)$ are related by [37, p. 62, Eq. (26.88)]:

$$\nu(-q) = q\omega(q^2) + \frac{E_4^3}{E_2^2}.$$

Let $p_\omega(n)$ denote the number of partitions of n in which each odd part is less than twice the smallest part and let $p_\nu(n)$ denote the number of partitions of n in which the parts are distinct and all odd parts are less than twice the smallest part. It was shown by Andrews, Dixit and Yee [8] that

$$\sum_{n=1}^{\infty} p_\omega(n)q^n = q\omega(q) \tag{1.6.3}$$

and

$$\sum_{n=1}^{\infty} p_\nu(n)q^n = \nu(-q).$$

By (1.6.1) and (1.6.3), it is clear that $p_\omega(n) = a_\omega(n-1)$, and hence, Waldherr's congruences (1.6.2) can be recast as

$$p_\omega(40n+28) \equiv p_\omega(40n+36) \equiv 0 \pmod{5}. \tag{1.6.4}$$

Andrews, Passary, Sellers and Yee [12] found an elementary proof of the above congruences. They also proved several congruences modulo 2 and infinite families of congruences modulo 4 and modulo 8 for $p_\omega(n)$ and $p_\nu(n)$. Motivated by the works in [8, 12], Wang [67] and Cui, Gu and Hao [34] found many new congruences satisfied by $p_\omega(n)$ and $p_\nu(n)$ modulo 11 and modulo powers of 2 and 3. In particular, Wang [67] derived the following exact generating functions:

$$\sum_{n=0}^{\infty} p_\nu(2n)q^n = \frac{E_2^3}{E_1^2}$$

and

$$\sum_{n=0}^{\infty} p_\omega(8n+4)q^n = 4\frac{E_2^{10}}{E_1^9}. \tag{1.6.5}$$

In fact, the first identity was proved earlier by Hirschhorn and Sellers [48, Eq. (9)] while proving some results for the so-called 1-shell totally symmetric plane partitions, first introduced by Blecher [23]. It is to be noted that

$$p_\nu(2n) = f(6n+1), \tag{1.6.6}$$

where $f(n)$ counts the number of 1-shell totally symmetric plane partitions of n. We refer to [23] and [48] for definition and further details. Xia [70] proved that

$$\sum_{n=0}^{\infty} f(30n+25)q^n = \sum_{n=0}^{\infty} p_\nu(10n+8)q^n = 5\frac{E_2^2E_5^2E_{10}}{E_1^4}, \tag{1.6.7}$$

which is the exact generating function of $p_\nu(10n+8)$, and from which the congruence (1.6.12) below, proved by Andrews, Dixit and Yee [8], follows immediately. Xia [70] also proved that

$$f(750n+625) \equiv 0 \pmod{25},$$

which is clearly equivalent to

$$p_\nu(250n+208) \equiv 0 \pmod{25}. \tag{1.6.8}$$

In the third chapter of our thesis, we find exact representations of the generating functions of $p_\nu(50n+8)$ and $p_\omega(40n+12)$, and deduce several congruences.

The smallest parts function $\mathrm{spt}(n)$, counting the total number of appearances of the smallest parts in all partitions of n, was introduced by Andrews [3], and the function has received great attention since its introduction. For example, see [6, 9, 25, 36, 39, 51, 52, 53].

Andrews, Dixit and Yee [8] studied the associated smallest parts functions $\mathrm{spt}_\omega(n)$ and $\mathrm{spt}_\nu(n)$, which count the number of smallest parts in the partitions enumerated by $p_\omega(n)$ and $p_\nu(n)$, respectively. Of course, $\mathrm{spt}_\nu(n) = p_\nu(n)$. They proved the congruences

$$\mathrm{spt}_\omega(5n+3) \equiv 0 \pmod 5, \tag{1.6.9}$$

$$\mathrm{spt}_\omega(10n+7) \equiv 0 \pmod 5, \tag{1.6.10}$$

$$\mathrm{spt}_\omega(10n+9) \equiv 0 \pmod 5 \tag{1.6.11}$$

and

$$\mathrm{spt}_\nu(10n+8) = p_\nu(10n+8) \equiv 0 \pmod 5, \tag{1.6.12}$$

where the first congruence was also established by Garvan and Jennings-Shaffer [41]. In fact, Garvan and Jennings-Shaffer [41] introduced a crank-type function that explains the congruence (1.6.9). The asymptotic behavior of that crank function was studied by Jang and Kim [50]. As mentioned earlier, (1.6.12) immediately follows from (1.6.7), a fact not possibly noticed by the authors of [8]. In [7], Andrews, Dixit, Schultz and Yee studied the overpartition analogue of $p_\omega(n)$, namely, $\overline{p}_\omega(n)$, which counts the number of overpartitions of n such that all odd parts are less than twice the smallest part and in which the smallest part is always overlined. They also studied $\overline{\mathrm{spt}}_\omega(n)$, the number of smallest parts in the overpartitions of n in which the smallest part is always overlined and all odd parts are less than twice the smallest part. They found several congruences modulo 2, 3, 4, 5 and 6 for $\overline{p}_\omega(n)$ and $\overline{\mathrm{spt}}_\omega(n)$. They [7, Problem 1] also raised the question of relating the generating function of $\overline{p}_\omega(n)$ to modular forms. Recently, the question was answered in affirmative by Bringmann, Jennings-Shaffer and Mahlburg [24].

Recently, Wang [67] and Cui, Gu and Hao [34] also found many new congruences satisfied by $\overline{p}_\omega(n)$, $\mathrm{spt}_\omega(n)$ and $\overline{\mathrm{spt}}_\omega(n)$ modulo powers of 2 and 3. In particular, Wang [67] derived the following exact generating functions:

$$\sum_{n=0}^{\infty} \mathrm{spt}_\omega(2n+1)q^n = \frac{E_2^8}{E_1^5} \tag{1.6.13}$$

and

$$\sum_{n=0}^{\infty} \overline{\text{spt}}_{\omega}(2n+1)q^n = \frac{E_2^9}{E_1^6}. \tag{1.6.14}$$

We note that congruences (1.6.10) and (1.6.11) and the congruence

$$\text{spt}_{\omega}(10n+3) \equiv 0 \pmod 5, \tag{1.6.15}$$

can be easily deduced from (1.6.13).

Wang [67] offered the following interesting conjecture.

Conjecture 1.6.1. *For any integers $k \geq 1$ and $n \geq 0$,*

$$\text{spt}_{\omega}\left(2 \cdot 5^{2k-1}n + \frac{7 \cdot 5^{2k-1}+1}{12}\right) \equiv 0 \pmod{5^{2k-1}}$$

and

$$\text{spt}_{\omega}\left(2 \cdot 5^{2k}n + \frac{11 \cdot 5^{2k}+1}{12}\right) \equiv 0 \pmod{5^{2k}}.$$

The cases $k = 1$ and 2 of the above congruences are (1.6.15),

$$\text{spt}_{\omega}(50n+23) \equiv 0 \pmod{5^2}, \tag{1.6.16}$$

$$\text{spt}_{\omega}(250n+73) \equiv 0 \pmod{5^3} \tag{1.6.17}$$

and

$$\text{spt}_{\omega}(1250n+573) \equiv 0 \pmod{5^4}. \tag{1.6.18}$$

In the third chapter of our thesis, we find the exact generating functions of $\text{spt}_{\omega}(10n+3)$ and $\text{spt}_{\omega}(50n+23)$ and deduce some new congruences.

There are several congruences for $\overline{\text{spt}}_{\omega}(n)$ modulo 11 and powers of 2 and 3 (For example, see [7, 34, 67]). But for modulo 5, to our knowledge, the following congruence, found by Andrews, Dixit, Schultz and Yee [7], is the only available one:

$$\overline{\text{spt}}_{\omega}(10n+6) \equiv 0 \pmod 5.$$

In the third chapter, we present the exact generating function of $\overline{\mathrm{spt}}_\omega\,(10n+5)$ and easily deduce congruences (1.6.15), (1.6.16), and infinite families of congruences.

In the same chapter, we also find exact generating function and new congruences for the coefficients of the following second order mock theta function of Andrews [2]:

$$\eta(q) := \sum_{n\geq 0} \frac{q^{n(n-1}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}.$$

In this chapter also we employ identities involving $R(q)$ and $R(q^2)$.

1.7 Partition identities and congruences by using relations involving $R(q)$, $R(q^3)$ and $R(q^4)$

In Chapter 4, we use identities involving $R(q)$, $R(q^3)$ and $R(q^4)$ to find generating functions and congruences modulo 5 for some partition functions.

Let $p_3(n)$ denote the number of 2-color partitions of n where one of the colors appears only in parts that are multiples of 3. For example, $p_3(6) = 16$, where the relevant partitions are (6), $(6')$, $(5,1)$, $(4,2)$, $(4,1,1)$, $(3,3)$, $(3,3')$, $(3',3')$, $(3,2,1)$, $(3',2,1)$, $(3,1,1,1)$, $(3',1,1,1)$, $(2,2,2)$, $(2,2,1,1)$, $(2,1,1,1,1)$, and $(1,1,1,1,1,1)$. Clearly, the generating function for $p_3(n)$ is given by

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{1}{E_1E_3}.$$

In [1], Ahmed, Baruah and Dastidar proved that

$$p_3(25n+21) \equiv 0 \pmod{5}. \tag{1.7.1}$$

In Chapter 4 of our thesis, we find the exact generating function for $p_3(5n+1)$ and deduce (1.7.1) as well as the congruence

$$\sum_{n=0}^{\infty} p_3(25n+21)q^n \equiv 10\left(\frac{E_{25}}{E_1^2E_3} + q^2\frac{E_{75}}{E_1E_3^2}\right) \pmod{25}.$$

In [72], Zhang and Shi studied the sixth order mock theta function $\beta(q)$, defined by

$$\beta(q) := \sum_{n=0}^{\infty} \frac{q^{3n^2+3n+1}}{(q;q^3)_{n+1}(q^2;q^3)_{n+1}}.$$

In particular, for $p_\beta(n)$, defined by

$$\beta(q) =: \sum_{n=0}^{\infty} p_\beta(n) q^n,$$

they proved that

$$\sum_{n=0}^{\infty} p_\beta(3n+1) q^n = \frac{E_3^3}{E_1^2}$$

and

$$\sum_{n=0}^{\infty} p_\beta(9n+5) q^n = 3\frac{E_3^6}{E_1^5}.$$

They also found the following congruences modulo 5:

$$p_\beta(15n+7) \equiv 0 \pmod{5},$$
$$p_\beta(45n+23) \equiv 0 \pmod{15}$$

and

$$p_\beta(45n+41) \equiv 0 \pmod{15}.$$

In our work, we also deduce the above congruences by finding the generating function representations involving $R(q)$ and $R(q^3)$.

A partition of a positive integer n is said to be 4-regular if none of its parts is divisible by 4. For example, $(5,3,2,2,1)$ is a 4-regular partition of 13 as none of its parts is divisible by 4.

If $b_4(n)$ denotes the number of 4-regular partitions of n, then the generating function for $b_4(n)$ is given by

$$\sum_{n=0}^{\infty} b_4(n) q^n = \frac{E_4}{E_1},$$

where $b_4(0) = 1$.

Since

$$\frac{E_4}{E_1} = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty},$$

it is clear that

$$b_4(n) = ped(n),$$

where $ped(n)$ counts the number of partitions of n wherein the even parts are distinct (and the odd parts are unrestricted). Arithmetical properties and many interesting congruences modulo 2, 3, 4, 6, 8 and 12 for $ped(n)$ were found by Andrews, Hirschhorn and Sellers [11], Chen [28], Cui and Gu [33], Hirschhorn and Sellers [49], Xia [69] and Merca [57].

In Chapter 4, we prove the following new generating function representation for $b_4(5n+3)$, equivalently, for $ped(5n+3)$, by employing relations involving $R(q)$ and $R(q^4)$.

$$\sum_{n=0}^{\infty} b_4(5n+3)q^n = 3\frac{E_2^2E_{10}^6}{E_1^5E_4E_{20}^2} + q\frac{E_2^4E_5^5E_{20}^3}{E_1^6E_4^2E_{10}^4} + 4q^2\frac{E_2^3E_{10}E_{20}^3}{E_1^5E_4^2}.$$

Similar technique can be used to find generating function representations of some other partition functions. We present with an example involving $a_4(n)$, the number of 4-core partitions of n. A t-core partition and $a_t(n)$, the number of t-core partitions of n, are defined below.

The Ferrers-Young diagram of a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ of n is an array of left-aligned nodes with π_i nodes in the i^{th} row. Let π'_j denote the number of nodes in column j in the Ferrers-Young diagram of π. The hook number of the (i, j) node in the Ferrers-Young diagram of π is denoted by $H(i, j) := \pi_i + \pi'_j - i - j + 1$. A partition of n is called a t-core partition (or simply a t-core) if none of the hook numbers is a multiple of t. For example, the Ferrers-Young diagram of the partition $\pi = (5, 2, 1)$ is given by:

$$\begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & & \\ \bullet & & & & \end{array}$$

The nodes $(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2)$ and $(3, 1)$ have hook numbers $7, 5, 3, 2, 1, 3, 1$ and 1 respectively. Therefore π is a 4-core. Obviously, it is a t-core for $t \geq 8$.

If $a_t(n)$ denotes the number of partitions of n that are t-cores, then the generating function for $a_t(n)$ is given by [40, Eq.(2.1)]

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{E_t^t}{E_1}.$$

In particular, if $a_4(n)$ denotes the number 4-core partitions of n, then

$$\sum_{n=0}^{\infty} a_4(n)q^n = \frac{E_4^4}{E_1}.$$

In Chapter 4, we present a new generating function representation for $a_4(5n)$.

1.8 Conjectural congruences of S. H. Chan on Appell-Lerch sums

Let $x, z \in \mathbb{C}^*$ with neither z nor xz an integral power of q. Following the definition given by Hickerson and Mortenson in [44, Definition 1.1], an Appell-Lerch sum $m(x, q, z)$ is a series of the form

$$m(x, q, z) := \frac{1}{(q, q/z, q; q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2} z^{n+1}}{1 - xzq^n}.$$

These sums were first studied in the nineteenth century by Appell [13, 14, 15] and then by Lerch [54]. But, in recent years, there has been considerable work on these sums and their connections to mock theta functions. We refer to [10, 26, 44, 45, 58, 59, 66, 74] for the details.

In his lost notebook [62, pp. 2, 4, 13, 17], Ramanujan recorded seven mock theta functions and eleven identities involving them. Andrews and Hickerson [10] proved these eleven identities and called the seven functions sixth order mock theta functions. Three of the sixth order mock theta functions are

$$\rho(q) := \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n+1)/2}}{(q; q^2)_{n+1}},$$

$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{(n+1)^2}}{(-q; q)_{2n+1}}$$

and

$$\lambda(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^n}{(-q;q)_n}.$$

On page 3 of his lost notebook [62], Ramanujan defines the function

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(q;q^2)_{n+1}^2},$$

and then states that

$$\rho(q) = 2q^{-1}\phi(q^3) + \frac{(q^2;q^2)_\infty^2 (-q^3;q^3)_\infty}{(q;q^2)_\infty^2 (q^3;q^3)_\infty}.$$

Choi [32] proved two analogous identities involving ϕ and the two functions μ and λ. The function $\phi(q)$ was also studied by Hikami [46].

Now, let $\sum_{n=1}^{\infty} a(n)q^n := \phi(q)$. S. H. Chan [26] proved several congruences for the coefficients $a(n)$ of the function ϕ modulo 2, 3, 4, 5, 7, and 27. In particular, S. H. Chan [26] proved the congruence

$$a(10n+9) \equiv 0 \pmod{5} \tag{1.8.1}$$

and conjectured ([26, Conjecture 7.1]) that, for any nonnegative integer n,

$$a(50n+19) \equiv a(50n+39) \equiv a(50n+49) \equiv 0 \pmod{25}. \tag{1.8.2}$$

In the fifth and final chapter of our thesis, we find the exact generating function of $a(10n+9)$ analogous to (1.1.4) and deduce the above congruences. Furthermore, we find the following new congruences:
For any nonnegative integer n, we have

$$a(1250n + 250r + 219) \equiv 0 \pmod{125}, \quad \text{for } r = 1, 3, 4.$$

In [26], S. H. Chan studied some other functions similar to ϕ and found congruences for them. In particular, he considered, for any integer $p \geq 2$ and $1 \leq j \leq p-1$ with p and j coprime, the Appell-Lerch sum

$$\sum_{n=0}^{\infty} a_{j,p}(n) q^n = \frac{1}{(q^j, q^{p-j}, q^p; q^p)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{pn(n+1)/2+jn+j}}{1-q^{pn+j}},$$

and proved that

$$\sum_{n=0}^{\infty} a_{j,p}(pn+(p-j)j)q^n = p\,\frac{E_p^4}{(q^j,q^{p-j};q^p)_\infty^2 E_1^3},$$

which readily implies the congruence

$$a_{j,p}(pn+(p-j)j) \equiv 0 \pmod{p}.$$

It is to be noted that $2a(n) = a_{1,2}(n)$.

In [27], S. H. Chan and Mao gave a generalization of $a_{j,p}$.

S. H. Chan [26, Conjecture 7.1] also presented the following conjectural congruences:

$$a_{1,6}(2n) \equiv 0 \pmod{2}, \tag{1.8.3}$$

$$a_{1,10}(2n) \equiv a_{3,10}(2n) \equiv 0 \pmod{2}, \tag{1.8.4}$$

$$a_{1,6}(6n+3) \equiv 0 \pmod{3}, \tag{1.8.5}$$

$$a_{1,3}(5n+3) \equiv a_{1,3}(5n+4) \equiv 0 \pmod{5}, \tag{1.8.6}$$

$$a_{1,10}(10n+5) \equiv 0 \pmod{5} \tag{1.8.7}$$

and

$$a_{3,10}(10n+5) \equiv 0 \pmod{5}. \tag{1.8.8}$$

Recently, Qu, Wang, and Yao [60] proved (1.8.3) and (1.8.4) by finding the following general congruence:

If j and k are positive integers with $1 \le j \le k-1$ and j odd, then for any nonnegative integer n,

$$a_{j,2k}(2n) \equiv 0 \pmod{2}.$$

They also proved (1.8.5) by finding the following identity analogous to (1.1.4):

$$\sum_{n=0}^{\infty} a_{1,6}(6n+3)q^n = 3\,\frac{E_2^3E_3^5}{E_1^6E_6}.$$

Congruences in (1.8.6) were proved by Ding and Xia [35].

In Chapter 5, we prove the remaining conjectural congruences (1.8.7) and (1.8.8) of S. H. Chan [26].