Chapter 4

Sums of squares and sums of triangular numbers-I

4.1 Introduction

As mentioned in the introductory chapter, here we prove the following seven conjectures of Sun [34].

Theorem 4.1.1. (Conjecture 2.9 in Sun [34]) *Let* $n \in \mathbb{N}^+$ *with* $n \equiv 0, 3, 5, 6, 7 \pmod{11}$. *Then*

$$48T(1,1,4,11;n) = N(1,1,4,11;8n+17). \tag{4.1.1}$$

Theorem 4.1.2. (Conjecture 2.10 in Sun [34]) *Let* $n \in \mathbb{N}^+$ *with* $n \equiv 0, 1, 2, 4, 7 \pmod{11}$. *Then*

$$48T(1,1,2,22;n) = N(1,1,2,22;8n+26). \tag{4.1.2}$$

Theorem 4.1.3. (Conjecture 2.12 in Sun [34]) *Let* $n \in \mathbb{N}^+$ *with* $n \equiv 1 \pmod{4}$. *Then*

$$32T(3,5,20,32;n) = N(3,5,20,32;8n+60) - 4N(3,5,20,32;2n+15). \tag{4.1.3}$$

Theorem 4.1.4. (Conjecture 2.13 in Sun [34]) *Let* $n \in \mathbb{N}^+$ *with* $n \equiv 1 \pmod 4$. *Then*

$$24T(1,6,15,18;n) = N(1,6,15,18;8n+40) - 3N(1,6,15,18;2n+10). \quad (4.1.4)$$

Theorem 4.1.5. (Conjecture 2.16 in Sun [34]) *Let* $n \in \mathbb{N}^+$ *with* $n \equiv 0 \pmod 4$. *Then*

$$24T(1,7,10,30;n) = N(1,7,10,30;8n+48) - 3N(1,7,10,30;2n+12). \quad (4.1.5)$$

Theorem 4.1.6. (Conjecture 2.17 in Sun [34]) *Let* $n \in \mathbb{N}^+$ *with* $n \equiv 3 \pmod 4$. *Then*

$$24T(1,10,15,30;n) = N(1,10,15,30;8n+56) - 3N(1,10,15,30;2n+14). \quad (4.1.6)$$

Theorem 4.1.7. (Conjecture 2.5 in Sun [34]) *Let* $n \in \mathbb{N}^+$ *with* $n \equiv 0,2 \pmod 8$. *Then*

$$4T(1,2,4,17;n) = N(1,2,4,17;n+3). \quad (4.1.7)$$

We employ elementary dissections of Ramanujan's theta functions to prove the first six theorems. We could not effectively use that method to prove the last theorem. So we prove that theorem by employing some elementary techniques. We would like to thank Mingyu Kim and Byeong Kweon Oh for their contribution towards the proof this theorem.

In Section 4.2, we state some preliminary identities. In the remaining sections, we prove Theorems 4.1.1–4.1.7.

4.2 Preliminary lemmas

In the following lemma, we record some well-known 2-dissections, 3-dissections and identities from Berndt's book [21, pp. 39, 40, 49, 114, and 115]. The Identities

(4.2.1), (4.2.2), (4.2.5)–(4.2.7) can also be found in a recent book by Hirschhorn [27, Eqs. (1.9.4), (1.10.1), (1.7.1), (10.7.3), and (10.7.6)], which contains many other interesting results.

Lemma 4.2.1. *We have*

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{4.2.1}$$

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2, \tag{4.2.2}$$

$$\phi(q)\psi(q^2) = \psi(q)^2, \tag{4.2.3}$$

$$\psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}), \tag{4.2.4}$$

$$f(-q)^3 = \phi(-q)^2\psi(q) = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}, \tag{4.2.5}$$

$$\phi(-q)^2 f(-q) = \sum_{k=-\infty}^{\infty}(6k+1)q^{k(3k+1)/2}, \tag{4.2.6}$$

and

$$\psi(q^2)f(-q)^2 = \sum_{k=-\infty}^{\infty}(3k+1)q^{k(3k+2)}. \tag{4.2.7}$$

Some useful 2-dissections are given in the following lemma.

Lemma 4.2.2. *We have*

$$\phi(q)\phi(q^3) = \phi(q^4)\phi(q^{12}) + 2q\psi(q^2)\psi(q^6) + 4q^4\psi(q^8)\psi(q^{24}), \tag{4.2.8}$$

$$\psi(q)\psi(q^3) = \psi(q^4)\phi(q^6) + q\phi(q^2)\psi(q^{12}), \tag{4.2.9}$$

$$\psi(q)\psi(q^7) = \psi(q^8)\phi(q^{28}) + q\psi(q^2)\psi(q^{14}) + q^6\phi(q^4)\psi(q^{56}), \tag{4.2.10}$$

$$\psi(q^3)\psi(q^5) = \psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\phi(q^4)\psi(q^{120}), \tag{4.2.11}$$

$$\phi(q^3)\phi(q^5) = \phi(-q^2)\phi(-q^{30}) + 2q^2\psi(q)\psi(q^{15}), \tag{4.2.12}$$

$$\begin{aligned}\phi(q)\phi(q^{15}) = {} & \phi(-q^6)\phi(-q^{10}) + 2q\psi(q^8)\phi(q^{60}) + 2q^4\psi(q^2)\psi(q^{30}) \\ & + 2q^{15}\phi(q^4)\psi(q^{120}).\end{aligned} \tag{4.2.13}$$

Proof. Identity (4.2.8) follows by setting $(\mu,\nu)=(2,1)$ in [21, p. 68, eq. (36.2)], and then employing (4.2.1), identities (4.2.9), (4.2.10) and (4.2.11) follow from (36.8) of [21, p. 69] by setting $(\mu,\nu)=(2,1)$, $(\mu,\nu)=(4,3)$ and $(\mu,\nu)=(4,1)$, respectively. Identity (5.2.11) is in [21, p. 377, Entry 9(ii)]. Finally, identity (4.2.13) follows from [21, p. 377, Entry 9(ii)] and (4.2.11). □

Lemma 4.2.3. *We have*

$$\phi(q^{22})\psi(q^4)+q^5\phi(q^2)\psi(q^{44})=f(-q)f(-q^{11})+q\psi(q)\psi(q^{11}). \tag{4.2.14}$$

Proof. With the aid of (4.2.1) and the identity (see Berndt's book [21, p. 365, eq. (7.5)])

$$\phi(q)\phi(q^{11})-\phi(-q)\phi(-q^{11})=4qf(-q^2)f(-q^{22})+4q^3\psi(q^2)\psi(q^{22}),$$

we have

$$\begin{aligned}\phi(q^{44})\psi(q^8)+q^{10}\phi(q^4)\psi(q^{88}) &= \left(\frac{\phi(q^{11})+\phi(-q^{11})}{2}\right)\left(\frac{\phi(q)-\phi(-q)}{4q}\right)\\ &\quad+q^{10}\left(\frac{\phi(q)+\phi(-q)}{4q}\right)\left(\frac{\phi(q^{11})-\phi(-q^{11})}{2}\right)\\ &=\frac{1}{4q}\left(\phi(q)\phi(q^{11})-\phi(-q)\phi(-q^{11})\right)\\ &=\frac{1}{4q}\left(4qf(-q^2)f(-q^{22})+4q^3\psi(q^2)\psi(q^{22})\right)\\ &=f(-q^2)f(-q^{22})+q^2\psi(q^2)\psi(q^{22}).\end{aligned}$$

Replacing q^2 by q we arrive at (4.2.14). □

Lemma 4.2.4. *We have*

$$\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20})-f(q,q^7)f(q^{45},q^{75})-q^7f(q^3,q^5)f(q^{15},q^{105})=0. \tag{4.2.15}$$

Proof. With the help of (4.2.4), we find that

$$f(q^2,q^{14})f(q^{90},q^{150})+q^{14}f(q^6,q^{10})f(q^{30},q^{210})$$

$$= \frac{1}{4q}\left(\psi(q)-\psi(-q)\right)\left(\psi(q^{15})+\psi(-q^{15})\right)+\frac{1}{2q}\left(\psi(q)+\psi(-q)\right)\left(\psi(q^{15})-\psi(-q^{15})\right)$$

$$= \frac{1}{2q}\left(\psi(q)\psi(q^{15})-\psi(-q)\psi(-q^{15})\right). \tag{4.2.16}$$

On the other hand, with the aid of (4.2.1) and (5.2.11), we find that

$$\phi(q^{20})\psi(q^{24})+q^2\phi(q^{12})\psi(q^{40})$$

$$= \frac{1}{8q^3}\left(\phi(q^5)+\phi(-q^5)\right)\left(\phi(q^3)-\phi(-q^3)\right)+\frac{1}{8q^3}\left(\phi(q^3)+\phi(-q^3)\right)\left(\phi(q^5)-\phi(-q^5)\right)$$

$$= \frac{1}{4q^3}\left(\phi(q^3)\phi(q^5)-\phi(-q^3)\phi(-q^5)\right)$$

$$= \frac{1}{2q}\left(\psi(q)\psi(q^{15})-\psi(-q)\psi(-q^{15})\right). \tag{4.2.17}$$

From (4.2.16) and (4.2.17), we arrive at

$$f(q^2,q^{14})f(q^{90},q^{150})+q^{14}f(q^6,q^{10})f(q^{30},q^{210}) = \phi(q^{20})\psi(q^{24})+q^2\phi(q^{12})\psi(q^{40}),$$

which is clearly equivalent to (4.2.15) with q^2 replaced by q. □

Lemma 4.2.5.

$$\psi(q)\psi(q^{15}) = \psi(q^6)\psi(q^{10})+q\phi(q^{20})\psi(q^{24})+q^3\phi(q^{12})\psi(q^{40}). \tag{4.2.18}$$

Proof. Identity (4.2.18) easily follows from (4.2.17) and the identity

$$\psi(q)\psi(q^{15})+\psi(-q)\psi(-q^{15}) = 2\psi(q^6)\psi(q^{10}),$$

in [21, p. 377, Entry 9]. □

4.3 Proof of Theorem 4.1.1

We have

$$\sum_{n=0}^{\infty} N(1,1,4,11;n)q^n = \phi(q)^2\phi(q^4)\phi(q^{11})$$

$$= \phi(q^4)\left(\phi(q^2)^2+4q\psi(q^4)^2\right)\left(\phi(q^{44})+2q^{11}\psi(q^{88})\right),$$

from which we extract

$$\sum_{n=0}^{\infty} N(1,1,4,11;2n+1)q^n = 4\phi(q^2)\phi(q^{22})\psi(q^2)^2 + 2q^5\phi(q^2)\phi(q)^2\psi(q^{44})$$
$$= 4\phi(q^2)\phi(q^{22})\psi(q^2)^2 + 2q^5\phi(q^2)\psi(q^{44})\left(\phi(q^2)^2 + 4q\psi(q^4)^2\right),$$

from which we further extract

$$\sum_{n=0}^{\infty} N(1,1,4,11;4n+1)q^n = 4\psi(q)^2\phi(q)\phi(q^{11}) + 8q^3\psi(q^2)^2\phi(q)\psi(q^{22})$$
$$= 4\phi(q)^2\phi(q^{11})\psi(q^2) + 8q^3\phi(q)\psi(q^2)^2\psi(q^{22})$$
$$= 4\left(\phi(q^2)^2 + 4q\psi(q^4)^2\right)\left(\phi(q^{44}) + 2q^{11}\psi(q^{88})\right)\psi(q^2)$$
$$+ 8q^3\left(\phi(q^4) + 2q\psi(q^8)\right)\psi(q^2)^2\psi(q^{22}),$$

where the second equality is due (4.2.3).

It follows that

$$\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n = 4\psi(q)\phi(q)^2\phi(q^{22}) + 32q^6\psi(q)\psi(q^2)^2\psi(q^{44})$$
$$+ 16q^2\psi(q)^2\psi(q^4)\psi(q^{11}),$$

which becomes

$$\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n - 16\sum_{n=0}^{\infty} T(1,1,4,11;n)q^{n+2}$$
$$= 4\phi(q)^2\phi(q^{22})\psi(q) + 32q^6\psi(q)\psi(q^2)^2\psi(q^{44})$$
$$= 4\left(\phi(-q)^2 + 8q\psi(q^4)^2\right)\phi(q^{22})\psi(q) + 32q^6\psi(q)\psi(q^2)^2\psi(q^{44})$$
$$= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(\phi(q^{22})\psi(q)\psi(q^4)^2 + q^5\psi(q)\psi(q^2)^2\psi(q^{44})\right)$$
$$= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(\phi(q^{22})\psi(q^4)^2 + q^5\phi(q^2)\psi(q^4)\psi(q^{44}))\right)\psi(q)$$
$$= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(\phi(q^{22})\psi(q^4) + q^5\phi(q^2)\psi(q^{44}))\right)\psi(q)\psi(q^4)$$
$$= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(f(-q)f(-q^{11}) + q\psi(q)\psi(q^{11})\right)\psi(q)\psi(q^4)$$
$$= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\psi(q^4)\psi(q)f(-q)f(-q^{11}) + 32q^2\psi(q)^2\psi(q^4)\psi(q^{11}),$$

and so,

$$\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n - 48q^2\sum_{n=0}^{\infty} T(1,1,4,11;n)q^n$$

$$= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\psi(q^4)f(-q^2)^2f(-q^{11})$$
$$= 4\phi(q^{22})f(-q)^3 + 32qf(-q^{11})\psi(q^4)f(-q^2)^2$$
$$= 4\phi(q^{22})\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(n^2+n)/2} + 32f(-q^{11})\sum_{n=-\infty}^{\infty}(3n+1)q^{6n^2+4n+1},$$

where the last equality is due to (4.2.5) and (4.2.7).

Now, it can be easily verified that $(n^2+n)/2 \equiv 0, 1, 3, 4, 6,$ or $10 \pmod{11}$ and $6n^2+4n+1 \equiv 0, 1, 3, 4, 6,$ or $10 \pmod{11}$. Therefore, extracting the terms involving q^n for $n \equiv 2, 5, 7, 8, 9 \pmod{11}$ in the above, we find that

$$\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n - 48\sum_{n=0}^{\infty} T(1,1,4,11;n)q^{n+2} = 0,$$

which readily implies that, for $n \equiv 0, 3, 5, 6, 7 \pmod{11}$,

$$N(1,1,4,11;8n+17) = 48T(1,1,4,11;n).$$

This completes the proof.

4.4 Proof of Theorem 4.1.2

We have

$$\sum_{n=0}^{\infty} N(1,1,2,22;n)q^n = \phi(q)^2\phi(q^2)\phi(q^{22})$$
$$= \left(\phi(q^2)^2 + 4q\psi(q^4)^2\right)\phi(q^2)\phi(q^{22}),$$

from which we extract

$$\sum_{n=0}^{\infty} N(1,1,2,22;2n)q^n = \phi(q)^3\phi(q^{11})$$
$$= \left(\phi(q^4) + 2q\psi(q^8)\right)^3\left(\phi(q^{44}) + 2q^{11}\psi(q^{88})\right).$$

It follows that

$$\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n = 6\phi(q)^2\psi(q^2)\phi(q^{11}) + 24q^3\phi(q)\psi(q^2)^2\psi(q^{22})$$

$$= 6\phi(q)\psi(q)^2\phi(q^{11}) + 24q^3\psi(q)^2\psi(q^2)\psi(q^{22}),$$

which implies that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n - 24\sum_{n=0}^{\infty} T(1,1,2,22;n)q^{n+3} \\
&= 6\phi(q)\psi(q)^2\phi(q^{11}) \\
&= 6\psi(q)^2\phi(-q)\phi(-q^{11}) + 24q^3\psi(q)^2\psi(q^2)\psi(q^{22}) + 24q\psi(q)^2 f(-q^2)f(-q^{22}),
\end{aligned}$$

and so,

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n - 48q^3\sum_{n=0}^{\infty} T(1,1,2,22;n)q^n \\
&= 6\psi(q)^2\phi(-q)\phi(-q^{11}) + 24q\psi(q)^2 f(-q^2)f(-q^{22}) \\
&= 6f(-q^2)^3\phi(-q^{11}) + 24q\psi(q^2)f(q)^2 f(-q^{22}) \\
&= 6\phi(-q^{11})\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n^2+n} + 24f(-q^{22})\sum_{n=-\infty}^{\infty}(3n+1)(-q)^{3n^2+2n+1}.
\end{aligned}$$

Since $n^2+n \equiv 0,1,2,6,8,9 \pmod{11}$ and $3n^2+2n+1 \equiv 0,1,2,6,8,9 \pmod{11}$, extracting the terms involving q^n for $n \equiv 3,4,5,7,10 \pmod{11}$ in the above, we find that

$$\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n - 48\sum_{n=0}^{\infty} T(1,1,2,22;n)q^{n+3} = 0.$$

Thus, for $n \equiv 0,1,2,4,7 \pmod{11}$, we have

$$N(1,1,2,22;8n+26) = 48T(1,1,2,22;n).$$

4.5 Proof of Theorem 4.1.3

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} N(3,5,20,32;n)q^n &= \phi(q^3)\phi(q^5)\phi(q^{20})\phi(q^{32}) \\
&= \left(\phi(q^{12}) + 2q^3\psi(q^{24})\right)\left(\phi(q^{20}) + 2q^5\psi(q^{40})\right)\phi(q^{20})\phi(q^{32}).
\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty} N(3,5,20,32;4n)q^n = \left(\phi(q^3)\phi(q^5)+4q^2\psi(q^6)\psi(q^{10})\right)\phi(q^5)\phi(q^8)$$
$$= \Bigg(\left(\phi(q^{12})+2q^3\psi(q^{24})\right)\left(\phi(q^{20})+2q^5\psi(q^{40})\right)+4q^2\psi(q^6)\psi(q^{10})\Bigg)$$
$$\times \left(\phi(q^{20})+2q^5\psi(q^{40})\right)\phi(q^8),$$

from which we extract

$$\sum_{n=0}^{\infty} N(3,5,20,32;8n+4)q^n$$
$$= 2q\phi(q^4)\phi(q^{10})^2\psi(q^{12})+4q^2\phi(q^4)\phi(q^6)\phi(q^{10})\psi(q^{20})+8q^6\phi(q^4)\psi(q^{12})\psi(q^{20})^2$$
$$+8q^3\psi(q^3)\psi(q^5)\phi(q^4)\psi(q^{20})$$
$$= 2q\phi(q^4)\phi(q^{10})^2\psi(q^{12})+4q^2\phi(q^4)\phi(q^6)\phi(q^{10})\psi(q^{20})+8q^6\phi(q^4)\psi(q^{12})\psi(q^{20})^2$$
$$+8q^3\left(\psi(q^8)\phi(q^{60})+q^3\psi(q^2)\psi(q^{30})+q^{14}\phi(q^4)\psi(q^{120})\right)\phi(q^4)\psi(q^{20}).$$

We further extract

$$\sum_{n=0}^{\infty} N(3,5,20,32;16n+4)q^n$$
$$= 4q\phi(q^2)\phi(q^3)\phi(q^5)\psi(q^{10})+8q^3\phi(q^2)\psi(q^6)\psi(q^{10})^2+8q^3\phi(q^2)\psi(q^{10})\psi(q)\psi(q^{15})$$
$$= 4q\phi(q^2)\psi(q^{10})\left(\phi(q^{12})+2q^3\psi(q^{24})\right)\left(\phi(q^{20})+2q^5\psi(q^{40})\right)+8q^3\phi(q^2)\psi(q^6)\psi(q^{10})^2$$
$$+8q^3\phi(q^2)\psi(q^{10})\left(f(q^6,q^{10})+qf(q^2,q^{14})\right)\left(f(q^{90},q^{150})+q^{15}f(q^{30},q^{210})\right),$$

from which we have

$$\sum_{n=0}^{\infty} N(3,5,20,32;32n+4)q^n$$
$$= 8q^2\phi(q)\psi(q^5)\phi(q^{10})\psi(q^{12})+8q^3\phi(q)\psi(q^5)\phi(q^6)\psi(q^{20})$$
$$+8q^2\phi(q)\psi(q^5)f(q,q^7)f(q^{45},q^{75})+8q^9\phi(q)\psi(q^5)f(q^3,q^5)f(q^{15},q^{105}). \quad (4.5.1)$$

Next, we have

$$\sum_{n=0}^{\infty} N(3,5,20,32;4n+1)q^n = 2q\phi(q^3)\psi(q^{10})\phi(q^5)\phi(q^8)$$

$$= 2q\left(\phi(q^{12}) + 2q^3\psi(q^{24})\right)\left(\phi(q^{20}) + 2q^5\psi(q^{40})\right)\phi(q^8)\psi(q^{10}),$$

from which it follows that

$$\sum_{n=0}^{\infty} N(3,5,20,32;8n+1)q^n = 4q^2\phi(q^4)\psi(q^5)\psi(q^{12})\phi(q^{10}) + 4q^3\phi(q^4)\psi(q^5)\phi(q^6)\psi(q^{20}). \tag{4.5.2}$$

Again,

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(3,5,20,32;n)q^n \\
&= \psi(q^3)\psi(q^5)\psi(q^{20})\psi(q^{32}) \\
&= \left(\psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\phi(q^4)\psi(q^{120})\right)\psi(q^{20})\psi(q^{32}),
\end{aligned}$$

from which we extract

$$\begin{aligned}
\sum_{n=0}^{\infty} T(3,5,20,32;2n+1)q^n &= q\psi(q)\psi(q^{15})\psi(q^{10})\psi(q^{16}) \\
&= q\big(\phi(-q^6)\phi(-q^{10}) + 2q\psi(q^8)\phi(q^{60}) + 2q^4\psi(q^2)\psi(q^{30}) \\
&\quad + 2q^{15}\phi(q^4)\psi(q^{120})\big)\psi(q^{10})\psi(q^{16}),
\end{aligned}$$

from which we further extract

$$\begin{aligned}
\sum_{n=0}^{\infty} T(3,5,20,32;4n+1)q^n &= q\psi(q^5)\psi(q^8)f(q,q^7)f(q^{45},q^{75}) \\
&\quad + q^8\psi(q^5)\psi(q^8)f(q^3,q^5)f(q^{15},q^{105}).
\end{aligned} \tag{4.5.3}$$

From (4.5.1), (4.5.2) and (4.5.3), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(3,5,20,32;32n+4)q^n - 4\sum_{n=0}^{\infty} N(3,5,20,32;8n+1)q^n \\
&- 32\sum_{n=0}^{\infty} T(3,5,20,32;4n+1)q^{n+2} \\
&= 8q^2\phi(q)\psi(q^5)\phi(q^{10})\psi(q^{12}) + 8q^3\phi(q)\psi(q^5)\phi(q^6)\psi(q^{20}) \\
&\quad + 8q^2\phi(q)\psi(q^5)f(q,q^7)f(q^{45},q^{75}) + 8q^9\phi(q)\psi(q^5)f(q^3,q^5)f(q^{15},q^{105}) \\
&\quad - 16q^2\phi(q^4)\psi(q^5)\psi(q^{12})\phi(q^{10}) - 16q^3\phi(q^4)\psi(q^5)\phi(q^6)\psi(q^{20})
\end{aligned}$$

$$
\begin{aligned}
&- 32q^3\psi(q^5)\psi(q^8)f(q,q^7)f(q^{45},q^{75}) - 32q^{10}\psi(q^5)\psi(q^8)f(q^3,q^5)f(q^{15},q^{105}) \\
&= 8q^2\psi(q^5)\left(\phi(q) - 2\phi(q^4)\right)\left(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20})\right) \\
&\quad + 8q^2\psi(q^5)\left(\phi(q^4) - 2q\psi(q^8)\right)\left(f(q,q^7)f(q^{45},q^{75}) + q^7 f(q^3,q^5)f(q^{15},q^{105})\right) \\
&= -8q^2\psi(q^5)\phi(-q)\left(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20})\right) \\
&\quad + 8q^2\psi(q^5)\phi(-q)\left(f(q,q^7)f(q^{45},q^{75}) + q^7 f(q^3,q^5)f(q^{15},q^{105})\right) \\
&= -8q^2\psi(q^5)\phi(-q)\big(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20}) - f(q,q^7)f(q^{45},q^{75}) \\
&\quad - q^7 f(q^3,q^5)f(q^{15},q^{105})\big) \\
&= 0,
\end{aligned}
$$

where the last equality is due to (4.2.15).

It follows that

$$N(3,5,20,32;32n+68) - 4N(3,5,20,32;8n+17) = 32T(3,5,20,32;4n+1).$$

This completes the proof.

4.6 Proof of Theorem 4.1.4

We have

$$
\begin{aligned}
\sum_{n=0}^{\infty} N(1,6,15,18;n)q^n &= \phi(q)\cdot\phi(q^{15})\cdot\phi(q^6)\phi(q^{18}) \\
&= \left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^{60}) + 2q^{15}\psi(q^{120})\right) \\
&\quad \times \left(\phi(q^{24})\phi(q^{72}) + 2q^6\psi(q^{12})\psi(q^{36}) + 4q^{24}\psi(q^{48})\psi(q^{144})\right),
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,6,15,18;4n)q^n \\
&= \left(\left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^{60}) + 2q^{15}\psi(q^{120})\right) + 4q^4\psi(q^2)\psi(q^{30})\right) \\
&\quad \times \left(\phi(q^6)\phi(q^{18}) + 4q^6\psi(q^{12})\psi(q^{36})\right). \qquad (4.6.1)
\end{aligned}
$$

We extract from here that

$$\sum_{n=0}^{\infty} N(1,6,15,18;8n)q^n$$
$$= \left(\phi(q^2)\phi(q^{30}) + 4q^8\psi(q^4)\psi(q^{60}) + 4q^2\psi(q)\psi(q^{15})\right)\left(\phi(q^3)\phi(q^9) + 4q^3\psi(q^6)\psi(q^{18})\right)$$
$$= \Big(\phi(q^2)\phi(q^{30}) + 4q^8\psi(q^4)\psi(q^{60}) + 4q^2\big(\psi(q^6)\psi(q^{10}) + q\phi(q^{20})\psi(q^{24})$$
$$+ q^3\phi(q^{12})\psi(q^{40})\big)\Big)\left(\phi(q^{12})\phi(q^{36}) + 6q^3\psi(q^6)\psi(q^{18}) + 4q^{12}\psi(q^{24})\psi(q^{72})\right),$$

from which we extract

$$\sum_{n=0}^{\infty} N(1,6,15,18;16n)q^n$$
$$= \left(\phi(q^6)\phi(q^{18}) + 4q^6\psi(q^{12})\psi(q^{36})\right)\left(\phi(q)\phi(q^{15}) + 4q^4\psi(q^2)\psi(q^{30}) + 4q\psi(q^3)\psi(q^5)\right)$$
$$+ 24q^3\psi(q^3)\psi(q^9)\left(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20})\right)$$
$$= \left(\phi(q^6)\phi(q^{18}) + 4q^6\psi(q^{12})\psi(q^{36})\right)\Bigg(\left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^{60}) + 2q^{15}\psi(q^{120})\right)$$
$$+ 4q^4\psi(q^2)\psi(q^{30}) + 4q\left(\psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\phi(q^4)\psi(q^{120})\right)\Bigg)$$
$$+ 24q^3\left(\psi(q^{12})\phi(q^{18}) + q^3\phi(q^6)\psi(q^{36})\right)\left(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20})\right),$$

from which we further extract

$$\sum_{n=0}^{\infty} N(1,6,15,18;32n+16)q^n$$
$$= 6\phi(q^3)\phi(q^9)\psi(q^4)\psi(q^{30}) + 6q^7\phi(q^3)\phi(q^9)\phi(q^2)\psi(q^{60}) + 24q^3\psi(q^6)\psi(q^{18})\psi(q^4)\psi(q^{30})$$
$$+ 24q^{10}\psi(q^6)\psi(q^{18})\phi(q^2)\psi(q^{60}) + 24q\phi(q^5)\psi(q^6)^2\phi(q^9) + 24q^3\phi(q^3)^2\psi(q^{10})\psi(q^{18}). \tag{4.6.2}$$

From (4.6.1), we also extract

$$\sum_{n=0}^{\infty} N(1,6,15,18;8n+4)q^n$$
$$= 2\phi(q^3)\phi(q^9)\psi(q^4)\psi(q^{30}) + 2q^7\phi(q^3)\phi(q^9)\phi(q^2)\psi(q^{60})$$
$$+ 8q^3\psi(q^6)\psi(q^{18})\psi(q^4)\psi(q^{30}) + 8q^{10}\psi(q^6)\psi(q^{18})\phi(q^2)\psi(q^{60}). \tag{4.6.3}$$

Again,

$$\sum_{n=0}^{\infty} T(1,6,15,18;n)q^n$$

$$
\begin{aligned}
&= \psi(q)\psi(q^{15}) \cdot \psi(q^6)\psi(q^{18}) \\
&= \left(\psi(q^6)\psi(q^{10}) + q\phi(q^{20})\psi(q^{24}) + q^3\phi(q^{12})\psi(q^{40})\right)\psi(q^6)\psi(q^{18}),
\end{aligned}
$$

from which we extract

$$
\begin{aligned}
&\sum_{n=0}^{\infty} T(1,6,15,18;2n+1)q^n \\
&= \psi(q^3)\psi(q^9)\left(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20})\right) \\
&= \left(\psi(q^{12})\phi(q^{18}) + q^3\phi(q^6)\psi(q^{36})\right)\left(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20})\right),
\end{aligned}
$$

from which it follows that

$$
\sum_{n=0}^{\infty} T(1,6,15,18;4n+1)q^n = \psi(q^6)^2\phi(q^9)\phi(q^5) + q^2\phi(q^3)^2\psi(q^{18})\psi(q^{10}). \tag{4.6.4}
$$

From (4.6.2), (4.6.3) and (4.6.4), we have

$$
\begin{aligned}
&2\sum_{n=0}^{\infty} N(1,6,15,18;32n+16)q^n - 6\sum_{n=0}^{\infty} N(1,6,15,18;8n+4)q^n \\
&= 48\sum_{n=0}^{\infty} T(1,6,15,18;4n+1)q^{n+1},
\end{aligned}
$$

and hence,

$$
N(1,6,15,18;32n+48) - 3N(1,6,15,18;8n+12) = 24T(1,6,15,18;4n+1).
$$

Thus we complete the proof.

4.7 Proof of Theorem 4.1.5

We have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;n)q^n \\
&= \phi(q)\phi(q^7)\phi(q^{10})\phi(q^{30}) \\
&= \left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^{28}) + 2q^7\psi(q^{56})\right) \\
&\quad \times \left(\phi(q^{40})\phi(q^{120}) + 2q^{10}\psi(q^{20})\psi(q^{60}) + 4q^{40}\psi(q^{80})\psi(q^{240})\right),
\end{aligned}
$$

from which it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;4n)q^n \\
&= \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right)\left(\phi(q)\phi(q^7) + 4q^2\psi(q^2)\psi(q^{14})\right) \\
&= \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right) \\
&\quad \times \left(\left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^{28}) + 2q^7\psi(q^{56})\right) + 4q^2\psi(q^2)\psi(q^{14})\right), \qquad (4.7.1)
\end{aligned}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;8n)q^n \\
&= \left(\phi(q^5)\phi(q^{15}) + 4q^5\psi(q^{10})\psi(q^{30})\right)\left(\phi(q^2)\phi(q^{14}) + 4q^4\psi(q^4)\psi(q^{28}) + 4q\psi(q)\psi(q^7)\right) \\
&= \left(\phi(q^{20})\phi(q^{60}) + 6q^5\psi(q^{10})\psi(q^{30}) + 4q^{20}\psi(q^{40})\psi(q^{120})\right) \\
&\quad \times \left(\phi(q^2)\phi(q^{14}) + 4q^4\psi(q^4)\psi(q^{28}) + 4q\psi(q^8)\phi(q^{28}) + 4q^2\psi(q^2)\psi(q^{14})\right. \\
&\quad \left. + 4q^7\phi(q^4)\psi(q^{56})\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;16n)q^n \\
&= \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right)\left(\phi(q)\phi(q^7) + 4q^2\psi(q^2)\psi(q^{14}) + 4q\psi(q)\psi(q^7)\right) \\
&\quad + 6\psi(q^5)\psi(q^{15})\left(4q^3\psi(q^4)\phi(q^{14}) + 4q^6\phi(q^2)\psi(q^{28})\right) \\
&= \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right)\left(\phi(q^4)\phi(q^{28}) + 6q\psi(q^8)\phi(q^{28})\right. \\
&\quad \left. + 6q^7\phi(q^4)\psi(q^{56}) + 4q^8\psi(q^8)\psi(q^{56}) + 8q^2\psi(q^2)\psi(q^{14})\right) \\
&\quad + 24q^3\left(\psi(q^{20})\phi(q^{30}) + q^5\phi(q^{10})\psi(q^{60})\right)\left(\psi(q^4)\phi(q^{14}) + q^3\phi(q^2)\psi(q^{28})\right),
\end{aligned}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;32n+16)q^n \\
&= 6\phi(q^5)\psi(q^4)\phi(q^{14})\phi(q^{15}) + 6q^3\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{28}) \\
&\quad + 24q^5\psi(q^4)\psi(q^{10})\phi(q^{14})\psi(q^{30}) + 24q^8\phi(q^2)\psi(q^{10})\psi(q^{28})\psi(q^{30}) \\
&\quad + 24q\psi(q^2)\phi(q^7)\psi(q^{10})\phi(q^{15}) + 24q^5\phi(q)\phi(q^5)\psi(q^{14})\psi(q^{30}). \qquad (4.7.2)
\end{aligned}$$

From (4.7.1), we also extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;8n+4)q^n \\
&= 2\psi(q^4)\phi(q^{14})\phi(q^5)\phi(q^{15}) + 2q^3\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{28}) \\
&\quad + 8q^5\psi(q^4)\psi(q^{10})\phi(q^{14})\psi(q^{30}) + 8q^8\phi(q^2)\psi(q^{10})\psi(q^{28})\psi(q^{30}). \qquad (4.7.3)
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(1,7,10,30;n)q^n \\
&= \psi(q)\psi(q^7)\psi(q^{10})\psi(q^{30}) \\
&= \left(\psi(q^8)\phi(q^{28}) + q\psi(q^2)\psi(q^{14}) + q^6\phi(q^4)\psi(q^{56})\right)\left(\psi(q^4)\phi(q^6) + q\phi(q^2)\psi(q^{12})\right),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(1,7,10,30;4n)q^n \\
&= \psi(q^2)\phi(q^7)\psi(q^{10})\phi(q^{15}) + q^4\phi(q)\phi(q^5)\psi(q^{14})\psi(q^{30}). \qquad (4.7.4)
\end{aligned}$$

From (4.7.2), (4.7.3) and (4.7.4), we have

$$\begin{aligned}
&2\sum_{n=0}^{\infty} N(1,7,10,30;32n+16)q^n - 6\sum_{n=0}^{\infty} N(1,7,10,30;8n+4)q^n \\
&= 48\sum_{n=0}^{\infty} T(1,7,10,30;4n)q^{n+1},
\end{aligned}$$

which implies that

$$N(1,7,10,30;32n+48) - 3N(1,7,10,30;8n+12) = 24T(1,7,10,30;4n).$$

Thus we finish the proof.

4.8 Proof of Theorem 4.1.6

We have

$$\sum_{n=0}^{\infty} N(1,10,15,30;n)q^n$$

$$
\begin{aligned}
&= \phi(q)\phi(q^{15}) \cdot \phi(q^{10})\phi(q^{30}) \\
&= \left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^{60}) + 2q^{15}\psi(q^{120})\right) \\
&\quad \times \left(\phi(q^{40})\phi(q^{120}) + 2q^{10}\psi(q^{20})\psi(q^{60}) + 4q^{40}\psi(q^{80})\psi(q^{240})\right),
\end{aligned}
$$

from which we extract

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;4n)q^n \\
&= \left(\phi(q)\phi(q^{15}) + 4q^4\psi(q^2)\psi(q^{30})\right)\left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right) \\
&= \left(\left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^{60}) + 2q^{15}\psi(q^{120})\right) + 4q^4\psi(q^2)\psi(q^{30})\right) \\
&\quad \times \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right), \qquad (4.8.1)
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;8n)q^n \\
&= \left(\phi(q^2)\phi(q^{30}) + 4q^8\psi(q^4)\psi(q^{60}) + 4q^2\psi(q)\psi(q^{15})\right) \\
&\quad \times \left(\phi(q^5)\phi(q^{15}) + 4q^5\psi(q^{10})\psi(q^{30})\right) \\
&= \left(\phi(q^2)\phi(q^{30}) + 4q^8\psi(q^4)\psi(q^{60}) + 4q^2\psi(q^6)\psi(q^{10})\right. \\
&\quad \left. + 4q^3\phi(q^{20})\psi(q^{24}) + 4q^5\phi(q^{12})\psi(q^{40})\right) \\
&\quad \times \left(\phi(q^{20})\phi(q^{60}) + 6q^5\psi(q^{10})\psi(q^{30}) + 4q^{20}\psi(q^{40})\psi(q^{120})\right).
\end{aligned}
$$

From the above we extract

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;16n)q^n \\
&= \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right) \\
&\quad \times \left(\phi(q)\phi(q^{15}) + 4q^4\psi(q^2)\psi(q^{30}) + 4q\psi(q^3)\psi(q^5)\right) \\
&\quad + 6q^2\psi(q^5)\psi(q^{15})\left(4q^2\phi(q^{10})\psi(q^{12}) + 4q^3\phi(q^6)\psi(q^{20})\right) \\
&= \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right) \\
&\quad \times \left(\phi(-q^6)\phi(-q^{10}) + 6q\psi(q^8)\phi(q^{60}) + 6q^4\psi(q^2)\psi(q^{30}) + 6q^{15}\phi(q^4)\psi(q^{120})\right. \\
&\quad \left. + 4q^4\psi(q^2)\psi(q^{30})\right) + 6q^2\left(\psi(q^{20})\phi(q^{30}) + q^5\phi(q^{10})\psi(q^{60})\right)
\end{aligned}
$$

$$\times \left(4q^2\phi(q^{10})\psi(q^{12}) + 4q^3\phi(q^6)\psi(q^{20})\right),$$

from which it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;32n+16)q^n \\
&= 6\psi(q^4)\phi(q^5)\phi(q^{15})\phi(q^{30}) + 6q^7\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{60}) \\
&\quad + 24q^5\psi(q^4)\psi(q^{10})\phi(q^{30})\psi(q^{30}) + 24q^{12}\phi(q^2)\psi(q^{10})\psi(q^{30})\psi(q^{60}) \\
&\quad + 24q^2\phi(q^3)\psi(q^{10})^2\phi(q^{15}) + 24q^4\phi(q^5)^2\psi(q^6)\psi(q^{30}).
\end{aligned} \tag{4.8.2}$$

Now, from (4.8.1) we also extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;8n+4)q^n \\
&= \left(2\psi(q^4)\phi(q^{30}) + 2q^7\phi(q^2)\psi(q^{60})\right)\left(\phi(q^5)\phi(q^{15}) + 4q^5\psi(q^{10})\psi(q^{30})\right) \\
&= 2\psi(q^4)\phi(q^5)\phi(q^{15})\phi(q^{30}) + 2q^7\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{60}) \\
&\quad + 8q^5\psi(q^4)\psi(q^{10})\phi(q^{30})\psi(q^{30}) + 8q^{12}\phi(q^2)\psi(q^{10})\psi(q^{30})\psi(q^{60}).
\end{aligned} \tag{4.8.3}$$

On the other hand, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} T(1,10,15,30;n)q^n &= \psi(q)\psi(q^{15}) \cdot \psi(q^{10})\psi(q^{30}) \\
&= \left(\psi(q^6)\psi(q^{10}) + q\phi(q^{20})\psi(q^{24}) + q^3\phi(q^{12})\psi(q^{40})\right) \\
&\quad \times \left(\psi(q^{40})\phi(q^{60}) + q^{10}\phi(q^{20})\psi(q^{120})\right),
\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} T(1,10,15,30;4n+3)q^n = \phi(q^3)\psi(q^{10})^2\phi(q^{15}) + q^2\phi(q^5)^2\psi(q^6)\psi(q^{30}). \tag{4.8.4}$$

From (4.8.2), (4.8.3) and (4.8.4), we have

$$\begin{aligned}
&2\sum_{n=0}^{\infty} N(1,10,15,30;32n+16)q^n - 6\sum_{n=0}^{\infty} N(1,10,15,30;8n+4)q^n \\
&= 48\sum_{n=0}^{\infty} T(1,10,15,30;4n+3)q^{n+2},
\end{aligned}$$

which readily implies that

$$N(1,7,10,30;32n+80) - 3N(1,7,10,30;8n+20) = 24T(1,7,10,30;4n+3).$$

This completes the proof of Theorem 4.1.6.

4.9 Proof of Theorem 4.1.7

Our proof of Theorem 4.1.7 is quite different from the proofs of Theorems 4.1.1 – 4.1.6. In fact, the method is quite similar to that of [30], which considers the ternary case of Sun's conjectures in [35].

For a quaternary quadratic form $f(x, y, z, w)$ and a positive integer n, we define

$$R(f, n) = \{(x, y, z, w) \in \mathbb{Z}^4 : f(x, y, z, w) = n\} \quad \text{and} \quad r(f, n) = |R(f, n)|.$$

At first, we state and prove a proposition.

Proposition 4.9.1. *For any positive integer* $n \equiv 3, 5 \pmod 8$, *we have*

$$r(x^2 + 2y^2 + 4z^2 + 17w^2, n) = r(2x^2 + 3y^2 + 4z^2 + 8w^2 + 2xy + 2yz + 2yw, n).$$

Proof of Proposition 4.9.1. Let

$$f = f(x, y, z, w) = x^2 + 2y^2 + 4z^2 + 17w^2,$$
$$g = g(x, y, z, w) = 2x^2 + 3y^2 + 4z^2 + 8w^2 + 2xy + 2yz + 2yw.$$

First, we consider the case when n is a positive integer congruent to 3 (mod 8). Note that if $(x, y, z, w) \in R(f, n)$, then $x \not\equiv w \pmod 2$. Furthermore, one may easily show that if $(x, y, z, w) \in R(f, n)$ and $x \equiv 1 \pmod 2$, then $2x + 2y - 2z - 3w \equiv 0 \pmod 4$. Since $f(-x, y, z, w) = f(x, y, z, w)$, the map $\eta_1 : R(f, n) \to R(f, n)$ defined by

$$\eta_1(x, y, z, w) = (-x, y, z, w),$$

is a well defined bijective map. Hence we have

$$\begin{aligned} &|\{(x, y, z, w) \in R(f, n) : x \equiv 1 \pmod 2, 2x + 2y - 2z - 3w \equiv 0 \pmod 8\}| \\ &= |\{(x, y, z, w) \in R(f, n) : x \equiv 1 \pmod 2, 2x + 2y - 2z - 3w \equiv 4 \pmod 8\}|, \end{aligned}$$

which implies that

$$|\{(x, y, z, w) \in R(f, n) : x \equiv 1 \pmod 2\}|$$

$$= 2|\{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2, 2x+2y-2z-3w \equiv 0 \pmod 8\}|.$$

Note that if $(x,y,z,w) \in R(f,n)$ and $x \equiv 0 \pmod 2$, then $x+6y-6z+6w \equiv 0 \pmod 4$. Since $f(x,y,z,-w) = f(x,y,z,w)$, the map $\eta_2 : R(f,n) \to R(f,n)$ defined by

$$\eta_2(x,y,z,w) = (x,y,z,-w),$$

is a well defined bijective map. Hence we have

$$\begin{aligned}&|\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2, x+6y-6z+6w \equiv 0 \pmod 8\}|\\ &= |\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2, x+6y-6z+6w \equiv 4 \pmod 8\}|,\end{aligned}$$

which implies that

$$\begin{aligned}&|\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2\}|\\ &= 2|\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2, x+6y-6z+6w \equiv 0 \pmod 8\}|.\end{aligned}$$

Now, if we define

$$\begin{aligned}F_1 &= \{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2, 2x+2y-2z-3w \equiv 0 \pmod 8\},\\ F_2 &= \{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2, x+6y-6z+6w \equiv 0 \pmod{16}\},\\ F_3 &= \{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2, x+6y-6z+6w \equiv 8 \pmod{16}\},\end{aligned}$$

then we have

$$r(f,n) = 2(|F_1| + |F_2| + |F_3|).$$

Now, we analyze the set $R(g,n)$. First, we note that $y \equiv 1 \pmod 2$ for any $(x,y,z,w) \in R(g,n)$. Since $g(x+y,-y,-z,-w) = g(x,y,z,w)$, the map $\eta_3 : R(g,n) \to R(g,n)$ defined by

$$\eta_3(x,y,z,w) = (x+y,-y,-z,-w)$$

is a well defined bijective map. Therefore, we have

$$r(g,n) = 2|\{(x,y,z,w) \in R(g,n) : x \equiv 0 \pmod 2\}|.$$

One may easily check that for $(x,y,z,w)\in R(g,n)$, if $x\equiv 0 \pmod 2$, then $x-z+w\equiv 0 \pmod 4$. Furthermore, if $x-z+w\equiv 4 \pmod 8$, then $7x-4y+9z-w\equiv 0 \pmod 8$. Thus if we define

$$G_1=\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\quad x-z+w\equiv 0 \pmod 8\},$$
$$G_2=\left\{(x,y,z,w)\in R(g,n): \begin{array}{r} x\equiv 0 \pmod 2,\quad x-z+w\equiv 4 \pmod 8,\\ 7x-4y+9z-w\equiv 8 \pmod{16}\end{array}\right\},$$
$$G_3=\left\{(x,y,z,w)\in R(g,n): \begin{array}{r} x\equiv 0 \pmod 2,\quad x-z+w\equiv 4 \pmod 8,\\ 7x-4y+9z-w\equiv 0 \pmod{16}\end{array}\right\},$$

then the set $\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2\}$ is a disjoint union of G_1, G_2 and G_3. Hence we have

$$r(g,n)=2(|G_1|+|G_2|+|G_3|).$$

Now, for $j=1,2,3$, we define maps $\phi_j: G_j\to F_j$ by

$$\phi_1(x,y,z,w)=\frac{1}{8}\begin{pmatrix}4&8&-4&-12\\-2&-8&-6&-10\\-3&0&-5&5\\-2&0&2&-2\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix},$$
$$\phi_2(x,y,z,w)=\frac{1}{16}\begin{pmatrix}2&24&-2&18\\-10&-8&-6&22\\-3&-4&-13&-11\\-4&0&4&-4\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix},$$
$$\phi_3(x,y,z,w)=\frac{1}{16}\begin{pmatrix}6&-8&-6&-42\\2&-8&14&2\\7&12&9&-1\\4&0&-4&4\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}.$$

It is easy to check that all of them are well defined bijective maps. Therefore, we have

$$r(f,n)=2(|F_1|+|F_2|+|F_3|)=2(|G_1|+|G_2|+|G_3|)=r(g,n).$$

Next, we consider the case when n is a positive integer congruent to $5 \pmod 8$. Note that if $(x,y,z,w)\in R(f,n)$ and $x\equiv 1 \pmod 2$, then $2x+2y-2z+5w\equiv$

$0 \pmod 4$. Since $f(-x,y,z,w)=f(x,y,z,w)$, we have

$$\begin{aligned}&|\{(x,y,z,w)\in R(f,n): x\equiv 1 \pmod 2,\quad 2x+2y-2z+5w\equiv 0 \pmod 8\}|\\ &=|\{(x,y,z,w)\in R(f,n): x\equiv 1 \pmod 2,\quad 2x+2y-2z+5w\equiv 4 \pmod 8\}|,\end{aligned}$$

which implies that

$$\begin{aligned}&|\{(x,y,z,w)\in R(f,n): x\equiv 1 \pmod 2\}|\\ &=2|\{(x,y,z,w)\in R(f,n): x\equiv 1 \pmod 2,\ 2x+2y-2z+5w\equiv 0 \pmod 8\}|.\end{aligned}$$

For $(x,y,z,w)\in R(f,n)$, if $x\equiv 0 \pmod 2$, then we have $w\equiv 1 \pmod 2$ and $x+6y-6z+6w\equiv 0 \pmod 4$. Since $f(x,y,z,-w)=f(x,y,z,w)$, we have

$$\begin{aligned}&|\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2,\ x+6y-6z+6w\equiv 0 \pmod 8\}|\\ &=|\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2,\ x+6y-6z+6w\equiv 4 \pmod 8\}|,\end{aligned}$$

which implies that

$$\begin{aligned}&|\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2\}|\\ &=2|\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2,\ x+6y-6z+6w\equiv 0 \pmod 8\}|.\end{aligned}$$

Thus, if we define

$$\begin{aligned}X_1&=\{(x,y,z,w)\in R(f,n): x\equiv 1 \pmod 2,\ 2x+2y-2z+5w\equiv 0 \pmod 8\},\\ X_2&=\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2,\ x+6y-6z+6w\equiv 0 \pmod{16}\},\\ X_3&=\{(x,y,z,w)\in R(f,n): x\equiv 0 \pmod 2,\ x+6y-6z+6w\equiv 8 \pmod{16}\},\end{aligned}$$

then we have

$$r(f,n)=2(|X_1|+|X_2|+|X_3|).$$

Now, we analyze the set $R(g,n)$. One may check the followings;

(i) if $(x,y,z,w)\in R(g,n)$ and $x\equiv 0 \pmod 2$, then $x+y+z-w\equiv 0 \pmod 4$;

(ii) if $(x,y,z,w)\in R(g,n)$ and $x\equiv 1 \pmod 2$, then $x-z+w\equiv 0 \pmod 4$.

Since $g(x+y,-y,-z,-w)=g(x,y,z,w)$, we have

$$\begin{aligned}&|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\}|\\&=|\{(x,y,z,w)\in R(g,n): x\equiv 1 \pmod 2,\ x-z+w\equiv 0 \pmod 8\}|\end{aligned}$$

and

$$\begin{aligned}&|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 4 \pmod 8\}|\\&=|\{(x,y,z,w)\in R(g,n): x\equiv 1 \pmod 2,\ x-z+w\equiv 4 \pmod 8\}|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}r(g,n)=&2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2\}|\\=&2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\}|\\&+2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 4 \pmod 8\}|\\=&2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\}|\\&+2|\{(x,y,z,w)\in R(g,n): x\equiv 1 \pmod 2,\ x-z+w\equiv 4 \pmod 8\}|.\end{aligned}$$

One may easily show that for $(x,y,z,w)\in R(g,n)$, if $x\equiv 1 \pmod 2$ and $x-z+w\equiv 4 \pmod 8$, then $7x-4y+9z-w\equiv 0 \pmod 8$. Thus if we define

$$\begin{aligned}Y_1&=\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\quad x+y+z-w\equiv 0 \pmod 8\},\\Y_2&=\left\{(x,y,z,w)\in R(g,n): \begin{array}{r}x\equiv 1 \pmod 2,\quad x-z+w\equiv 4 \pmod 8,\\ 7x-4y+9z-w\equiv 8 \pmod{16}\end{array}\right\},\\Y_3&=\left\{(x,y,z,w)\in R(g,n): \begin{array}{r}x\equiv 1 \pmod 2,\quad x-z+w\equiv 4 \pmod 8,\\ 7x-4y+9z-w\equiv 0 \pmod{16}\end{array}\right\},\end{aligned}$$

then, we have

$$r(g,n)=2(|Y_1|+|Y_2|+|Y_3|).$$

For $j=1,2,3$, if we define maps $\psi_j: Y_j\to X_j$ by

$$\psi_1(x,y,z,w)=\frac{1}{8}\begin{pmatrix}4&-4&4&12\\-2&6&6&10\\-3&-3&5&-5\\-2&-2&-2&2\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix},$$

$$\psi_2(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 2 & 24 & -2 & 18 \\ -10 & -8 & -6 & 22 \\ -3 & -4 & -13 & -11 \\ -4 & 0 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

$$\psi_3(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 6 & -8 & -6 & -42 \\ 2 & -8 & 14 & 2 \\ 7 & 12 & 9 & -1 \\ 4 & 0 & -4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

then one may check that they are all bijective. Therefore, we have

$$r(f,n) = 2(|X_1| + |X_2| + |X_3|) = 2(|Y_1| + |Y_2| + |Y_3|) = r(g,n),$$

which completes the proof.

□

Now we are in a position to prove Theorem 4.1.7.

Proof of Theorem 4.1.7. Let

$$\begin{aligned} f &= f(x,y,z,w) = x^2 + 2y^2 + 4z^2 + 17w^2, \\ g &= g(x,y,z,w) = 2x^2 + 3y^2 + 4z^2 + 8w^2 + 2xy + 2yz + 2yw, \\ h_1 &= h_1(x,y,z,w) = 2x^2 + 4y^2 + 4z^2 + 6w^2 + 2xw + 2yw + 4zw, \\ h_2 &= h_2(x,y,z,w) = x^2 + 2y^2 + 2z^2 + 9w^2 + 2zw. \end{aligned}$$

First, note that

$$\begin{aligned} &|\{(x,y,z,w) \in R(h_1, 2n+6) : w \equiv 0 \pmod 2\}| \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(x,y,z,2w) = 2n+6\right\}\right| \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 : 2 \cdot h_2(x+w, z+w, y, w) = 2n+6\right\}\right| = r(h_2, n+3). \end{aligned}$$

Next, for $(x,y,z,w) \in R(h_1, 2n+6)$, if $w \equiv 1 \pmod 2$, then $y \equiv 0 \pmod 2$. Hence we have

$$|\{(x,y,z,w) \in R(h_1, 2n+6) : x \equiv 0 \pmod 2,\ y \equiv 0 \pmod 2,\ w \equiv 1 \pmod 2\}|$$

$$= |\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(2x,2y,z,w) = 2n+6\}|$$
$$= |\{(x,y,z,w) \in \mathbb{Z}^4 : 2\cdot g(z,w,x,y) = 2n+6\}| = r(g,n+3).$$

Finally, since $h_1(w-2x,2y,z,w) = 2\cdot g(z,w,x-w,y)$, we have,

$$|\{(x,y,z,w) \in R(h_1,2n+6) : x \equiv 1 \pmod 2,\ y \equiv 0 \pmod 2,\ w \equiv 1 \pmod 2\}|$$
$$= \left|\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(w-2x,2y,z,w) = 2n+6\}\right|$$
$$= r(g,n+3).$$

Therefore, we have

$$r(h_1,2n+6) = r(h_2,n+3) + 2r(g,n+3), \tag{4.9.1}$$

for any nonnegative even integer n.

By Proposition 4.9.1 and (4.9.1), we arrive at

$$2\cdot r(f,n+3) = r(h_1,2n+6) - r(h_2,n+3) \text{ for any } n \equiv 0,2 \pmod 8. \tag{4.9.2}$$

Now, if $8x^2+y^2+2z^2+9w^2-4xw \equiv 0 \pmod 4$, then $y \equiv z \equiv w \pmod 2$. Since $h_1(y,x,z,-w) = 4x^2+2y^2+4z^2+6w^2-2xw-2yw-4zw$, we have

$$|\{(x,y,z,w) \in R(f,8n+24) : x \equiv w \pmod 4\}|$$
$$= r((w-4x)^2+2y^2+4z^2+17w^2, 8n+24)$$
$$= r(8x^2+y^2+2z^2+9w^2-4xw, 4n+12)$$
$$= r(8x^2+(w-2y)^2+2(w-2z)^2+9w^2-4xw, 4n+12)$$
$$= r(4x^2+2y^2+4z^2+6w^2-2xw-2yw-4zw, 2n+6)$$
$$= r(h_1,2n+6).$$

Now, if $x^2+8y^2+16z^2+17w^2 = 8n+24$, then $x \equiv w \equiv 0 \pmod 2$. Since $2\cdot h_2(y,z,x,-w) = (w-2x)^2+2y^2+4z^2+17w^2$, we see that

$$|\{(x,y,z,w) \in \mathbb{Z}^4 : x^2+8y^2+16z^2+17w^2 = 8n+24\}|$$
$$= |\{(x,y,z,w) \in \mathbb{Z}^4 : 4x^2+8y^2+16z^2+68w^2 = 8n+24\}|$$

$$= r(x^2 + 2y^2 + 4z^2 + 17w^2, 2n + 6) = r((w - 2x)^2 + 2y^2 + 4z^2 + 17w^2, 2n + 6)$$
$$= r(h_2, n + 3).$$

From these equalities and (4.9.2), we have

$$2 \cdot r(f, n+3) = |\{(x, y, z, w) \in R(f, 8n + 24) : x \equiv w \pmod 4\}| - \left|\left\{(x, y, z, w) \in \mathbb{Z}^4 : x^2 + 8y^2 + 16z^2 + 17w^2 = 8n + 24\right\}\right|, \tag{4.9.3}$$

for any $n \equiv 0, 2 \pmod 8$.

Now, as $\frac{x(x-1)}{2} = \frac{(-x+1)(-x)}{2}$, we see that

$$16T(1, 2, 4, 17; n)$$
$$= 16\left|\left\{(x, y, z, w) \in \mathbb{N}^4 : \frac{x(x+1)}{2} + 2\frac{y(y+1)}{2} + 4\frac{z(z+1)}{2} + 17\frac{w(w+1)}{2} = n\right\}\right|$$
$$= \left|\left\{(x, y, z, w) \in \mathbb{Z}^4 : \frac{x(x+1)}{2} + 2\frac{y(y+1)}{2} + 4\frac{z(z+1)}{2} + 17\frac{w(w+1)}{2} = n\right\}\right|$$
$$= |\{(x, y, z, w) \in \mathbb{Z}^4 : (2x+1)^2 + 2(2y+1)^2 + 4(2z+1)^2 + 17(2w+1)^2 = 8n + 24\}|$$
$$= |\{(x, y, z, w) \in R(f, 8n + 24) : xyzw \equiv 1 \pmod 2\}|.$$

Note that if $x^2 + 2y^2 + 4z^2 + 17w^2 = 8n + 24$, then

$$(x^2, 2y^2, 4z^2, 17w^2) \equiv (1, 2, 4, 1), (0, 0, 0, 0), (4, 0, 0, 4), (4, 0, 4, 0) \text{ or } (0, 0, 4, 4) \pmod 8.$$

From this and (4.9.3), we may easily deduce that

$$8T(1, 2, 4, 17; n) = |\{(x, y, z, w) \in R(f, 8n + 24) : x \equiv w \pmod 4,\ y \equiv z \equiv 1 \pmod 2\}|$$
$$= |\{(x, y, z, w) \in R(f, 8n + 24) : x \equiv w \pmod 4\}|$$
$$- |\{(x, y, z, w) \in R(f, 8n + 24) : y \equiv z \equiv 0 \pmod 2\}|$$
$$= |\{(x, y, z, w) \in R(f, 8n + 24) : x \equiv w \pmod 4\}|$$
$$- \left|\left\{(x, y, z, w) \in \mathbb{Z}^4 : x^2 + 8y^2 + 16z^2 + 17w^2 = 8n + 24\right\}\right|$$
$$= 2 \cdot r(f, n + 3),$$

which is equivalent to (4.1.7). This completes the proof. □