Chapter 3

Generating functions and congruences for some partition functions related to mock theta functions

3.1 Introduction

In sequel to Chapter 2, in this chapter, by using relations between $R(q)$ and $R(q^2)$, we find several new exact generating functions for some partition functions related to Ramanujan/Watson mock theta functions as well as the associated smallest parts functions and deduce several new congruences modulo powers of 5. We refer to Section 1.6 of the introductory chapter of the thesis for the definitions of various partition functions appearing in this chapter.

We find the following representation of the generating function of $p_\nu(50n+8)$.

Theorem 3.1.1. *We have*

$$\sum_{n=0}^{\infty} p_\nu(50n+8)q^n = 5\left(\frac{E_2^5E_5^4}{E_1^6E_{10}^2} + 160q\frac{E_2^{11}E_5^4}{E_1^{14}} + 2000q^2\frac{E_2^{11}E_5^{10}}{E_1^{20}}\right). \tag{3.1.1}$$

We also deduce (1.6.8) and the following new congruence modulo 125.

Corollary 3.1.2. *For any nonnegative integer n, we have*

$$p_\nu(6250n+5208) \equiv 0 \pmod{125}.$$

From (1.6.6), Theorem 3.1.1 and Corollary 3.1.2, we also have the following new results on $f(n)$, the number of 1-shell totally symmetric plane partitions of n.

Theorem 3.1.3. *We have*

$$\sum_{n=0}^{\infty} f(150n+25)q^n = 5\left(\frac{E_2^5E_5^4}{E_1^6E_{10}^2} + 160q\frac{E_2^{11}E_5^4}{E_1^{14}} + 2000q^2\frac{E_2^{11}E_5^{10}}{E_1^{20}}\right).$$

Furthermore, for any nonnegative integer n,

$$f(18750n+15625) \equiv 0 \pmod{125}.$$

We see that Waldherr's congruences (1.6.4) can be easily deduced from (1.6.5).

In this chapter, we also find the following generating function of $p_\omega(40n+12)$.

Theorem 3.1.4. *We have*

$$\begin{aligned}\sum_{n=0}^{\infty} p_\omega(40n+12)q^n = 4\Big(9\frac{E_2^2E_5}{E_1^2} &+ 3975q\frac{E_2^3E_{10}^3}{E_1^5} + 207425q^2\frac{E_2^4E_{10}^6}{E_1^8E_5} \\ &+ 4229000q^3\frac{E_2^5E_{10}^9}{E_1^{11}E_5^2} + 44850000q^4\frac{E_2^6E_{10}^{12}}{E_1^{14}E_5^3} \\ &+ 274000000q^5\frac{E_2^7E_{10}^{15}}{E_1^{17}E_5^4} + 980000000q^6\frac{E_2^8E_{10}^{18}}{E_1^{20}E_5^5} \\ &+ 1920000000q^7\frac{E_2^9E_{10}^{21}}{E_1^{23}E_5^6} + 1600000000q^8\frac{E_2^{10}E_{10}^{24}}{E_1^{26}E_5^7}\Big). \qquad (3.1.2)\end{aligned}$$

We deduce the following interesting congruence recently proved by Xia [71].

Corollary 3.1.5. *For any nonnegative integers n and k, we have*

$$p_\omega\left(8\times 5^{2k+1}n + \frac{7\times 5^{2k+1}+1}{3}\right) \equiv (-1)^k p_\omega(40n+12) \pmod{5}. \qquad (3.1.3)$$

In this chapter, we find the following exact generating functions of $\mathrm{spt}_\omega(10n+3)$ and $\mathrm{spt}_\omega(50n+23)$.

Theorem 3.1.6. *We have*

$$\sum_{n=0}^{\infty} \mathrm{spt}_\omega(10n+3)q^n = 5\left(E_1^2E_5 + 6q\frac{E_2E_{10}^3}{E_1} + 25q\frac{E_2^8E_5^5}{E_1^{10}}\right) \qquad (3.1.4)$$

and

$$\sum_{n=0}^{\infty} \mathrm{spt}_{\omega}\left(50n+23\right)q^n$$
$$= 25\Bigg(E_1E_5^2 + 6q\frac{E_2E_5E_{10}^3}{E_1^2} + 25\left(2q\frac{E_1^3E_{10}^4}{E_2^4} + 25q^3\frac{E_1^3E_{10}^{10}}{E_2^{10}}\right)$$
$$+ 50\Bigg(\frac{E_1^{16}E_{10}^3}{E_2^{15}E_5} + 150q\frac{E_1^{13}E_{10}^6}{E_2^{14}E_5^2} + 5650q^2\frac{E_1^{10}E_{10}^9}{E_2^{13}E_5^3} + 101825q^3\frac{E_1^7E_{10}^{12}}{E_2^{12}E_5^4}$$
$$+ 1068125q^4\frac{E_1^4E_{10}^{15}}{E_2^{11}E_5^5} + 7042500q^5\frac{E_1E_{10}^{18}}{E_2^{10}E_5^6} + 29800000q^6\frac{E_{10}^{21}}{E_1^2E_2^9E_5^7}$$
$$+ 79000000q^7\frac{E_{10}^{24}}{E_1^5E_2^8E_5^8} + 120000000q^8\frac{E_{10}^{27}}{E_1^8E_2^7E_5^9} + 80000000q^9\frac{E_{10}^{30}}{E_1^{11}E_2^6E_5^{10}}\Bigg)$$
$$+ 625q\Bigg(63\frac{E_2^8E_5^6}{E_1^{11}} + 6500q\frac{E_2^8E_5^{12}}{E_1^{17}} + 196875q^2\frac{E_2^8E_5^{18}}{E_1^{23}} + 2343750q^3\frac{E_2^8E_5^{24}}{E_1^{29}}$$
$$+ 9765625q^4\frac{E_2^8E_5^{30}}{E_1^{35}}\Bigg)\Bigg). \tag{3.1.5}$$

Note that the congruences (1.6.15) and (1.6.16) follow trivially from the above theorem. Furthermore, we deduce (1.6.17) and (1.6.18).

We also present the following exact generating function of $\overline{\mathrm{spt}}_{\omega}\left(10n+5\right)$.

Theorem 3.1.7. *We have*

$$\sum_{n=0}^{\infty} \overline{\mathrm{spt}}_{\omega}\left(10n+5\right)q^n = 18E_1^2E_{10} + 720q\frac{E_2E_{10}^4}{E_1E_5} + 7625q^2\frac{E_2^2E_{10}^7}{E_1^4E_5^2}$$
$$+ 32500q^3\frac{E_2^3E_{10}^{10}}{E_1^7E_5^3} + 50000q^4\frac{E_2^4E_{10}^{13}}{E_1^{10}E_5^4}. \tag{3.1.6}$$

As a corollary, we deduce the following new infinite families of congruences.

Corollary 3.1.8. *For any nonnegative integers n and k, we have*

$$\overline{\mathrm{spt}}_{\omega}\left(5^{2k}\left(10n+5\right)\right) \equiv \overline{\mathrm{spt}}_{\omega}\left(10n+5\right) \pmod 5, \tag{3.1.7}$$

$$\overline{\mathrm{spt}}_{\omega}\left(5^{2k+2}\left(10n+3\right)\right) \equiv 0 \pmod 5 \tag{3.1.8}$$

and

$$\overline{\mathrm{spt}}_{\omega}\left(5^{2k+2}\left(10n+7\right)\right) \equiv 0 \pmod 5. \tag{3.1.9}$$

Andrews [2] claims that the second order mock theta functions

$$\eta(q) := \sum_{n\geq 0} \frac{q^{n(n-1}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} = \sum_{n\geq 0} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}$$

and

$$\xi(q) := \sum_{n\geq 0} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2},$$

may be related though the former does not appear in Ramanujan's Lost Notebook.

He also showed that

$$\eta(q) = \frac{(-q^2,q^2)_\infty}{(q^2,q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(n+1)}}{1-q^{2n+1}}.$$

Using this Gordon and McIntosh [42] proved that

$$\frac{\eta(q)+\eta(-q)}{2} = \frac{E_4^5}{E_2^4}.$$

If

$$\sum_{n=0}^{\infty} \mathfrak{b}(n)q^n := \eta(q),$$

then clearly,

$$\sum_{n=0}^{\infty} \mathfrak{b}(2n)q^n := \frac{E_2^5}{E_1^4}. \tag{3.1.10}$$

Chern and Wang [31] found the following infinite family of congruences modulo powers of 3.

Theorem 3.1.9. *For $\alpha \geq 1$ and $n \geq 0$, we have*

$$\mathfrak{b}(2\cdot 3^{2\alpha-1}n + \frac{3^{2\alpha}-1}{2}) \equiv 0 \pmod{3^\alpha}.$$

In this chapter, we find the following exact generating function for $\mathfrak{b}(10n+2)$.

Theorem 3.1.10. *We have*

$$\sum_{n=0}^{\infty} \mathfrak{b}(10n+2)q^n = 4\frac{E_2E_5}{E_1} + 125q\frac{E_2^2E_{10}^3}{E_1^4} + 900q^2\frac{E_2^3E_{10}^6}{E_1^7E_5} + 2000q^3\frac{E_2^4E_{10}^9}{E_1^{10}E_5^2}. \tag{3.1.11}$$

In the process of our proof, we also find the following congruences.

Corollary 3.1.11. *For $\alpha \geq 1$ and $n \geq 0$, we have*

$$\mathfrak{b}(10n+6) \equiv \mathfrak{b}(10n+8) \equiv 0 \pmod{5} \tag{3.1.12}$$

and

$$\mathfrak{b}(2 \cdot 5^{2\alpha+1}n + \frac{5^{2\alpha+1}-1}{2}) \equiv 3^{\alpha}\mathfrak{b}(10n+2) \pmod{25}. \tag{3.1.13}$$

We organize this chapter as follows. In Section 3.2, we state two useful lemmas. In Section 3.3, we deduce Waldherr's congruences (1.6.4) from the generating function of $p_\omega(8n+4)$ and in Section 3.4, we deduce two congruences for $\mathrm{spt}_\omega(10n+7)$ and $\mathrm{spt}_\omega(10n+9)$ from the generating function of $\mathrm{spt}_\omega(2n+1)$. In Section 3.5, we find the exact generating function for $p_\nu(50n+8)$. We also deduce congruences for $p_\nu(250n+208)$ and $p_\nu(6250n+5208)$. In Section 3.6, we find the exact generating function for $p_\omega(40n+12)$ and deduce the infinite family of congruences given by Corollary 3.1.5. In Section 3.7, we find the exact generating function for $\mathrm{spt}_\omega(10n+3)$ and $\mathrm{spt}_\omega(50n+23)$ and deduce congruences for $\mathrm{spt}_\omega(250n+73)$ and $\mathrm{spt}_\omega(1250n+573)$. In Section 3.8, we find the exact generating function for $\overline{\mathrm{spt}}_\omega(10n+5)$ and deduce some infinite family of congruences given by Corollary 3.1.8. In the final section of this chapter, we prove Theorem 3.1.10 and the congruences in Corollary 3.1.11.

The contents of Sections 3.3–3.8 have been submitted to *International Journal of Number Theory* [18].

3.2 Two useful lemmas

In the following two lemmas we recall the famous Euler's pentagonal number theorem and a well-known identity on the Rogers-Ramanujan continued fraction $\mathcal{R}(q)$ from Berndt's book [20, p. 12 and p. 164].

Lemma 3.2.1. *We have*

$$E_1 = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k+1)/2}. \tag{3.2.1}$$

Lemma 3.2.2. *If* $R(q) = q^{1/5}\mathcal{R}(q)$, *then*

$$11q + \frac{E_1^6}{E_5^6} = R(q)^5 - \frac{q^2}{R(q)^5}. \tag{3.2.2}$$

3.3 Deduction of Waldherr's congruences (1.6.4) from (1.6.5)

Using (2.4.2) in (1.6.5), we have

$$\sum_{n=0}^{\infty} p_\omega(8n+4)q^n \equiv 4\frac{E_{10}^2 E_1}{E_5^2} \pmod 5,$$

which can be rewritten with the aid of famous Euler's pentagonal number theorem (3.2.1), as

$$\sum_{n=0}^{\infty} p_\omega(8n+4)q^n \equiv 4\frac{E_{10}^2}{E_5^2} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k+1)/2} \pmod 5.$$

Since $k(3k+1)/2 \equiv 0,\ 1,\ \text{or } 2 \pmod 5$ only, equating the coefficients of q^{5n+r}, $r = 3, 4$ from both sides of the above, we easily arrive at (1.6.4).

3.4 Deduction of (1.6.10), (1.6.11) and (1.6.15) from (1.6.13)

Taking congruences modulo 5 in (1.6.13) and using (2.4.2), we have

$$\sum_{n=0}^{\infty} \mathrm{spt}_\omega(2n+1)q^n \equiv \frac{E_{10}E_2^3}{E_5} \pmod 5. \tag{3.4.1}$$

Employing Jacobi's identity (2.2.10) in (3.4.1), we have

$$\sum_{n=0}^{\infty} \mathrm{spt}_\omega(2n+1)q^n \equiv \frac{E_{10}}{E_5} \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)} \pmod 5.$$

Since $k(k+1) \equiv 0,\ 1,$ or $2 \pmod 5$, equating the coefficients of q^{5n+r}, $r = 3, 4$ from both sides of the above, we easily arrive at (1.6.10) and (1.6.11). Furthermore, we note that $k(k+1) \equiv 1 \pmod 5$ only when $k \equiv 2 \pmod 5$, that is, only when $2k+1 \equiv 0 \pmod 5$. Therefore, equating the coefficients of q^{5n+1} from both sides of the above we arrive at (1.6.15) which is, in fact, contained in (1.6.9).

3.5 Proofs of Theorem 3.1.1, (1.6.8) and Corollary 3.1.2

Proof of Theorem 3.1.1. Employing (2.2.5) successively in (1.6.7), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} p_\nu(10n+8)q^n &= 5\left(\frac{E_{10}^3}{E_5^2} + 4q\frac{E_2E_{10}^6}{E_1^3E_5^3}\right)\\ &= 5\left(\frac{E_{10}^3}{E_5^2} + 4q\frac{E_1E_{10}^8}{E_2E_5^7} + 16q^2\frac{E_{10}^{11}}{E_1^2E_5^8}\right).\end{aligned}$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n}, and then replacing q^5 by q, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p_\nu(50n+8)q^n = 5\frac{E_2^3}{E_1^2} &+ 20q\frac{E_2^2E_5E_{10}^5}{E_1^7}\left(2\left(R(q)R(q^2)^2 - \frac{q^2}{R(q)R(q^2)^2}\right)\right.\\ &\left.- \left(\frac{R(q^2)^3}{R(q)} + q^2\frac{R(q)}{R(q^2)^3}\right) - 5q\right) + 400q\frac{E_2^{11}E_5^{10}}{E_1^{20}}\\ &\times\left(2\left(R(q)^5 - \frac{q^2}{R(q)^5}\right) + 3q\right).\end{aligned}$$

Employing (3.2.2), (2.2.1) and (2.2.3) in the above, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p_\nu(50n+8)q^n &= 5\frac{E_2^3}{E_1^2} + 20q\frac{E_2^3E_5^6}{E_1^8} - 60q^2\frac{E_2^2E_5E_{10}^5}{E_1^7} - 80q^3\frac{E_2E_{10}^{10}}{E_1^6E_5^4} + 800q\frac{E_2^{11}E_5^4}{E_1^{14}}\\ &+ 10000q^2\frac{E_2^{11}E_5^{10}}{E_1^{20}}.\end{aligned}$$

We now reduce the above into the form (3.1.1) with the aid of (2.2.5) and (2.2.6) as shown below:

$$\sum_{n=0}^{\infty} p_\nu(50n+8)q^n = 5\frac{E_2^3}{E_1^2} + 20q\frac{E_2^3E_5E_{10}^3}{E_1^7}\left(\frac{E_5^5}{E_1E_{10}^3} + q\frac{E_{10}^2}{E_2}\right)$$

$$
\begin{aligned}
&- 80q^2\frac{E_2^2E_{10}^8}{E_1^6E_5^4}\left(\frac{E_5^5}{E_1E_{10}^3}+q\frac{E_{10}^2}{E_2}\right)+800q\frac{E_2^{11}E_5^4}{E_1^{14}}\\
&+10000q^2\frac{E_2^{11}E_5^{10}}{E_1^{20}}\\
=&\ 5\frac{E_2^3}{E_1^2}+20q\frac{E_2^6E_5^3E_{10}}{E_1^9}-80q^2\frac{E_2^5E_{10}^6}{E_1^8E_5^2}+800q\frac{E_2^{11}E_5^4}{E_1^{14}}\\
&+10000q^2\frac{E_2^{11}E_5^{10}}{E_1^{20}}\\
=&\ 5\frac{E_2^3}{E_1^2}+20q\frac{E_2^6E_{10}^4}{E_1^5E_5^2}\left(\frac{E_5^5}{E_1^4E_{10}^3}-4q\frac{E_{10}^2}{E_1^3E_2}\right)+800q\frac{E_2^{11}E_5^4}{E_1^{14}}\\
&+10000q^2\frac{E_2^{11}E_5^{10}}{E_1^{20}}\\
=&\ 5\frac{E_2^3}{E_1^2}+20q\frac{E_2^4E_{10}^3}{E_1^5E_5}+800q\frac{E_2^{11}E_5^4}{E_1^{14}}+10000q^2\frac{E_2^{11}E_5^{10}}{E_1^{20}}\\
=&\ 5\frac{E_2^5E_5^4}{E_1^6E_{10}^2}+800q\frac{E_2^{11}E_5^4}{E_1^{14}}+10000q^2\frac{E_2^{11}E_5^{10}}{E_1^{20}}.
\end{aligned}
$$

□

Proof of (1.6.8). Employing (2.4.2) in (3.1.1), then using (1.1.1), we have

$$\sum_{n=0}^{\infty}p_\nu(50n+8)q^n \equiv 5\ \frac{E_5^3}{E_1E_{10}} \equiv 5\ \frac{E_5^3}{E_{10}}\ \sum_{k=0}^{\infty}p(k)q^k \pmod{25}.$$

Extracting the coefficients of q^{5n+4}, and then using (1.1.4), we easily arrive at (1.6.8). □

Proof of Corollary 3.1.2. From (3.1.1), we have

$$\sum_{n=0}^{\infty}p_\nu(50n+8)q^n \equiv 5\left(\frac{E_2^5E_5^4}{E_1^6E_{10}^2}+160q\frac{E_2^{11}E_5^4}{E_1^{14}}\right) \pmod{125}.$$

Employing (2.2.7) in the above, we see that

$$
\begin{aligned}
\sum_{n=0}^{\infty}p_\nu(50n+8)q^n &\equiv 5\left(\frac{E_5^3}{E_1E_{10}}+165q\frac{E_2E_5^2E_{10}^2}{E_1^4}\right)\\
&\equiv 5\left(\frac{E_5^3}{E_1E_{10}}+165qE_1E_2E_5E_{10}^2\right) \pmod{125}.
\end{aligned}
$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+4} from both sides, dividing by q^4, then replacing q^5 by q, and then using (1.1.4), we obtain

$$\sum_{n=0}^{\infty} p_\nu(250n+208)q^n \equiv 25\left(\frac{E_5^5}{E_1^3E_2}+33E_1E_2^2E_5E_{10}\right)$$
$$\equiv 25\left(E_5^4\cdot\frac{E_1^2}{E_2}+33E_1E_2^2E_5E_{10}\right) \pmod{125}. \tag{3.5.1}$$

Employing (1.4.1) and (1.4.3) in (3.5.1), extracting the terms involving q^{5n} from both sides, and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} p_\nu(1250n+208)q^n$$
$$\equiv 25\left(\frac{E_1^4E_5^2}{E_{10}}+33E_1E_2E_5E_{10}^2\left(R(q)R(q^2)^2+q-\frac{q^2}{R(q)R(q^2)^2}\right)\right)$$
$$\equiv 25\left(\frac{E_5^3}{E_1E_{10}}+33E_1E_2E_5E_{10}^2\left(R(q)R(q^2)^2+q-\frac{q^2}{R(q)R(q^2)^2}\right)\right) \pmod{125}. \tag{3.5.2}$$

Now, from [62, p. 56] ([4, p. 35, Entry 1.8.2]), we note that if $k=\dfrac{q}{R(q)R^2(q^2)}$ and $k\leq\sqrt{5}-2$, then

$$\frac{\psi^2(q)}{q\psi^2(q^5)}=\frac{1+k-k^2}{k},$$

which can be seen to be equivalent to

$$R(q)R(q^2)^2+q-\frac{q^2}{R(q)R(q^2)^2}=\frac{\psi^2(q)}{\psi^2(q^5)}=\frac{E_2^4E_5^2}{E_1^2E_{10}^4}, \tag{3.5.3}$$

where the last equality is by (1.2.2).

Using (3.5.3) in (3.5.2) and then applying (2.4.2), we obtain

$$\sum_{n=0}^{\infty} p_\nu(1250n+208)q^n \equiv 25\times 34\ \frac{E_5^3}{E_1E_{10}}$$
$$\equiv 25\times 34\ \frac{E_5^3}{E_{10}}\sum_{n=0}^{\infty}p(n)q^n \pmod{125}.$$

Equating the coefficients of q^{5n+4} from both sides of the above, and then applying (1.1.4), we arrive at

$$p_\nu(6250n+5208)\equiv 0 \pmod{125},$$

to finish the proof. □

3.6 Proofs of Theorem 3.1.4 and Corollary 3.1.5

Proof of Theorem 3.1.4. Employing (2.2.7) in (1.6.5) successively, we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_\omega(8n+4)q^n &= 4\left(\frac{E_2^5E_{10}}{E_1^4E_5} + 5q\frac{E_2^6E_{10}^4}{E_1^7E_5^2}\right)\\ &= 4\left(\frac{E_1E_{10}^2}{E_5^2} + 5q\frac{E_2E_{10}^5}{E_1^2E_5^3} + 5q\left(\frac{E_2E_{10}^5}{E_1^2E_5^3} + 5q\frac{E_2^2E_{10}^8}{E_1^5E_5^4}\right)\right)\\ &= 4\left(\frac{E_1E_{10}^2}{E_5^2} + 10q\frac{E_2E_{10}^5}{E_1^2E_5^3} + 25q^2\frac{E_2^2E_{10}^8}{E_1^5E_5^4}\right).\end{aligned}$$

Now applying (2.2.5) in the above successively, we see that

$$\begin{aligned}&\sum_{n=0}^{\infty} p_\omega(8n+4)q^n\\ &= 4\left(\frac{E_1E_{10}^2}{E_5^2} + 10q\left(\frac{E_1^2E_{10}^7}{E_2E_5^7} + 4q\frac{E_{10}^{10}}{E_1E_5^8}\right) + 25q^2\left(\frac{E_{10}^{10}}{E_1E_5^8} + 4q\frac{E_2E_{10}^{13}}{E_1^4E_5^9}\right)\right)\\ &= 4\left(\frac{E_1E_{10}^2}{E_5^2} + 10q\frac{E_1^2E_{10}^7}{E_2E_5^7} + 65q^2\frac{E_{10}^{10}}{E_1E_5^8} + 100q^3\frac{E_{10}^{15}}{E_2E_5^{13}} + 400q^4\frac{E_{10}^{18}}{E_1^3E_5^{14}}\right).\end{aligned}$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+1}, dividing both sides by q, and then replacing q^5 by q, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} p_\omega(40n+12)q^n\\ &= -4\frac{E_2^2E_5}{E_1^2} + 40\frac{E_2E_5^2E_{10}^5}{E_1^7}\left(R(q)^2R(q^2)^4 + \frac{q^4}{R(q)^2R(q^2)^4}\right) + q\Bigg(1300\frac{E_2^{10}E_5^5}{E_1^{14}}\\ &+ 14400\frac{E_2^{18}E_5^{15}}{E_1^{32}}\left(R(q)^{10} + \frac{q^4}{R(q)^{10}}\right) - 160\frac{E_2E_5^2E_{10}^5}{E_1^7}\Bigg(R(q)R(q^2)^2\\ &- \frac{q^2}{R(q)R(q^2)^2}\Bigg) + 80\frac{E_2E_5^2E_{10}^5}{E_1^7}\left(\frac{R(q^2)^3}{R(q)} + q^2\frac{R(q)}{R(q^2)^3}\right)\Bigg) + q^2\Bigg(2000\frac{E_2^9E_{10}^5}{E_1^{13}}\\ &- 200\frac{E_2E_5^2E_{10}^5}{E_1^7} + 283200\frac{E_2^{18}E_5^{15}}{E_1^{32}}\left(R(q)^5 - \frac{q^2}{R(q)^5}\right) - 120\frac{E_2E_5^2E_{10}^5}{E_1^7}\Bigg(\frac{R(q)^2}{R(q^2)}\\ &- \frac{R(q^2)}{R(q)^2}\Bigg)\Bigg) + 113600q^3\frac{E_2^{18}E_5^{15}}{E_1^{32}}.\end{aligned}$$

With the aid of (2.2.1) – (2.2.4), the above reduces to

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_\omega(40n+12)q^n \\
&= -4\frac{E_2^2E_5}{E_1^2} + 40\frac{E_2^3E_5^{12}}{E_1^9E_{10}^5} - 80q\frac{E_2^2E_5^7}{E_1^8} - 280q^2\frac{E_2E_5^2E_{10}^5}{E_1^7} - 160q^3\frac{E_{10}^{10}}{E_1^6E_5^3} \\
&\quad + 1300q\frac{E_2^{10}E_5^5}{E_1^{14}} + 2000q^2\frac{E_2^9E_{10}^5}{E_1^{13}} + 14400q\frac{E_2^{20}E_5^{25}}{E_1^{34}E_{10}^{10}} + 398400q^2\frac{E_2^{19}E_5^{20}}{E_1^{33}E_{10}^5} \\
&\quad + 1736000q^3\frac{E_2^{18}E_5^{15}}{E_1^{32}} + 3648000q^4\frac{E_2^{17}E_5^{10}E_{10}^5}{E_1^{31}} + 7296000q^5\frac{E_2^{16}E_5^5E_{10}^{10}}{E_1^{30}} \\
&\quad + 3686400q^6\frac{E_2^{15}E_{10}^{15}}{E_1^{29}} + 3686400q^7\frac{E_2^{14}E_{10}^{20}}{E_1^{28}E_5^5}.
\end{aligned}$$

With the help of (2.2.7), the above can be shown to be equivalent to (3.1.2). This completes the proof. □

Proof of Corollary 3.1.5. Taking congruences modulo 5 on both sides of (3.1.2), we have

$$\sum_{n=0}^{\infty} p_\omega(40n+12)q^n \equiv \frac{E_2^2E_5}{E_1^2} \pmod 5. \tag{3.6.1}$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+2}, dividing both sides by q^2, and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} p_\omega(200n+92)q^n \equiv 3\frac{E_2^2E_5^2}{E_1^3} \equiv 3E_1^2E_2^2E_5 \pmod 5,$$

where the last congruence is by (2.4.2). Once again invoking (1.4.1) in the above, extracting the terms involving q^{5n+1}, dividing both sides by q, and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty} p_\omega(1000n+292)q^n \equiv -\frac{E_1^8E_{10}}{E_2^3E_5} \equiv -\frac{E_2^2E_5}{E_1^2} \pmod 5. \tag{3.6.2}$$

From (3.6.1) and (3.6.2), we see that

$$p_\omega(1000n+292) = p_\omega(40(25n+7)+12) \equiv -p_\omega(40n+12) \pmod 5.$$

Iterating the above congruence as shown by Xia [71], one can easily arrive at (3.1.3) to finish the proof.

□

3.7 Proofs of Theorem 3.1.6, (1.6.17) and (1.6.18)

Proof of Theorem 3.1.6. Employing (2.2.7) in (1.6.13), we find that

$$\sum_{n=0}^{\infty} \operatorname{spt}_{\omega}(2n+1)\, q^n = \frac{E_2^3 E_{10}}{E_5} + 5q\frac{E_1^2 E_{10}^5}{E_2 E_5^3} + 25q^2\frac{E_{10}^8}{E_1 E_5^4}.$$

Applying (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+1}, dividing both sides by q, replacing q^5 by q, and then proceeding as in the previous section, we obtain (3.1.4).

With the aid of (2.2.7), we can rewrite (3.1.4) as

$$\sum_{n=0}^{\infty} \operatorname{spt}_{\omega}(10n+3)\, q^n = 5\Bigg(E_1^2 E_5 + 6q\frac{E_2 E_{10}^3}{E_1} + 25q\frac{E_5^3 E_{10}^2}{E_2^2} + 250q^2\frac{E_5^2 E_{10}^5}{E_1^3 E_2} + 625q^3\frac{E_5 E_{10}^8}{E_1^6}\Bigg). \tag{3.7.1}$$

Now, let $[q^{5n+r}]\{F(q)\}$, $r = 0, 1, \ldots, 4$ denote the terms after extracting the terms involving q^{5n+r}, dividing by q^r and then replacing q^5 by q.

With the aid of (1.4.1)–(1.4.3) and Lemmas 2.2.1 and 2.2.2, omitting details, we find that

$$[q^{5n+2}]\left\{E_1^2 E_5 + 6q\frac{E_2 E_{10}^3}{E_1}\right\} = 5\left(E_1 E_5^2 + 6q\frac{E_2 E_5 E_{10}^3}{E_1^2}\right),$$

$$[q^{5n+2}]\left\{25q\frac{E_5^3 E_{10}^2}{E_2^2}\right\} = 125\left(2q\frac{E_1^3 E_{10}^4}{E_2^4} + 25q^3\frac{E_1^3 E_{10}^{10}}{E_2^{10}}\right),$$

$$[q^{5n+2}]\left\{250q^2\frac{E_5^2 E_{10}^5}{E_1^3 E_2}\right\}$$

$$= 250\Bigg(\frac{E_1^{16} E_{10}^3}{E_2^{15} E_5} + 150q\frac{E_1^{13} E_{10}^6}{E_2^{14} E_5^2} + 5650q^2\frac{E_1^{10} E_{10}^9}{E_2^{13} E_5^3} + 101825q^3\frac{E_1^7 E_{10}^{12}}{E_2^{12} E_5^4} + 1068125q^4\frac{E_1^4 E_{10}^{15}}{E_2^{11} E_5^5} + 7042500q^5\frac{E_1 E_{10}^{18}}{E_2^{10} E_5^6} + 29800000q^6\frac{E_{10}^{21}}{E_1^2 E_2^9 E_5^7} + 79000000q^7\frac{E_{10}^{24}}{E_1^5 E_2^8 E_5^8} + 120000000q^8\frac{E_{10}^{27}}{E_1^8 E_2^7 E_5^9} + 80000000q^9\frac{E_{10}^{30}}{E_1^{11} E_2^6 E_5^{10}}\Bigg)$$

and

$$[q^{5n+2}]\left\{625q^3\frac{E_5 E_{10}^8}{E_1^6}\right\}$$

$$= 3125q\left(63\frac{E_2^8E_5^6}{E_1^{11}} + 6500q\frac{E_2^8E_5^{12}}{E_1^{17}} + 196875q^2\frac{E_2^8E_5^{18}}{E_1^{23}} + 2343750q^3\frac{E_2^8E_5^{24}}{E_1^{29}}\right.$$
$$\left. + 9765625q^4\frac{E_2^8E_5^{30}}{E_1^{35}}\right).$$

Invoking the above in (3.7.1), we obtain (3.1.5), as desired. □

Proof of (1.6.17). Taking congruences modulo 125 on both sides of (3.1.5), we have

$$\sum_{n=0}^{\infty} \mathrm{spt}_{\omega}\,(50n+23)\,q^n \equiv 25\left(E_1E_5^2 + q\frac{E_2E_5E_{10}^3}{E_1^2}\right) \pmod{125}.$$

Again employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+1}, dividing both sides by q, replacing q^5 by q and then proceeding as in the earlier sections, we find that

$$\sum_{n=0}^{\infty} \mathrm{spt}_{\omega}\,(250n+73)\,q^n \equiv 25\left(-E_1^2E_5 + \frac{E_2^5E_5^2}{E_1^3E_{10}} + 20q\frac{E_2^6E_5E_{10}^2}{E_1^6} + 80q^2\frac{E_2^7E_{10}^5}{E_1^9}\right)$$
$$\equiv 25\left(-E_1^2E_5 + \frac{E_2^5E_5^2}{E_1^3E_{10}}\right) \pmod{125}.$$

By (2.2.7), the above becomes

$$\sum_{n=0}^{\infty} \mathrm{spt}_{\omega}\,(250n+73)\,q^n \equiv 125q\frac{E_2E_{10}^3}{E_1} \pmod{125},$$

from which (1.6.17) is apparent. □

Proof of (1.6.18). Taking congruences modulo 625 on both sides of (3.1.5), we have

$$\sum_{n=0}^{\infty} \mathrm{spt}_{\omega}\,(50n+23)\,q^n \equiv 25\left(E_1E_5^2 + 6q\frac{E_2E_5E_{10}^3}{E_1^2}\right) \pmod{625}.$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+1}, dividing both sides by q, replacing q^5 by q and then proceeding as in the earlier sections, we obtain

$$\sum_{n=0}^{\infty} \mathrm{spt}_{\omega}\,(250n+73)\,q^n \equiv 25\left(- E_1^2E_5 + 6\frac{E_2^5E_5^2}{E_1^3E_{10}} + 120q\frac{E_2^6E_5E_{10}^2}{E_1^6}\right.$$

$$+ 480q^2\frac{E_2^7E_{10}^5}{E_1^9}\Bigg) \pmod{625}.$$

Employing (2.2.7) and (2.4.2) in the above, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{spt}_{\omega}\,(250n+73)\,q^n &\equiv 25\Bigg(-E_1^2E_5+6\left(E_1^2E_5+5q\frac{E_2E_{10}^3}{E_1}\right)\\ &\quad+120q\left(\frac{E_2E_{10}^3}{E_1}+5q\frac{E_2^2E_{10}^6}{E_1^4E_5}\right)\\ &\quad+480q^2\left(\frac{E_2^2E_{10}^6}{E_1^4E_5}+5q\frac{E_2^3E_{10}^9}{E_1^7E_5^2}\right)\Bigg)\\ &\equiv 125\left(E_1^2E_5+216q^2\frac{E_2^2E_{10}^6}{E_1^4E_5}\right)\\ &\equiv 125\left(E_1^2E_5+q^2\frac{E_1E_2^2E_{10}^6}{E_5^2}\right) \pmod{625}.\end{aligned}$$

Again employing (1.4.1) in the above, extracting the terms involving q^{5n+2}, dividing both sides by q^2, replacing q^5 by q, and then proceeding as before, we find that

$$\sum_{n=0}^{\infty}\mathrm{spt}_{\omega}\,(1250n+573)\,q^n \equiv 125\left(-E_1E_5^2+\frac{E_2^{10}E_5^3}{E_1^4E_{10}^2}\right) \pmod{625}. \tag{3.7.2}$$

Since by (2.4.2),

$$-E_1E_5^2+\frac{E_2^{10}E_5^3}{E_1^4E_{10}^2} \equiv 0 \pmod{5},$$

from (3.7.2), we readily arrive at (1.6.18) to finish the proof. □

3.8 Proofs of Theorem 3.1.7 and Corollary 3.1.8

Proof of Theorem 3.1.7. Employing (2.2.5) – (2.2.7) in (1.6.14), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\mathrm{spt}}_{\omega}\,(2n+1)\,q^n &= \frac{E_2^4E_{10}}{E_1E_5}+5q\frac{E_2^5E_{10}^4}{E_1^4E_5^2}\\ &= E_2E_5^2+q\frac{E_1E_{10}^5}{E_5^3}+5q\left(\frac{E_1E_{10}^5}{E_5^3}+5q\frac{E_2E_{10}^8}{E_1^2E_5^4}\right)\\ &= E_2E_5^2+6q\frac{E_1E_{10}^5}{E_5^3}+25q^2\left(\frac{E_1^2E_{10}^{10}}{E_2E_5^8}+4q\frac{E_{10}^{13}}{E_1E_5^9}\right)\end{aligned}$$

$$= E_2E_5^2 + 6q\frac{E_1E_{10}^5}{E_5^3} + 25q^2\frac{E_1^2E_{10}^{10}}{E_2E_5^8} + 100q^3\frac{E_{10}^{13}}{E_1E_5^9}.$$

Applying (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+2}, dividing both sides by q^2, replacing q^5 by q and then using (2.3.4) – (2.3.12), we find that

$$\sum_{n=0}^{\infty}\overline{\mathrm{spt}}_{\omega}\,(10n+5)\,q^n = -E_1^2E_{10} - 6\frac{E_2^5E_5}{E_1^3} + 25\frac{E_2^{10}E_5^2}{E_1^8E_{10}} + 500q\frac{E_2^{13}E_5^5}{E_1^{15}}.$$

We now complete the proof of Theorem 3.1.7 by transforming the above into (3.1.6) with the aid of (2.2.5) and (2.2.7) as given below:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{\mathrm{spt}}_{\omega}\,(10n+5)\,q^n \\
&= -E_1^2E_{10} - 6\left(E_1^2E_{10} + 5q\frac{E_2E_{10}^4}{E_1E_5}\right) + 25\left(\frac{E_2^5E_5}{E_1^3} + 5q\frac{E_2^6E_{10}^3}{E_1^6}\right) \\
&\quad + 500q\left(\frac{E_2^8E_5^4E_{10}}{E_1^{10}} + 5q\frac{E_2^9E_5^3E_{10}^4}{E_1^{13}}\right) \\
&= -7E_1^2E_{10} - 30q\frac{E_2E_{10}^4}{E_1E_5} + 25\left(E_1^2E_{10} + 5q\frac{E_2E_{10}^4}{E_1E_5}\right) + 125q\left(\frac{E_2E_{10}^4}{E_1E_5} + 5q\frac{E_2^2E_{10}^7}{E_1^4E_5^2}\right) \\
&\quad + 500q\left(\frac{E_2^3E_5^3E_{10}^2}{E_1^5} + 5q\frac{E_2^4E_5^2E_{10}^5}{E_1^8}\right) + 2500q^2\left(\frac{E_2^4E_5^2E_{10}^5}{E_1^8} + 5q\frac{E_2^5E_5E_{10}^8}{E_1^{11}}\right) \\
&= 18E_1^2E_{10} + 220q\frac{E_2E_{10}^4}{E_1E_5} + 625q^2\frac{E_2^2E_{10}^7}{E_1^4E_5^2} + 500q\frac{E_2^3E_5^3E_{10}^2}{E_1^5} + 5000q^2\frac{E_2^4E_5^2E_{10}^5}{E_1^8} \\
&\quad + 12500q^3\frac{E_2^5E_5E_{10}^8}{E_1^{11}} \\
&= 18E_1^2E_{10} + 220q\frac{E_2E_{10}^4}{E_1E_5} + 625q^2\frac{E_2^2E_{10}^7}{E_1^4E_5^2} + 500q\left(\frac{E_2E_{10}^4}{E_1E_5} + 4q\frac{E_2^2E_{10}^7}{E_1^4E_5^2}\right) \\
&\quad + 5000q^2\left(\frac{E_2^2E_{10}^7}{E_1^4E_5^2} + 4q\frac{E_2^3E_{10}^{10}}{E_1^7E_5^3}\right) + 12500q^3\left(\frac{E_2^3E_{10}^{10}}{E_1^7E_5^3} + 4q\frac{E_2^4E_{10}^{13}}{E_1^{10E_5^4}}\right) \\
&= 18E_1^2E_{10} + 720q\frac{E_2E_{10}^4}{E_1E_5} + 7625q^2\frac{E_2^2E_{10}^7}{E_1^4E_5^2} + 32500q^3\frac{E_2^3E_{10}^{10}}{E_1^7E_5^3} + 50000q^4\frac{E_2^4E_{10}^{13}}{E_1^{10}E_5^4}.
\end{aligned}$$

□

Proof of Corollary 3.1.8. Taking congruences modulo 5 on both sides of (3.1.6) and then employing (2.4.2), we have

$$\sum_{n=0}^{\infty}\overline{\mathrm{spt}}_{\omega}\,(10n+5)\,q^n \equiv 3E_1^2E_{10} \pmod 5. \tag{3.8.1}$$

Using (1.4.1) in the above, extracting the terms involving q^{5n+2}, dividing by q^2, replacing q^5 by q, we see that

$$\sum_{n=0}^{\infty} \overline{\text{spt}}_{\omega}(50n+25)\, q^n \equiv 2E_2E_5^2 \pmod 5.$$

Once again using (1.4.1) in the above, extracting the terms involving q^{5n+1}, q^{5n+3}, and q^{5n+2}, in turn, we find that

$$\overline{\text{spt}}_{\omega}(10(25n+7)+5) \equiv 0 \pmod 5, \tag{3.8.2}$$

$$\overline{\text{spt}}_{\omega}(10(25n+17)+5) \equiv 0 \pmod 5, \tag{3.8.3}$$

and

$$\sum_{n=0}^{\infty} \overline{\text{spt}}_{\omega}(250n+125)\, q^n \equiv 3E_1^2E_{10} \pmod 5. \tag{3.8.4}$$

From (3.8.1) and (3.8.4), we have

$$\overline{\text{spt}}_{\omega}(250n+125) \equiv \overline{\text{spt}}_{\omega}(10n+5) \pmod 5.$$

Iterating the above, we easily deduce (3.1.7).

Employing (3.8.2) and (3.8.3) in (3.1.7), we immediately arrive at (3.1.8) and (3.1.9). Thus, we complete the proof.

□

3.9 Proofs of Theorem 3.1.10 and Corollary 3.1.11

Proof of (3.1.11). From (3.1.10) and (2.2.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{b}(2n)q^n &= \frac{E_2^5}{E_1^4} \\ &= \frac{E_1E_{10}}{E_5} + 5q\frac{E_2E_{10}^4}{E_1^2E_5^2}.\end{aligned} \tag{3.9.1}$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+1}, dividing both sides of the resulting identity by q and then replacing q^5 by q, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{b}(10n+2)q^n \\
&= -\frac{E_2E_5}{E_1} + 5\frac{E_2^4E_5^{10}E_{10}}{E_1^{14}}\bigg(\Big(R(q)^8R(q^2) - \frac{q^4}{R(q)^8R(q^2)}\Big) + q\Big(-10\Big(R(q)^5 - \frac{q^2}{R(q)^5}\Big) \\
&\quad - \Big(\frac{R(q)^7}{R(q^2)} + q^2\frac{R(q^2)}{R(q)^7}\Big) + 16\Big(R(q)^3R(q^2) + \frac{q^2}{R(q)^3R(q^2)}\Big)\Big) \\
&\quad - q^2\Big(15 + 27\Big(\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2}\Big)\Big)\bigg).
\end{aligned}$$

Employing (2.3.5), (2.3.7), (2.3.8), (2.3.10) and (2.3.11) in the above, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{b}(10n+2)q^n \\
&= -\frac{E_2E_5}{E_1} + 5\frac{E_2^6E_5^{20}}{E_1^{16}E_{10}^9} + 50q\frac{E_2^5E_5^{15}}{E_1^{15}E_{10}^4} - 75q^2\frac{E_2^4E_5^{10}E_{10}}{E_1^{14}} - 600q^3\frac{E_2^3E_5^5E_{10}^6}{E_1^{13}} \\
&\quad - 800q^4\frac{E_2^2E_{10}^{11}}{E_1^{12}} - 320q^5\frac{E_2E_{10}^{16}}{E_1^{11}E_5^5} \\
&= -\frac{E_2E_5}{E_1} + 5\left(\frac{E_5^5}{E_1^4E_{10}^3} - 4q\frac{E_{10}^2}{E_1^3E_2}\right)\left(\frac{E_5^5}{E_1E_{10}^3} + q\frac{E_{10}^2}{E_2}\right)\left(\frac{E_2^6E_5^{10}}{E_1^{11}E_{10}^3}\right. \\
&\quad \left. + 13q\frac{E_2^5E_5^5E_{10}^2}{E_1^{10}} + 28q^2\frac{E_2^4E_{10}^7}{E_1^9} + 16q^3\frac{E_2^3E_{10}^{12}}{E_1^8E_5^5}\right)
\end{aligned}$$

Using (2.2.5) and (2.2.6), the above can be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{b}(10n+2)q^n \\
&= -\frac{E_2E_5}{E_1} + 5\left(\frac{E_2^7E_5^{13}}{E_1^{13}E_{10}^6} + 13q\frac{E_2^6E_5^8}{E_1^{12}E_{10}} + 28q^2\frac{E_2^5E_5^3E_{10}^4}{E_1^{11}} + 16q^3\frac{E_2^4E_{10}^9}{E_1^{10}E_5^2}\right)
\end{aligned}$$

Using (2.2.7) successively in the above, we arrive at (3.1.11).

Proofs of (3.1.12) *and* (3.1.13). From (3.9.1), we have

$$\sum_{n=0}^{\infty}\mathfrak{b}(2n)q^n \equiv E_1\frac{E_{10}}{E_5} \pmod 5.$$

Clearly, using (1.4.1) in the above, we arrive at the congruences (3.1.12).

Now, taking congruences modulo 25 on both sides of (3.1.11), we have

$$\sum_{n=0}^{\infty} \mathfrak{b}(10n+2)q^n \equiv 4\frac{E_2E_5}{E_1} \pmod{25}.$$

Employing (1.4.1) and (1.4.2) in the above, extracting the terms involving q^{5n+1}, dividing both sides by q, and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \mathfrak{b}(50n+12)q^n \equiv 4\frac{E_2^2E_5^3}{E_1^3E_{10}} \pmod{25}.$$

We use (1.4.1) and (1.4.2) again in the above, extract the terms involving q^{5n+1}, divide both sides by q, and then replace q^5 by q, to get

$$\sum_{n=0}^{\infty} \mathfrak{b}(250n+62)q^n \equiv 12\frac{E_2E_5}{E_1} \equiv 3\sum_{n=0}^{\infty} \mathfrak{b}(10n+2)q^n \pmod{25}.$$

Thus, we have

$$\mathfrak{b}(250n+62) \equiv 3\mathfrak{b}(10n+2) \pmod{25}.$$

Iterating the above congruence, we arrive at (3.1.13).