Chapter 2

Generating functions and congruences for partitions into distinct parts

2.1 Introduction

Recall from Section 1.5 that $Q(n)$ denotes the number of partitions of n into distinct parts. In this chapter, we find the exact generating functions for $Q(5n+1)$, $Q(25n+1)$ and $Q(125n+26)$ given by Theorem 2.1.1. We also deduce some congruences modulo powers of 5.

Theorem 2.1.1. *For any nonnegative integer n, we have*

$$\sum_{n=0}^{\infty} Q(5n+1)q^n = \frac{E_2^2E_5^3}{E_1^4E_{10}}, \tag{2.1.1}$$

$$\sum_{n=0}^{\infty} Q(25n+1)q^n = \frac{E_2^3E_5^4}{E_1^5E_{10}^2} + 160q\ \frac{E_2^4E_{10}E_5^3}{E_1^8} + 2800q^2\frac{E_2^5E_5^2E_{10}^4}{E_1^{11}} + 16000q^3\frac{E_2^6E_5E_{10}^7}{E_1^{14}} + 32000q^4\frac{E_2^7E_{10}^{10}}{E_1^{17}} \tag{2.1.2}$$

and

$$\sum_{n=0}^{\infty} Q(125n+26)q^n$$

$$= 33\times 5\frac{E_2^2E_5^3}{E_1^4E_{10}} + 1039573\times 2^2\times 5\ q\ \frac{E_2^3E_5^2E_{10}^2}{E_1^7} + 84358511\times 2^4\times 5^2q^2\frac{E_2^4E_5E_{10}^5}{E_1^{10}}$$

$$
\begin{aligned}
&+1519417629\times 2^{6}\times 5^{3}q^{3}\frac{E_2^{5}E_{10}^{8}}{E_1^{13}}+57468885219\times 2^{8}\times 5^{3}q^{4}\frac{E_2^{6}E_{10}^{11}}{E_1^{16}E_5}\\
&+239126250621\times 2^{10}\times 5^{4}q^{5}\frac{E_2^{7}E_{10}^{14}}{E_1^{19}E_5^{2}}+493702983\times 2^{20}\times 5^{6}q^{6}\frac{E_2^{8}E_{10}^{17}}{E_1^{22}E_5^{3}}\\
&+57851635449\times 2^{16}\times 5^{7}q^{7}\frac{E_2^{9}E_{10}^{20}}{E_1^{25}E_5^{4}}+155363323153\times 2^{17}\times 5^{8}q^{8}\frac{E_2^{10}E_{10}^{23}}{E_1^{28}E_5^{5}}\\
&+99443868167\times 2^{22}\times 5^{8}q^{9}\frac{E_2^{11}E_{10}^{26}}{E_1^{31}E_5^{6}}+1277863945093\times 2^{20}\times 5^{9}q^{10}\frac{E_2^{12}E_{10}^{29}}{E_1^{34}E_5^{7}}\\
&+82117001559\times 2^{23}\times 5^{11}q^{11}\frac{E_2^{13}E_{10}^{32}}{E_1^{37}E_5^{8}}+85675198911\times 2^{24}\times 5^{12}q^{12}\frac{E_2^{14}E_{10}^{35}}{E_1^{40}E_5^{9}}\\
&+916288433\times 2^{29}\times 5^{14}q^{13}\frac{E_2^{15}E_{10}^{38}}{E_1^{43}E_5^{10}}+32357578059\times 2^{29}\times 5^{13}q^{14}\frac{E_2^{16}E_{10}^{41}}{E_1^{46}E_5^{11}}\\
&+2366343709\times 2^{33}\times 5^{14}q^{15}\frac{E_2^{17}E_{10}^{44}}{E_1^{49}E_5^{12}}+57370733\times 2^{36}\times 5^{16}q^{16}\frac{E_2^{18}E_{10}^{47}}{E_1^{52}E_5^{13}}\\
&+22998577\times 2^{37}\times 5^{17}q^{17}\frac{E_2^{19}E_{10}^{50}}{E_1^{55}E_5^{14}}+30309607\times 2^{36}\times 5^{18}q^{18}\frac{E_2^{20}E_{10}^{53}}{E_1^{58}E_5^{15}}\\
&+20313321\times 2^{38}\times 5^{18}q^{19}\frac{E_2^{21}E_{10}^{56}}{E_1^{61}E_5^{16}}+2181069\times 2^{40}\times 5^{19}q^{20}\frac{E_2^{22}E_{10}^{59}}{E_1^{64}E_5^{17}}\\
&+18319\times 2^{43}\times 5^{21}q^{21}\frac{E_2^{23}E_{10}^{62}}{E_1^{67}E_5^{18}}+29\times 2^{48}\times 5^{23}q^{22}\frac{E_2^{24}E_{10}^{65}}{E_1^{70}E_5^{19}}\\
&+521\times 2^{46}\times 5^{22}q^{23}\frac{E_2^{25}E_{10}^{68}}{E_1^{73}E_5^{20}}+37\times 2^{49}\times 5^{22}q^{24}\frac{E_2^{26}E_{10}^{71}}{E_1^{76}E_5^{21}}+2^{50}\times 5^{23}q^{25}\frac{E_2^{27}E_{10}^{74}}{E_1^{79}E_5^{22}}.
\end{aligned}
\tag{2.1.3}
$$

When $j=1$ and $r=3$ and 4 in (1.5.3), then

$$Q\left(125n+76\right)\equiv 0 \pmod 5 \tag{2.1.4}$$

and

$$Q\left(125n+101\right)\equiv 0 \pmod 5. \tag{2.1.5}$$

With the aid of Theorem 2.1.1, we deduce the above two congruences as well as the following two congruences.

For any nonnegative integer n, we have

$$Q(625n+276)\equiv 0 \pmod{25} \tag{2.1.6}$$

and

$$Q(625n + 401) \equiv 0 \pmod{25}. \tag{2.1.7}$$

Note that congruences (1.5.4), (2.1.4)—(2.1.7) do not hold for higher powers of the modulus 5 as $Q(26) = 165$, $Q(201) = 517361670$, $Q(101) = 483330$, $Q(276) = 33888946600$ and $Q(625 + 401) = 17771036379080545908200$.

We organize the chapter in the following way. Section 2.2 includes a few useful lemmas. In Section 2.3, we find the exact generating functions for $Q(5n + 1)$, $Q(25n + 1)$ and $Q(125n + 26)$ and in the remaining two sections, we deduce the cases $j = 1$ and 2 of (1.5.2) and (2.1.4)–(2.1.7).

The contents of this chapter appeared in *International Journal of Number Theory* [16].

It is worthwhile to mention that motivated by our work [16], Chern and Hirschhorn [29] found an elementary proof of (1.5.2) whereas Chern and Tang [30] found elementary proofs of some congruences modulo 25 for the so-called broken k-diamond partitions earlier proved by Tang [65] by using the theory of modular forms.

2.2 Preliminary lemmas

Recall that $R(q) = q^{1/5}\mathcal{R}(q)$, where $\mathcal{R}(q)$ is the Rogers-Ramanujan continued fraction defined in Section 1.3. Some useful identities involving $R(q)$ and $R(q^2)$ are stated in the following lemma.

Lemma 2.2.1. *We have*

$$R(q)R(q^2)^2 - \frac{q^2}{R(q)R(q^2)^2} = \frac{E_2E_5^5}{E_1E_{10}^5}, \tag{2.2.1}$$

$$\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2} = 4q\frac{E_1E_{10}^5}{E_2E_5^5}, \tag{2.2.2}$$

$$\frac{R(q^2)^3}{R(q)} + q^2\frac{R(q)}{R(q^2)^3} = \frac{E_2E_5^5}{E_1E_{10}^5} + 4q^2\frac{E_1E_{10}^5}{E_2E_5^5} - 2q \tag{2.2.3}$$

and

$$R(q)^3R(q^2)+\frac{q^2}{R(q)^3R(q^2)}=\frac{E_2E_5^5}{E_1E_{10}^5}+4q^2\frac{E_1E_{10}^5}{E_2E_5^5}+2q. \tag{2.2.4}$$

Proof. From [62, p. 56] ([4, p. 35, Entry 1.8.2]), we note that if $k=\dfrac{q}{R(q)R(q^2)^2}$ and $k\leq\sqrt{5}-2$, then

$$\frac{\psi^2(q)}{q\psi^2(q^5)}=\frac{1+k-k^2}{k}.$$

Identity (2.2.1) now follows immediately. Identity (2.2.2) follows from the well-known identity

$$\frac{q}{R(q)R(q^2)^2}=\frac{R(q)^2-R(q^2)}{R(q)^2+R(q^2)}$$

of Ramanujan [20, p. 166] and (2.2.1).

Next, it is easy to see from the above identity that

$$\frac{R(q^2)^3}{R(q)}=R(q)R(q^2)^2-q\frac{R(q^2)}{R(q)^2}-q$$

and

$$q\frac{R(q)}{R(q^2)^3}=\frac{R(q)^2}{R(q^2)}-\frac{q}{R(q)R(q^2)^2}-1.$$

Therefore,

$$\frac{R(q^2)^3}{R(q)}+q^2\frac{R(q)}{R(q^2)^3}=\left(R(q)R(q^2)^2-\frac{q^2}{R(q)R(q^2)^2}\right)+q\left(\frac{R(q)^2}{R(q^2)}-\frac{R(q^2)}{R(q)^2}\right)-2q,$$

which, by (2.2.1) and (2.2.2), implies (2.2.3).

Finally, multiplying (2.2.1) and (2.2.2), and then using (2.2.3), we arrive at (2.2.4). □

Using various theta function identities, we can find several relations between E_1, E_2, E_5 and E_{10}. A few are listed in the following lemma.

Lemma 2.2.2. *We have*

$$\frac{E_5^5}{E_1^4E_{10}^3}=\frac{E_5}{E_2^2E_{10}}+4q\frac{E_{10}^2}{E_1^3E_2}, \tag{2.2.5}$$

$$\frac{E_2^3E_5^2}{E_1^2E_{10}^2}=\frac{E_5^5}{E_1E_{10}^3}+q\frac{E_{10}^2}{E_2} \tag{2.2.6}$$

and

$$\frac{E_2^3E_5^2}{E_1^5E_{10}^2}=\frac{E_5}{E_2^2E_{10}}+5q\frac{E_{10}^2}{E_1^3E_2}. \tag{2.2.7}$$

Proof. Recall from [19, Entry 10, p. 262] the following identities of Ramanujan:

$$\varphi^2(q)-\varphi^2(q^5)=4qf(q,q^9)f(q^3,q^7) \tag{2.2.8}$$

and

$$\psi^2(q)-\psi^2(q^5)=f(q,q^4)f(q^2,q^3). \tag{2.2.9}$$

Replacing q by $-q$ in (2.2.8), and then using (1.2.1) and (1.2.3), we arrive at (2.2.5).

On the other hand, using (1.2.2) and (1.2.3) in (2.2.9), we have (2.2.6).

Now, multiplying (2.2.6) by $1/E_1^3$ and then subtracting (2.2.5) from the resulting identity, we get (2.2.7). □

In the next lemma, we state Jacobi's identity.

Lemma 2.2.3. (Berndt [20, Theorem 1.3.9 and Corollary 1.3.22]) *We have*

$$E_1^3=\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}. \tag{2.2.10}$$

2.3 Proof of Theorem 2.1.1

Proof of (2.1.1). We can write (1.5.1) as

$$\sum_{n=0}^{\infty}Q(n)q^n=\frac{E_2}{E_1}. \tag{2.3.1}$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+1}, dividing both sides of the resulting identity by q, and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} Q(5n+1)q^n = \frac{E_5^5 E_{10}}{E_1^6}\left(\left(R(q)^3 R(q^2) + \frac{q^2}{R(q)^3 R(q^2)}\right) + q\left(-5 - 2\left(\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2}\right)\right)\right). \tag{2.3.2}$$

Using (2.2.2) and (2.2.4) in the above identity, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} Q(5n+1)q^n &= \frac{E_2 E_5^{10}}{E_1^7 E_{10}^4} - 3q\frac{E_5^5 E_{10}}{E_1^6} - 4q^2\frac{E_{10}^6}{E_1^5 E_2}\\ &= \frac{E_2 E_{10}^2}{E_1^2}\left(\frac{E_5^5}{E_1^4 E_{10}^3} - 4q\frac{E_{10}^2}{E_1^3 E_2}\right)\left(\frac{E_5^5}{E_1 E_{10}^3} + q\frac{E_{10}^2}{E_2}\right).\end{aligned}$$

We use (2.2.5) and (2.2.6) in the above to arrive at (2.1.1).

Proof of (2.1.2). With the aid of (2.2.5), (2.1.1) can be rewritten as

$$\sum_{n=0}^{\infty} Q(5n+1)q^n = \frac{E_{10}}{E_5} + 4q\frac{E_2 E_{10}^4}{E_1^3 E_5^2}.$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n} and then replacing q^5 by q, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} Q(25n+1)q^n\\ &= \frac{E_2}{E_1} + 4\frac{E_2^4 E_5^{15} E_{10}}{E_1^{20}}\Bigg(q\Bigg(51\left(R(q)^8 R(q^2) - \frac{q^4}{R(q)^8 R(q^2)}\right) - 9\left(R(q)^{10} + \frac{q^4}{R(q)^{10}}\right)\\ &\quad - \left(\frac{R(q)^{12}}{R(q^2)} - \frac{q^4 R(q^2)}{R(q)^{12}}\right)\Bigg) + q^2\Bigg(153\left(R(q)^3 R(q^2) + \frac{q^2}{R(q)^3 R(q^2)}\right)\\ &\quad - 177\left(R(q)^5 - \frac{q^2}{R(q)^5}\right) - 78\left(\frac{R(q)^7}{R(q^2)} + \frac{q^2 R(q^2)}{R(q)^7}\right)\Bigg)\\ &\quad - q^3\left(71 + 219\left(\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2}\right)\right)\Bigg).\end{aligned} \tag{2.3.3}$$

Now, in Lemma 2.2.1, if we set

$$a_1 := R(q)R(q^2)^2 - \frac{q^2}{R(q)R(q^2)^2} = \frac{E_2 E_5^5}{E_1 E_{10}^5}, \tag{2.3.4}$$

$$a_2 := \frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2} = 4q\frac{E_1E_{10}^5}{E_2E_5^5}, \tag{2.3.5}$$

$$a_3 := \frac{R(q^2)^3}{R(q)} + q^2\frac{R(q)}{R(q^2)^3} = \frac{E_2E_5^5}{E_1E_{10}^5} + 4q^2\frac{E_1E_{10}^5}{E_2E_5^5} - 2q \tag{2.3.6}$$

and

$$a_4 := R(q)^3R(q^2) + \frac{q^2}{R(q)^3R(q^2)} = \frac{E_2E_5^5}{E_1E_{10}^5} + 4q^2\frac{E_1E_{10}^5}{E_2E_5^5} + 2q, \tag{2.3.7}$$

then

$$a_5 := R(q)^5 - \frac{q^2}{R(q)^5} = a_1 + a_2 \cdot a_4, \tag{2.3.8}$$

$$a_6 := R(q)^{10} + \frac{q^4}{R(q)^{10}} = a_5^2 + 2q^2, \tag{2.3.9}$$

$$a_7 := R(q)^8R(q^2) - \frac{q^4}{R(q)^8R(q^2)} = a_4 \cdot a_5 - q^2a_2, \tag{2.3.10}$$

$$a_8 := \frac{R(q)^7}{R(q^2)} + q^2\frac{R(q^2)}{R(q)^7} = a_2 \cdot a_5 + a_4 \tag{2.3.11}$$

and

$$a_9 := \frac{R(q)^{12}}{R(q^2)} - q^4\frac{R(q^2)}{R(q)^{12}} = a_2 \cdot a_6 + a_7. \tag{2.3.12}$$

Employing (2.3.5), (2.3.7), (2.3.8)–(2.3.12) in (2.3.3), we arrive at

$$\sum_{n=0}^{\infty} Q(25n+1)q^n = \frac{E_2}{E_1} + 164q\frac{E_2^{10}E_5^{15}}{E_1^{20}E_{10}^5} + 816q^2\frac{E_2^9E_5^{10}}{E_1^{19}} + 1664q^3\frac{E_2^8E_5^5E_{10}^5}{E_1^{18}} + 1536q^4\frac{E_2^7E_{10}^{10}}{E_1^{17}} + 1024q^5\frac{E_2^6E_{10}^{15}}{E_1^{16}E_5^5}.$$

Simplifying the above with the aid of (2.2.5), we find that

$$\sum_{n=0}^{\infty} Q(25n+1)q^n = \frac{E_2}{E_1} + 164q\frac{E_2^2E_{10}^3}{E_1^4E_5} + 3440q^2\frac{E_2^3E_{10}^6}{E_1^7E_5^2} + 27200q^3\frac{E_2^4E_{10}^9}{E_1^{10}E_5^3} + 96000q^4\frac{E_2^5E_{10}^{12}}{E_1^{13}E_5^4} + 128000q^5\frac{E_2^6E_{10}^{15}}{E_1^{16}E_5^5}. \tag{2.3.13}$$

Using (2.2.5) once again in the above, we deduce (2.1.2).

Proof of (2.1.3). Note that, with the aid of (2.3.1), we can rewrite (2.3.13) as

$$\sum_{n=0}^{\infty} Q(25n+1)q^n = \sum_{n=0}^{\infty} Q(n)q^n + 164q\frac{E_2^2E_{10}^3}{E_1^4E_5} + 3440q^2\frac{E_2^3E_{10}^6}{E_1^7E_5^2} + 27200q^3\frac{E_2^4E_{10}^9}{E_1^{10}E_5^3}$$

$$+ 96000q^4\frac{E_2^5E_{10}^{12}}{E_1^{13}E_5^4} + 128000q^5\frac{E_2^6E_{10}^{15}}{E_1^{16}E_5^5}.$$

Employing (1.4.1) and (1.4.2) in the above, using Wolfram's **Mathematica** [68], and then extracting the terms involving q^{5n+1}, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} Q(125n+26)q^n \\
&= \sum_{n=0}^{\infty} Q(5n+1)q^n + 164\frac{E_2^3E_5^{20}E_{10}^2}{E_1^{25}}\bigg(R(q)^{16}R(q^2)^2 + \frac{q^8}{R(q)^{16}R(q^2)^2}\bigg) \\
&\quad + q\ C_1(R(q),R(q^2)) + q^2C_2(R(q),R(q^2)) + q^3C_3(R(q),R(q^2)) \\
&\quad + q^4C_4(R(q),R(q^2)) + q^5C_5(R(q),R(q^2)) + q^6C_6(R(q),R(q^2)) \\
&\quad + q^7C_7(R(q),R(q^2)) + q^8C_8(R(q),R(q^2)) + q^9C_9(R(q),R(q^2)) \\
&\quad + q^{10}C_{10}(R(q),R(q^2)) + q^{11}C_{11}(R(q),R(q^2)) + q^{12}C_{12}(R(q),R(q^2)) \\
&\quad + q^{13}C_{13}(R(q),R(q^2)) + q^{14}C_{14}(R(q),R(q^2)) + q^{15}C_{15}(R(q),R(q^2)) \\
&\quad + q^{16}C_{16}(R(q),R(q^2)),
\end{aligned} \tag{2.3.14}$$

where $C_j(R(q),R(q^2))$, $j=1,2,\ldots,16$, are functions involving certain expressions of $R(q)$, $R(q^2)$ and the terms

$$z_1 := \frac{E_2^3E_5^{20}E_{10}^2}{E_1^{25}},\ z_2 := \frac{E_2^6E_5^{35}E_{10}^3}{E_1^{44}},\ z_3 := \frac{E_2^9E_5^{50}E_{10}^4}{E_1^{63}},$$
$$z_4 := \frac{E_2^{12}E_5^{65}E_{10}^5}{E_1^{82}} \text{ and } z_5 := \frac{E_2^{15}E_5^{80}E_{10}^6}{E_1^{101}}.$$

For example,

$$\begin{aligned}
&C_1(R(q),R(q^2)) \\
&= 34768z_1\Big(R(q)^{11}R(q^2)^2 - \frac{q^6}{R(q)^{11}R(q^2)^2}\Big) - 13120z_1\Big(R(q)^{13}R(q^2) + \frac{q^6}{R(q)^{13}R(q^2)}\Big) \\
&\quad - 656z_1\Big(R(q)^{15} - \frac{q^6}{R(q)^{15}}\Big) + 1685600z_2\Big(R(q)^{24}R(q^2)^3 - \frac{q^{12}}{R(q)^{24}R(q^2)^3}\Big) \\
&\quad - 3612z_2\Big(R(q)^{26}R(q^2)^2 + \frac{q^{12}}{R(q)^{26}R(q^2)^2}\Big) + 8976000z_3\Big(R(q)^{37}R(q^2)^4 \\
&\quad - \frac{q^{18}}{R(q)^{37}R(q^2)^4}\Big) - 1088000z_3\Big(R(q)^{39}R(q^2)^3 + \frac{q^{18}}{R(q)^{39}R(q^2)^3}\Big) \\
&\quad + 9984000z_4\Big(R(q)^{50} + \frac{q^{20}}{R(q)^{50}}\Big)\Big(R(q^2)^5 - \frac{q^4}{R(q^2)^5}\Big)
\end{aligned}$$

$$- 480000z_4\Big(R(q)^{52}R(q^2)^4 + \frac{q^{24}}{R(q)^{52}R(q^2)^4}\Big)$$
$$+ 2048000z_5\Big(R(q)^{63}R(q^2)^6 - \frac{q^{30}}{R(q)^{63}R(q^2)^6}\Big),$$

$$C_2(R(q),R(q^2))$$
$$= 2408000z_2\Big(R(q)^{25} - \frac{q^{10}}{R(q)^{25}}\Big) - 68552z_1\Big(R(q)^{10} + \frac{q^4}{R(q)^{10}}\Big)$$
$$- 136000z_3\Big(R(q)^{40} + \frac{q^{16}}{R(q)^{40}}\Big) + 2296z_1\Big(\frac{R(q)^{14}}{R(q^2)^2} + q^4\frac{R(q^2)^2}{R(q)^{14}}\Big)$$
$$+ 34440z_1\Big(\frac{R(q)^{12}}{R(q^2)} - q^4\frac{R(q^2)}{R(q)^{12}}\Big) - 360800z_1\Big(R(q)^8R(q^2) - \frac{q^4}{R(q)^8R(q^2)}\Big)$$
$$+ 14144000z_3\Big(R(q)^{38}R(q^2) - \frac{q^{16}}{R(q)^{38}R(q^2)}\Big) + 305040z_1\Big(R(q)^6R(q^2)^2$$
$$+ \frac{q^4}{R(q)^6R(q^2)^2}\Big) - 103458000z_2\Big(R(q)^{21}R(q^2)^2 - \frac{q^{10}}{R(q)^{21}R(q^2)^2}\Big)$$
$$+ 77792000z_3\Big(R(q)^{36}R(q^2)^2 + \frac{q^{16}}{R(q)^{36}R(q^2)^2}\Big) + 12480000z_4\Big(R(q)^{51}R(q^2)^2$$
$$- \frac{q^{22}}{R(q)^{51}R(q^2)^2}\Big) + 165550000z_2\Big(R(q)^{19}R(q^2)^3 + \frac{q^{10}}{R(q)^{19}R(q^2)^3}\Big)$$
$$- 2004640000z_3\Big(R(q)^{34}R(q^2)^3 - \frac{q^{16}}{R(q)^{34}R(q^2)^3}\Big) + 305760000z_4\Big(R(q)^{49}R(q^2)^3$$
$$+ \frac{q^{22}}{R(q)^{49}R(q^2)^3}\Big) + 1280000z_5\Big(R(q)^{64}R(q^2)^3 - \frac{q^{28}}{R(q)^{64}R(q^2)^3}\Big)$$
$$+ 4458080000z_3\Big(R(q)^{32}R(q^2)^4 + \frac{q^{16}}{R(q)^{32}R(q^2)^4}\Big) - 7051200000z_4\Big(R(q)^{47}R(q^2)^4$$
$$- \frac{q^{22}}{R(q)^{47}R(q^2)^4}\Big) + 175104000z_5\Big(R(q)^{62}R(q^2)^4 + \frac{q^{28}}{R(q)^{62}R(q^2)^4}\Big)$$
$$- 4961280000z_5\Big(R(q)^{60} + \frac{q^{24}}{R(q)^{60}}\Big)\Big(R(q^2)^5 - \frac{q^4}{R(q^2)^5}\Big)$$
$$+ 19589120000z_5\Big(R(q)^{58}R(q^2)^6 + \frac{q^{28}}{R(q)^{58}R(q^2)^6}\Big),$$

and

$$C_{16}(R(q),R(q^2))$$
$$= 640000z_5\Bigg(2720684890067507 + 446739837578448\Big(\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2}\Big)$$
$$- 2974526668496760\Big(\frac{R(q)^4}{R(q^2)^2} + \frac{R(q^2)^2}{R(q)^4}\Big) - 1409182323856880\Big(\frac{R(q)^6}{R(q^2)^3}$$

$$-\frac{R(q^2)^3}{R(q)^6}\Big)+1792652008639068\Big(\frac{R(q)^8}{R(q^2)^4}+\frac{R(q^2)^4}{R(q)^8}\Big)$$
$$+351860419894488\Big(\frac{R(q)^{12}}{R(q^2)^6}+\frac{R(q^2)^6}{R(q)^{12}}\Big)\Bigg).$$

The expressions for the remaining C_j's are similar in nature. We observe that, as in (2.3.8)–(2.3.12), the expressions $\Big(R(q)^{13}R(q^2)+\frac{q^6}{R(q)^{13}R(q^2)}\Big)$, $\Big(R(q)^{24}R(q^2)^3-\frac{q^{12}}{R(q)^{24}R(q^2)^3}\Big)$, $\Big(\frac{R(q)^{12}}{R(q^2)}-q^4\frac{R(q^2)}{R(q)^{12}}\Big)$, etc., appearing in $C_j(R(q),R(q^2))$ with $j=1,2,\cdots,16$, can be transformed into terms involving E_1, E_2, E_5 and E_{10}. Employing these expressions and also using (2.1.1) in (2.3.14), we obtain

$$\sum_{n=0}^{\infty}Q(125n+26)q^n=\frac{E_2^2E_5^3}{E_1^4E_{10}}+\sum_{m=1}^{5}D_m(E_1,E_2,E_5,E_{10}), \tag{2.3.15}$$

where $D_m(E_1,E_2,E_5,E_{10})=:D_m$ are certain expressions involving E_1, E_2, E_5, and E_{10}. For example, we have

$$D_1=\frac{164}{E_1^{29}E_2^3E_5^{10}E_{10}^{18}}\Big(E_2E_5^5-4qE_1E_{10}^5\Big)^2\Big(E_2E_5^5+qE_1E_{10}^5\Big)^4$$
$$\times\Big(E_2^4E_5^{20}+144qE_1E_2^3E_5^{15}E_{10}^5+976q^2E_1^2E_2^2E_5^{10}E_{10}^{10}+1024q^3E_1^3E_2E_5^5E_{10}^{15}$$
$$+2816q^4E_1^4E_{10}^{20}\Big),$$

which, by (2.2.5) and (2.2.6), gives

$$D_1=164\frac{E_2^{15}E_5^{20}}{E_1^{25}E_{10}^{10}}+23616q\frac{E_2^{14}E_5^{15}}{E_1^{24}E_{10}^5}+160064q^2\frac{E_2^{13}E_5^{10}}{E_1^{23}}+167936q^3\frac{E_2^{12}E_5^5E_{10}^5}{E_1^{22}}$$
$$+461824q^4\frac{E_2^{11}E_{10}^{10}}{E_1^{21}}.$$

With the aid of (2.2.5) again, the above can be further reduced to

$$D_1=41\times2^2\frac{E_2^7E_5^4}{E_1^9E_{10}^2}+41\times2^7\times5\ q\ \frac{E_2^8E_5^3E_{10}}{E_1^{12}}+287\times2^6\times5^2q^2\frac{E_2^9E_5^2E_{10}^4}{E_1^{15}}$$
$$+41\times2^9\times5^3q^3\frac{E_2^{10}E_5E_{10}^7}{E_1^{18}}+41\times2^{10}\times5^3q^4\frac{E_2^{11}E_{10}^{10}}{E_1^{21}}. \tag{2.3.16}$$

In a similar fashion, we can reduce the remaining D_j's. The final forms are given by

$$D_2=3311\times2^4\times5^2q\ \frac{E_2^8E_5^3E_{10}}{E_1^{12}}+17243\times2^6\times5^3q^2\frac{E_2^9E_5^2E_{10}^4}{E_1^{15}}$$

$$+ 70563 \times 2^{9} \times 5^{3} q^{3} \frac{E_2^{10} E_5 E_{10}^{7}}{E_1^{18}} + 56201 \times 2^{11} \times 5^{4} q^{4} \frac{E_2^{11} E_{10}^{10}}{E_1^{21}}$$
$$+ 51041 \times 2^{12} \times 5^{5} q^{5} \frac{E_2^{12} E_{10}^{13}}{E_1^{24} E_5} + 7009 \times 2^{15} \times 5^{6} q^{6} \frac{E_2^{13} E_{10}^{16}}{E_1^{27} E_5^{2}}$$
$$+ 1161 \times 2^{17} \times 5^{7} q^{7} \frac{E_2^{14} E_{10}^{19}}{E_1^{30} E_5^{3}} + 43 \times 2^{18} \times 5^{9} q^{8} \frac{E_2^{15} E_{10}^{22}}{E_1^{33} E_5^{4}}$$
$$+ 43 \times 2^{20} \times 5^{8} q^{9} \frac{E_2^{16} E_{10}^{25}}{E_1^{36} E_5^{5}}, \tag{2.3.17}$$

$$D_3 = 493 \times 2^{7} \times 5^{3} q \frac{E_2^{8} E_5^{3} E_{10}}{E_1^{12}} + 404651 \times 2^{6} \times 5^{3} q^{2} \frac{E_2^{9} E_5^{2} E_{10}^{4}}{E_1^{15}}$$
$$+ 477513 \times 2^{10} \times 5^{4} q^{3} \frac{E_2^{10} E_5 E_{10}^{7}}{E_1^{18}} + 2024479 \times 2^{11} \times 5^{5} q^{4} \frac{E_2^{11} E_{10}^{10}}{E_1^{21}}$$
$$+ 1225887 \times 2^{14} \times 5^{6} q^{5} \frac{E_2^{12} E_{10}^{13}}{E_1^{24} E_5} + 3812981 \times 2^{14} \times 5^{7} q^{6} \frac{E_2^{13} E_{10}^{16}}{E_1^{27} E_5^{2}}$$
$$+ 126769 \times 2^{20} \times 5^{8} q^{7} \frac{E_2^{14} E_{10}^{19}}{E_1^{30} E_5^{3}} + 477377 \times 2^{21} \times 5^{8} q^{8} \frac{E_2^{15} E_{10}^{22}}{E_1^{33} E_5^{4}}$$
$$+ 32283 \times 2^{25} \times 5^{9} q^{9} \frac{E_2^{16} E_{10}^{25}}{E_1^{36} E_5^{5}} + 200311 \times 2^{22} \times 5^{10} q^{10} \frac{E_2^{17} E_{10}^{28}}{E_1^{39} E_5^{6}}$$
$$+ 6817 \times 2^{26} \times 5^{11} q^{11} \frac{E_2^{18} E_{10}^{31}}{E_1^{42} E_5^{7}} + 1241 \times 2^{27} \times 5^{12} q^{12} \frac{E_2^{19} E_{10}^{34}}{E_1^{45} E_5^{8}}$$
$$+ 17 \times 2^{31} \times 5^{13} q^{13} \frac{E_2^{20} E_{10}^{37}}{E_1^{48} E_5^{9}} + 17 \times 2^{30} \times 5^{13} q^{14} \frac{E_2^{21} E_{10}^{40}}{E_1^{51} E_5^{10}}, \tag{2.3.18}$$

$$D_4 = 297 \times 2^{8} \times 5^{3} q \frac{E_2^{8} E_5^{3} E_{10}}{E_1^{12}} + 94677 \times 2^{8} \times 5^{4} q^{2} \frac{E_2^{9} E_5^{2} E_{10}^{4}}{E_1^{15}}$$
$$+ 876531 \times 2^{8} \times 5^{6} q^{3} \frac{E_2^{10} E_5 E_{10}^{7}}{E_1^{18}} + 19851417 \times 2^{10} \times 5^{6} q^{4} \frac{E_2^{11} E_{10}^{10}}{E_1^{21}}$$
$$+ 9716589 \times 2^{12} \times 5^{8} q^{5} \frac{E_2^{12} E_{10}^{13}}{E_1^{24} E_5} + 74934057 \times 2^{14} \times 5^{8} q^{6} \frac{E_2^{13} E_{10}^{16}}{E_1^{27} E_5^{2}}$$
$$+ 198625479 \times 2^{17} \times 5^{8} q^{7} \frac{E_2^{14} E_{10}^{19}}{E_1^{30} E_5^{3}} + 19089573 \times 2^{22} \times 5^{9} q^{8} \frac{E_2^{15} E_{10}^{22}}{E_1^{33} E_5^{4}}$$
$$+ 88112979 \times 2^{21} \times 5^{10} q^{9} \frac{E_2^{16} E_{10}^{25}}{E_1^{36} E_5^{5}} + 19495419 \times 2^{24} \times 5^{11} q^{10} \frac{E_2^{17} E_{10}^{28}}{E_1^{39} E_5^{6}}$$
$$+ 26806929 \times 2^{24} \times 5^{12} q^{11} \frac{E_2^{18} E_{10}^{31}}{E_1^{42} E_5^{7}} + 899751 \times 2^{29} \times 5^{13} q^{12} \frac{E_2^{19} E_{10}^{34}}{E_1^{45} E_5^{8}}$$
$$+ 3769011 \times 2^{29} \times 5^{13} q^{13} \frac{E_2^{20} E_{10}^{37}}{E_1^{48} E_5^{9}} + 1220661 \times 2^{30} \times 5^{14} q^{14} \frac{E_2^{21} E_{10}^{40}}{E_1^{51} E_5^{10}}$$
$$+ 75021 \times 2^{33} \times 5^{15} q^{15} \frac{E_2^{22} E_{10}^{43}}{E_1^{54} E_5^{11}} + 13551 \times 2^{34} \times 5^{16} q^{16} \frac{E_2^{23} E_{10}^{46}}{E_1^{57} E_5^{12}}$$

$$+ 849 \times 2^{36} \times 5^{17} q^{17} \frac{E_2^{24} E_{10}^{49}}{E_1^{60} E_5^{13}} + 33 \times 2^{38} \times 5^{18} q^{18} \frac{E_2^{25} E_{10}^{52}}{E_1^{63} E_5^{14}}$$
$$+ 3 \times 2^{40} \times 5^{18} q^{19} \frac{E_2^{26} E_{10}^{55}}{E_1^{66} E_5^{15}} \tag{2.3.19}$$

and

$$D_5 = 2^{14} \times 5^3 q \frac{E_2^8 E_5^3 E_{10}}{E_1^{12}} + 11809 \times 2^{11} \times 5^4 q^2 \frac{E_2^9 E_5^2 E_{10}^4}{E_1^{15}}$$
$$+ 638351 \times 2^{12} \times 5^5 q^3 \frac{E_2^{10} E_5 E_{10}^7}{E_1^{18}} + 90526083 \times 2^{10} \times 5^6 q^4 \frac{E_2^{11} E_{10}^{10}}{E_1^{21}}$$
$$+ 200444429 \times 2^{13} \times 5^7 q^5 \frac{E_2^{12} E_{10}^{13}}{E_1^{24} E_5} + 67223049 \times 2^{18} \times 5^8 q^6 \frac{E_2^{13} E_{10}^{16}}{E_1^{27} E_5^2}$$
$$+ 4865982433 \times 2^{17} \times 5^8 q^7 \frac{E_2^{14} E_{10}^{19}}{E_1^{30} E_5^3} + 12697324153 \times 2^{18} \times 5^9 q^8 \frac{E_2^{15} E_{10}^{22}}{E_1^{33} E_5^4}$$
$$+ 3120926511 \times 2^{22} \times 5^{10} q^9 \frac{E_2^{16} E_{10}^{25}}{E_1^{36} E_5^5} + 9535583059 \times 2^{22} \times 5^{11} q^{10} \frac{E_2^{17} E_{10}^{28}}{E_1^{39} E_5^6}$$
$$+ 2890743287 \times 2^{25} \times 5^{12} q^{11} \frac{E_2^{18} E_{10}^{31}}{E_1^{42} E_5^7} + 1412243351 \times 2^{27} \times 5^{13} q^{12} \frac{E_2^{19} E_{10}^{34}}{E_1^{45} E_5^8}$$
$$+ 350983729 \times 2^{32} \times 5^{13} q^{13} \frac{E_2^{20} E_{10}^{37}}{E_1^{48} E_5^9} + 914358843 \times 2^{31} \times 5^{14} q^{14} \frac{E_2^{21} E_{10}^{40}}{E_1^{51} E_5^{10}}$$
$$+ 244451427 \times 2^{33} \times 5^{15} q^{15} \frac{E_2^{22} E_{10}^{43}}{E_1^{54} E_5^{11}} + 107206157 \times 2^{34} \times 5^{16} q^{16} \frac{E_2^{23} E_{10}^{46}}{E_1^{57} E_5^{12}}$$
$$+ 9595689 \times 2^{37} \times 5^{17} q^{17} \frac{E_2^{24} E_{10}^{49}}{E_1^{60} E_5^{13}} + 173707 \times 2^{42} \times 5^{18} q^{18} \frac{E_2^{25} E_{10}^{52}}{E_1^{63} E_5^{14}}$$
$$+ 802073 \times 2^{41} \times 5^{18} q^{19} \frac{E_2^{26} E_{10}^{55}}{E_1^{66} E_5^{15}} + 14423 \times 2^{43} \times 5^{20} q^{20} \frac{E_2^{27} E_{10}^{58}}{E_1^{69} E_5^{16}}$$
$$+ 487 \times 2^{46} \times 5^{21} q^{21} \frac{E_2^{28} E_{10}^{61}}{E_1^{72} E_5^{17}} + 93 \times 2^{46} \times 5^{22} q^{22} \frac{E_2^{29} E_{10}^{64}}{E_1^{75} E_5^{18}}$$
$$+ 7 \times 2^{49} \times 5^{22} q^{23} \frac{E_2^{30} E_{10}^{67}}{E_1^{78} E_5^{19}} + 2^{50} \times 5^{22} q^{24} \frac{E_2^{31} E_{10}^{70}}{E_1^{81} E_5^{20}}. \tag{2.3.20}$$

Employing (2.3.16)–(2.3.20) in (2.3.15), we have

$$\sum_{n=0}^{\infty} Q(125n+26) q^n$$
$$= \frac{E_2^2 E_5^3}{E_1^4 E_{10}} + 41 \times 2^2 \frac{E_2^7 E_5^4}{E_1^9 E_{10}^2} + 259883 \times 2^4 \times 5\ q\ \frac{E_2^8 E_5^3 E_{10}}{E_1^{12}}$$
$$+ 21024657 \times 2^6 \times 5^2 q^2 \frac{E_2^9 E_5^2 E_{10}^4}{E_1^{15}} + 374598243 \times 2^8 \times 5^3 q^3 \frac{E_2^{10} E_5 E_{10}^7}{E_1^{18}}$$
$$+ 13898973501 \times 2^{10} \times 5^3 q^4 \frac{E_2^{11} E_{10}^{10}}{E_1^{21}} + 703835241 \times 2^{16} \times 5^5 q^5 \frac{E_2^{12} E_{10}^{13}}{E_1^{24} E_5}$$

$$+ 7195412487 \times 2^{16} \times 5^{6} q^{6} \frac{E_2^{13} E_{10}^{16}}{E_1^{27} E_5^{2}} + 25328111481 \times 2^{17} \times 5^{7} q^{7} \frac{E_2^{14} E_{10}^{19}}{E_1^{30} E_5^{3}}$$
$$+ 16254401459 \times 2^{20} \times 5^{8} q^{8} \frac{E_2^{15} E_{10}^{22}}{E_1^{33} E_5^{4}} + 316503465373 \times 2^{20} \times 5^{8} q^{9} \frac{E_2^{16} E_{10}^{25}}{E_1^{36} E_5^{5}}$$
$$+ 24034011993 \times 2^{23} \times 5^{10} q^{10} \frac{E_2^{17} E_{10}^{28}}{E_1^{39} E_5^{6}} + 29041494783 \times 2^{24} \times 5^{11} q^{11} \frac{E_2^{18} E_{10}^{31}}{E_1^{42} E_5^{7}}$$
$$+ 884901627 \times 2^{30} \times 5^{12} q^{12} \frac{E_2^{19} E_{10}^{34}}{E_1^{45} E_5^{8}} + 2811638911 \times 2^{29} \times 5^{13} q^{13} \frac{E_2^{20} E_{10}^{37}}{E_1^{48} E_5^{9}}$$
$$+ 1143711469 \times 2^{33} \times 5^{13} q^{14} \frac{E_2^{21} E_{10}^{40}}{E_1^{51} E_5^{10}} + 15282903 \times 2^{37} \times 5^{15} q^{15} \frac{E_2^{22} E_{10}^{43}}{E_1^{54} E_5^{11}}$$
$$+ 26804927 \times 2^{36} \times 5^{16} q^{16} \frac{E_2^{23} E_{10}^{46}}{E_1^{57} E_5^{12}} + 19192227 \times 2^{36} \times 5^{17} q^{17} \frac{E_2^{24} E_{10}^{49}}{E_1^{60} E_5^{13}}$$
$$+ 555869 \times 2^{38} \times 5^{19} q^{18} \frac{E_2^{25} E_{10}^{52}}{E_1^{63} E_5^{14}} + 1604149 \times 2^{40} \times 5^{18} q^{19} \frac{E_2^{26} E_{10}^{55}}{E_1^{66} E_5^{15}}$$
$$+ 14423 \times 2^{43} \times 5^{20} q^{20} \frac{E_2^{27} E_{10}^{58}}{E_1^{69} E_5^{16}} + 487 \times 2^{46} \times 5^{21} q^{21} \frac{E_2^{28} E_{10}^{61}}{E_1^{72} E_5^{17}}$$
$$+ 93 \times 2^{46} \times 5^{22} q^{22} \frac{E_2^{29} E_{10}^{64}}{E_1^{75} E_5^{18}} + 7 \times 2^{49} \times 5^{22} q^{23} \frac{E_2^{30} E_{10}^{67}}{E_1^{78} E_5^{19}} + 2^{50} \times 5^{22} q^{24} \frac{E_2^{31} E_{10}^{70}}{E_1^{81} E_5^{20}}. \tag{2.3.21}$$

Using (2.2.7) in the second and third terms of the right side of (2.3.21), we see that

$$41 \times 2^2 \frac{E_2^7 E_5^4}{E_1^9 E_{10}^2} = 41 \times 2^2 \frac{E_2^2 E_5^3}{E_1^4 E_{10}} + 5 \times 41 \times 2^2 q \frac{E_2^3 E_5^2 E_{10}^2}{E_1^7}$$

and

$$259883 \times 2^4 \times 5\ q\ \frac{E_2^8 E_5^3 E_{10}}{E_1^{12}} = 259883 \times 2^4 \times 5\ q\ \frac{E_2^3 E_5^2 E_{10}^2}{E_1^7} + 259883 \times 2^4 \times 5^2 q^2 \frac{E_2^4 E_5 E_{10}^5}{E_1^{10}},$$

respectively. We do the same for the remaining terms of the right side of (2.3.21), except the first term. Hence, we arrive at (2.1.3), to finish the proof of Theorem 2.1.1.

2.4 Proof of (1.5.2) for $j = 1$ and 2

Note that the case $j = 1$ of (1.5.2) immediately follows from (2.1.3). We also notice from (2.1.3) that

$$\sum_{n=0}^{\infty} Q(5^3 n + 5^2 + 1) q^n = 165 \frac{E_2^2 E_5^3}{E_1^4 E_{10}} + 20791460 q \, \frac{E_2^3 E_5^2 E_{10}^2}{E_1^7} + A(q), \tag{2.4.1}$$

where $A(q)$ denotes the expression from the third term onwards of the right side of the equality in (2.1.3). Note that all the coefficients of $A(q)$ are multiples of 5^2. Furthermore, from (2.1.1) and (2.1.3), it is clear that the coefficients of q^{25n+5} in the first term on the right side of the above equality are multiples of 5^2. We show that the coefficients of q^{5n}, and hence q^{25n+5}, in the second term are also multiples of 5^2. To that end, define $\alpha(n)$ by

$$\sum_{n=1}^{\infty} \alpha(n) q^n = q \, \frac{E_2^3 E_5^2 E_{10}^2}{E_1^7}.$$

Since by the binomial theorem,

$$E_1^5 \equiv E_5 \pmod{5}, \tag{2.4.2}$$

we see that

$$\sum_{n=1}^{\infty} \alpha(n) q^n \equiv q E_1^3 E_2^3 E_{10}^2 \pmod{5}. \tag{2.4.3}$$

Applying Jacobi's identity (2.2.10), we can rewrite (2.4.3) as

$$\sum_{n=1}^{\infty} \alpha(n) q^n \equiv E_{10}^2 \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{\ell+k} (2\ell+1)(2k+1) q^{\ell(\ell+1)/2 + k(k+1) + 1} \pmod{5}. \tag{2.4.4}$$

Note that the exponents $\ell(\ell+1)/2 + k(k+1) + 1 \equiv 0 \pmod{5}$ only when both ℓ and k are congruent to 2 (mod 5), and hence, the coefficients $(2\ell+1)(2k+1) \equiv 0 \pmod{5}$. Thus, from (2.4.4), we have

$$\alpha(5n) \equiv 0 \pmod{5}.$$

Now, comparing the coefficients of q^{25n+5} from both sides of (2.4.1), we find that

$$Q(5^5 n + 5^4 + 5^2 + 1) \equiv 0 \pmod{5^2},$$

which is the case $j = 2$ of (1.5.2).

Remark 2.4.1. *The proof of* (1.5.2) *for all* $j \geq 1$ *might follow from the generating functions in Theorem 2.1.1. But it seems that the calculations are too lengthy and tedious even if we use* ***Mathematica****. It would be interesting to find a simpler way of deducing* (1.5.2) *from Theorem 2.1.1.*

2.5 Proofs of (2.1.4)–(2.1.7)

Proofs of (2.1.4) *and* (2.1.5)*.* With the aid of (2.2.5) we can transform (2.3.13) into

$$\sum_{n=0}^{\infty} Q(25n+1)q^n = \frac{E_2^3E_5^4}{E_1^5E_{10}^2} + 160q\frac{E_2^4E_5^3E_{10}}{E_1^8} + 2800q^2\frac{E_2^5E_5^2E_4^{10}}{E_1^{11}} + 16000q^3\frac{E_2^6E_5E_{10}^7}{E_1^{14}} + 32000q^4\frac{E_2^7E_{10}^{10}}{E_1^{17}},$$

which, by (2.4.2) gives

$$\sum_{n=0}^{\infty} Q(25n+1)q^n \equiv \frac{E_2^3E_5^3}{E_{10}^2} \pmod 5.$$

With the aid of (2.2.10), we have

$$\sum_{n=0}^{\infty} Q(25n+1)q^n \equiv \frac{E_5^3}{E_{10}^2}\sum_{\ell=0}^{\infty}(-1)^\ell(2\ell+1)q^{\ell(\ell+1)} \pmod 5.$$

As none of the exponents of q on the right side of the above congruence are of the form $5n+3$ or $5n+4$, we conclude that

$$Q(25(5n+3)+1) \equiv 0 \pmod 5$$

and

$$Q(25(5n+4)+1) \equiv 0 \pmod 5,$$

which are clearly (2.1.4) and (2.1.5), respectively.

Proofs of (2.1.6) *and* (2.1.7)*.* Applying (2.2.5) in (2.1.3), we have

$$\sum_{n=0}^{\infty} Q(125n+26)q^n = 165\frac{E_2^4E_5^7}{E_1^8E_{10}^3} + B(q),$$

where $B(q)$ consists of all other terms with coefficients congruent to 0 modulo 5^2. Therefore,

$$\sum_{n=0}^{\infty} Q(125n+26)q^n \equiv 165\frac{E_2^4E_5^7}{E_1^8E_{10}^3} \pmod{5^2},$$

which by (2.4.2) reduces to

$$\sum_{n=0}^{\infty} Q(125n+26)q^n \equiv 165\frac{E_1^2E_5^5}{E_2E_{10}^2} \pmod{5^2}.$$

With of aid (1.2.1), we can rewrite the above as

$$\sum_{n=0}^{\infty} Q(125n+26)q^n \equiv 165\frac{E_5^5}{E_{10}^2}\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2} \pmod{5^2}$$

Since the exponents of q on the right side of the above congruence cannot be of the forms $5n+2$ and $5n+3$, we obtain

$$Q(125(5n+2)+26) \equiv 0 \pmod{5^2}$$

and

$$Q(125(5n+3)+26) \equiv 0 \pmod{5^2},$$

which are equivalent to (2.1.6) and (2.1.7).

Remark 2.5.1. *At the beginning of Section 2.3, we employed* (1.4.1) *and* (1.4.2) *in* (2.3.1) *to arrive at* (2.3.2). *Proceeding in a similar way, we obtain*

$$\begin{aligned}
\sum_{n=0}^{\infty} Q(5n)q^n =&\frac{E_5^5E_{10}}{E_1^6}\Big(R(q)^4R(q^2) - q\Big(3R(q)+\frac{R(q)^3}{R(q^2)}+3\frac{R(q^2)}{R(q)}\Big)\\
&- q^2\Big(\frac{1}{R(q)^4}+\frac{2}{R(q)^2R(q^2)}\Big)\Big),\\
\sum_{n=0}^{\infty} Q(5n+2)q^n =&\frac{E_5^5E_{10}}{E_1^6}\Big(\Big(-R(q)^4+2R(q)^2R(q^2)\Big)\\
&+ q\Big(\frac{3}{R(q)}-3\frac{R(q)}{R(q^2)}-\frac{R(q^2)}{R(q)^3}\Big)-\frac{q^2}{R(q)^4R(q^2)}\Big),\\
\sum_{n=0}^{\infty} Q(5n+3)q^n =&\frac{E_5^5E_{10}}{E_1^6}\Big(\Big(-R(q)^3+3R(q)R(q^2)\Big)
\end{aligned}$$

$$+ q\Big(- \frac{2}{R(q)^2} - \frac{5}{R(q^2)} + \frac{R(q^2)}{R(q)^4}\Big)\Big)$$

and

$$\sum_{n=0}^{\infty} Q(5n+4)q^n = \frac{E_5^5 E_{10}}{E_1^6}\Big(\Big(- 2R(q)^2 - \frac{R(q)^4}{R(q^2)} + 5R(q^2)\Big) + q\Big(\frac{1}{R(q)^3} + \frac{3}{R(q)R(q^2)}\Big)\Big).$$

We note that the expressions involving $R(q)$ *and* $R(q^2)$ *appearing in the above could not be expressed in terms of* E_1, E_2, E_5 *and* E_{10} *as in* (2.3.2) *because of the non-availability of the expressions similar to those in Lemma 2.2.1 and* (2.3.8)–(2.3.12). *Therefore, in Theorem 2.1.1, we considered only the case* $Q(5n+1)$ *among* $Q(5n+i)$ *where* $0 \leq i \leq 4$. *For the same reason, we included only the cases* $Q(25n+1)$ *and* $Q(125n+26)$.