Chapter 5

Sums of squares and sums of triangular numbers-II

5.1 Introduction

As mentioned in our introductory chapter, Sun [35] proposed seven more open conjectures (Conjecture 6.1 – Conjecture 6.7). The five conjectures on the ternary case are proved by Kim and Oh [30] by an elementary method. Xia and Yan [43] proved Conjecture 6.1 – Conjecture 6.6. In this chapter, we give alternative proofs of three of the conjectures of Sun [35] that were proved by Xia and Yan [43] and also present a simple proof of Conjecture 6.7. Furthermore, we prove some new relations between $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$.

In the following four theorems, we state the conjectures of Sun [35] that we prove here.

Theorem 5.1.1. (Conjecture 6.4 in Sun [35]) *Let* $n \in \mathbb{N}^+$ *with* $n \equiv 0, 2 \pmod 3$. *Then,*

$$T(1,1,27;n) = \frac{1}{16}(N(1,1,27;4(8n+29)) - N(1,1,27;8n+29)). \tag{5.1.1}$$

Theorem 5.1.2. (Conjecture 6.5 in Sun [35]) *Let* $n \in \mathbb{N}^+$. *If*

$(a,b,c,d) = (1,1,1,2), (1,1,1,3), (1,1,1,4), (1,1,1,5), (1,1,1,6), (1,1,2,2), (1,1,2,3),$

$$(1,1,2,4),(1,1,3,3),(1,1,3,9),(1,1,6,9),(1,2,2,2),(1,2,2,3),(1,3,3,3),$$
$$(1,3,3,6),(1,3,6,6),(1,3,9,9),(1,6,9,9),(2,3,3,3),$$

then,

$$T(a,b,c,d;n) = \frac{1}{96}\big(N(a,b,c,d;4(8n+a+b+c+d)) - N(a,b,c,d;8n+a+b+c+d)\big). \tag{5.1.2}$$

Theorem 5.1.3. (Conjecture 6.6 in Sun [35]) *Let $n \in \mathbb{N}^+$. Then,*

$$T(1,1,1,7;n) = \frac{1}{48}\left(N(1,1,1,7;16n+20) - N(1,1,1,7;4n+5)\right) \tag{5.1.3}$$
$$= \frac{1}{56}\left(N(1,1,1,7;32n+40) - 2N(1,1,1,7;8n+10)\right) \tag{5.1.4}$$

and

$$T(1,7,7,7;n) = \frac{1}{48}\left(N(1,7,7,7;16n+44) - N(1,7,7,7;4n+11)\right) \tag{5.1.5}$$
$$= \frac{1}{56}\left(N(1,7,7,7;32n+88) - 2N(1,7,7,7;8n+22)\right). \tag{5.1.6}$$

Theorem 5.1.4. (Conjecture 6.7 in Sun [35]) *Let $n \in \mathbb{N}^+$. Then, n is represented by $\frac{x(x-1)}{2} + \frac{y(y-1)}{2} + 6\frac{z(z-1)}{2}$ if and only if $n \not\equiv 2 \cdot 3^{2r-1} - 1 \pmod{3^{2r}}$ for $r = 1,2,3....$*

In the remaining theorems, we state our new results.

Theorem 5.1.5. *Let $n \in \mathbb{N}^+$ and $\ell, k \in \mathbb{N} \cup \{0\}$. Then, we have*

$$\frac{1}{16}N(1,1,8\ell+4,8k+1;8(n+\ell+k)+7) = T(1,1,8\ell+4,8k+1;n) \tag{5.1.7}$$

and

$$\frac{1}{16}N(1,1,8\ell+4,8k+5;8(n+\ell+k)+11) = T(1,1,8\ell+4,8k+5;n). \tag{5.1.8}$$

Theorem 5.1.6. *Let $n \in \mathbb{N}^+$. Then,*

$$\frac{1}{24}N(1,3,10,14;8n+28) - \frac{1}{8}N(1,3,10,14;2n+7) = T(1,3,10,14;n). \tag{5.1.9}$$

Theorem 5.1.7. *Let $n \in \mathbb{N}^+$. Then,*

$$\frac{1}{24}N(1,2,3,6;8n+12) - \frac{1}{8}N(1,2,3,6;2n+3) = T(1,2,3,6;n). \tag{5.1.10}$$

Theorem 5.1.8. *Let $n \in \mathbb{N}^+$. Then,*

$$\frac{1}{32}N(1,1,4,14;8n+19) = T(1,2,2,14;n).$$

Theorem 5.1.9. *Let $n \in \mathbb{N}^+$. Then, for any positive odd integer a, we have*

$$\frac{1}{16}N(a,a,a,9a;8(n+a)+4) - \frac{1}{16}N(a,a,a,9a;2(n+a)+1) = T(a,a,a,9a;n)$$

and, for any $k \in \mathbb{N}$,

$$N(a,a,a,9a;4n) = N(a,a,a,9a;4^k n).$$

When $a = 1$, it is worthwhile to note from [37, Theorem 4.4] that

$$t(1,1,1,9;n) = \begin{cases} 4\sigma(2n+3) + 12\sigma\left(\dfrac{2n+3}{9}\right), & \text{if } n \equiv 0 \pmod 3; \\ 8\sigma(2n+3), & \text{if } n \equiv 1 \pmod 3; \\ 4(\sigma(2n+3) - c(2n+3)), & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

where

$$\sigma(n) = \sum_{d|n} d$$

and

$$c(n) = \frac{1}{3} \sum_{\substack{4n=k^2+3\ell^2 \ (h,\ell\in\mathbb{Z}) \\ k\equiv 2 \pmod 3,\ \ell\equiv k+2 \pmod 4}} (-1)^k k.$$

In Section 5.2, we state some preliminary lemmas. In Sections 5.3 – 5.6, we prove Theorems 5.1.1 – 5.1.4, respectively. The remaining sections are devoted to proving Theorems 5.1.5 – 5.1.9.

5.2 Preliminary lemmas

Some preliminary identities of $f(a,b)$ are given in the following lemma.

Lemma 5.2.1. (Berndt [21, pp. 45–46, Entries 29 and 30]) *We have*

$$f(a,b)+f(-a,-b)=2f(a^3b,ab^3), \tag{5.2.1}$$

$$f(a,b)-f(-a,-b)=2af\left(\frac{b}{a},a^5b^3\right), \tag{5.2.2}$$

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d)=2f(ac,bd)f(ad,bc), \tag{5.2.3}$$

$$f(a,b)f(c,d)-f(-a,-b)f(-c,-d)=2af\left(\frac{b}{c},ac^2d\right)f\left(\frac{b}{d},acd^2\right). \tag{5.2.4}$$

Setting $a=b=q$ in (5.2.1) – (5.2.4), it follows that (Berndt [21, p. 40, Entry 25])

$$\phi(q)+\phi(-q)=2\phi(q^4), \tag{5.2.5}$$

$$\phi(q)-\phi(-q)=4q\psi(q^8), \tag{5.2.6}$$

$$\phi(q)^2+\phi(-q)^2=2\phi(q^2)^2, \tag{5.2.7}$$

$$\phi(q)^2-\phi(-q)^2=8q\psi(q^4)^2. \tag{5.2.8}$$

In the next lemma, we recall the 3-dissections of $\phi(q)$ and $\psi(q)$ from Berndt's book [21, p. 49, Corollary].

Lemma 5.2.2. *We have*

$$\phi(q)=\phi(q^9)+2qf(q^3,q^{15}), \tag{5.2.9}$$

$$\psi(q)=f(q^3,q^6)+q\psi(q^9). \tag{5.2.10}$$

Lemma 5.2.3. *The following identity holds:*

$$\psi(q)\psi(q^{15})=\psi(q^6)\psi(q^{10})+q\phi(q^{20})\psi(q^{24})+q^3\phi(q^{12})\psi(q^{40}). \tag{5.2.11}$$

Proof. To prove, identity (5.2.11), we first note, by (4.2.1), that

$$\phi(q^3)\phi(q^5)-\phi(-q^3)\phi(-q^5)=4q^3\left(\phi(q^{20})\psi(q^{24})+q^2\phi(q^{12})\psi(q^{40})\right).$$

But, by [21, p. 377, Entry 9(iii)], we also have

$$\phi(q^3)\phi(q^5) - \phi(-q^3)\phi(-q^5) = 2q^2 \left(\psi(q)\psi(q^{15}) - \psi(-q)\psi(-q^{15})\right).$$

From the above two identities,

$$\psi(q)\psi(q^{15}) - \psi(-q)\psi(-q^{15}) = 2q \left(\phi(q^{20})\psi(q^{24}) + q^2\phi(q^{12})\psi(q^{40})\right).$$

On the other hand, from [21, p. 377, Entry 9(iv)], we recall that

$$\psi(q)\psi(q^{15}) + \psi(-q)\psi(-q^{15}) = 2\psi(q^6)\psi(q^{10}).$$

Adding the previous two identities, we readily arrive at (5.2.11). □

We end this section with the following lemma.

Lemma 5.2.4. *We have*

$$\phi(q^2)\phi(q^{14}) + 4q^4\psi(q^4)\psi(q^{28}) = \phi(q)\phi(q^7) - 2q\psi(-q)\psi(-q^7), \tag{5.2.12}$$

$$\psi(q)\psi(q^9) + \psi(-q)\psi(-q^9) = 4q^{10}\psi(q^{18})\psi(q^{72}) + 2f(q^6, q^{12})f(q^{24}, q^{48}). \tag{5.2.13}$$

Proof. Identity (5.2.12) is eq. (5.19) in [21, p. 473]. We now prove (5.2.13).

Multiplying both sides of (5.2.10) by $\psi(q^9)$, we have

$$f(q^3, q^6)\psi(q^9) = \psi(q)\psi(q^9) - q\psi(q^9)^2. \tag{5.2.14}$$

Replacing q by $-q$ in the above, and then adding with (5.2.14), we have

$$\begin{aligned}
&\psi(q)\psi(q^9) + \psi(-q)\psi(-q^9)\\
&= q(\psi(q^9)^2 - \psi(-q^9)^2) + \frac{1}{2}(f(1, q^9)f(q^3, q^6) + f(1, -q^9)f(-q^3, q^6)),\\
&= q\psi(q^{18})(\phi(q^9) - \phi(-q^9)) + \frac{1}{2}(f(1, q^9)f(q^3, q^6) + f(1, -q^9)f(-q^3, q^6))\\
&= 4q^{10}\psi(q^{18})\psi(q^{72}) + \frac{1}{2}(f(1, q^9)f(q^3, q^6) + f(1, -q^9)f(-q^3, q^6)).
\end{aligned} \tag{5.2.15}$$

where, we have used the facts that $f(1, q) = 2f(q, q^3) = 2\psi(q)$ and $\psi(q)^2 = \phi(q)\psi(q^2)$, and also used (5.2.6) with q replaced by q^9.

Now, setting $a = 1$, $b = q^9$, $c = q^3$ and $d = q^6$ in (5.2.3) and noting that $f(-1, a) = 0$, we have

$$f(1, q^9)f(q^3, q^6) = 2f(q^3, q^{15})f(q^6, q^{12}).$$

Therefore,

$$f(1, q^9)f(q^3, q^6) + f(1, -q^9)f(-q^3, q^6) = 2f(q^6, q^{12})\{f(q^3, q^{15}) + f(-q^3, -q^{15})\}.$$

Employing (5.2.1) in the above, we have

$$f(1, q^9)f(q^3, q^6) + f(1, -q^9)f(-q^3, q^6) = 4f(q^6, q^{12})f(q^{24}, q^{48}).$$

Using the above identity in (5.2.15), we readily arrive at (5.2.13) to finish the proof.

□

5.3 Proof of Theorem 5.1.1

First we present the proof of Theorem 5.1.1 for the case when $n \equiv 2 \pmod 3$ as it is much easier than the other case. For this case, (5.1.1) may be recast as

$$T(1, 1, 27; 3n + 2) = \frac{1}{16}(N(1, 1, 27; 96n + 180) - N(1, 1, 27; 24n + 45)). \qquad (5.3.1)$$

We now prove (5.3.1).

We have

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 1, 27; n)q^n &= \phi(q)^2\phi(q^{27}) \\ &= \left(\phi(q^9) + 2qf(q^3, q^{15})\right)^2 \phi(q^{27}), \end{aligned} \qquad (5.3.2)$$

from which we extract

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 1, 27; 3n)q^n &= \phi(q^3)^2\phi(q^9) \\ &= \left(\phi(q^{12}) + 2q^3\psi(q^{24})\right)^2 \left(\phi(q^{36}) + 2q^9\psi(q^{72})\right), \end{aligned}$$

from which we further extract

$$\sum_{n=0}^{\infty} N(1,1,27;3(4n+3))q^n = 4\phi(q^3)\phi(q^9)\psi(q^6) + 8q^3\psi(q^6)^2\psi(q^{18})$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,27;12n)q^n \\
&= \phi(q^3)^2\phi(q^9) + 8q^3\phi(q^3)\psi(q^6)\psi(q^{18}) \\
&= \left(\phi(q^{12}) + 2q^3\psi(q^{24})\right)^2 \left(\phi(q^{36}) + 2q^9\psi(q^{72})\right) \\
&\quad + 8q^3 \left(\phi(q^{12}) + 2q^3\psi(q^{24})\right) \left(\psi(q^{24})\varphi(q^{36}) + q^6\phi(q^{12})\psi(q^{72})\right).
\end{aligned}$$

From the last identity, we extract

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,1,27;12(4n+3))q^n &= 4\phi(q^3)\phi(q^9)\psi(q^6) + 8q^3\psi^2(q^6)\psi(q^{18}) \\
&\quad + 8\phi(q^3)\phi(q^9)\psi(q^6).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,27;12(4n+3))q^n - \sum_{n=0}^{\infty} N(1,1,27;3(4n+3))q^n \\
&= 8\phi(q^3)\phi(q^9)\psi(q^6) \\
&= 8\psi(q^6)\left(\phi(q^{12})\phi(q^{36}) + 2q^3\psi(q^6)\psi(q^{36}) + 4q^{12}\psi(q^{24})\psi(q^{72})\right),
\end{aligned}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,27;12(4(2n+3)+3))q^n - \sum_{n=0}^{\infty} N(1,1,27;3(4(2n+3)+3))q^n \\
&= 16\psi(q^3)^2\psi(q^9). \qquad (5.3.3)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{n=0}^{\infty} T(1,1,27;n)q^n &= \psi(q)^2\psi(q^{27}) \\
&= \left(f(q^3,q^6) + q\psi(q^9)\right)^2 \psi(q^{27}), \qquad (5.3.4)
\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} T(1,1,27;3n+2)q^n = \psi(q^3)^2\psi(q^9).$$

From the above identity and (5.3.3), we have

$$\sum_{n=0}^{\infty} N(1,1,27;12(4(2n+3)+3))q^n - \sum_{n=0}^{\infty} N(1,1,27;3(4(2n+3)+3))q^n$$
$$= 16\sum_{n=0}^{\infty} T(1,1,27;3n+2)q^n,$$

from which (5.3.1) is apparent. This completes the proof of the case when $n \equiv 2 \pmod 3$.

Now we prove Theorem 5.1.1 for the case when $n \equiv 0 \pmod 3$. The proof turns out to be a very involved one. For this case, (5.1.1) may be recast as

$$T(1,1,27;3n) = \frac{1}{16}(N(1,1,27;96n+116) - N(1,1,27;24n+29)). \tag{5.3.5}$$

Our aim is to prove (5.3.5).

At first, we extract from (5.3.4) that

$$\sum_{n=0}^{\infty} T(1,1,27;3n)q^n = f(q,q^2)^2\psi(q^9). \tag{5.3.6}$$

Again, it implies from (5.3.2) that

$$\sum_{n=0}^{\infty} N(1,1,27;3n+2)q^n = 4\phi(q^9)f(q,q^5)^2$$
$$= 4\left(\phi(q^{36}) + 2q^9\psi(q^{72})\right)\left(f(q^8,q^{16}) + qf(q^4,q^{20})\right)^2,$$

from which we extract

$$\sum_{n=0}^{\infty} N(1,1,27;12n+8)q^n = 4\phi(q^9)f(q,q^5)^2 + 16q^2\psi(q^{18})f(q^2,q^4)f(q,q^5).$$

Thus,

$$\sum_{n=0}^{\infty} N(1,1,27;12n+8)q^n - \sum_{n=0}^{\infty} N(1,1,27;3n+2)q^n$$

$$= 16q^2\psi(q^{18})f(q^2,q^4)f(q,q^5)$$
$$= 16q^2\psi(q^{18})f(q^2,q^4)\left(f(q^8,q^{16})+qf(q^4,q^{20})\right),$$

from which we extract

$$\sum_{n=0}^{\infty} N(1,1,27;24n+20)q^n - \sum_{n=0}^{\infty} N(1,1,27;6n+5)q^n$$
$$= 16q\psi(q^9)f(q,q^2)f(q^2,q^{10}).$$

With the help of (1.2.2) and (1.2.4), and manipulating the q-products, the above can be written as

$$\sum_{n=0}^{\infty} N(1,1,27;24n+20)q^n - \sum_{n=0}^{\infty} N(1,1,27;6n+5)q^n$$
$$= 16q\frac{f_2 f_{18}^2}{f_6} f(q^2,q^{10}) \times \frac{f_3^2}{f_1 f_9}. \tag{5.3.7}$$

Now we want to have a 2-dissection of the right side of the above. To that end, we notice that one of Ramanujan's famous forty identities for the Rogers-Ramanujan functions is (see [22, Entry 3.6, p. 8])

$$\frac{f_3^2}{f_1 f_9} = G(q)G(q^9) + q^2 H(q)H(q^9), \tag{5.3.8}$$

where the Rogers-Ramanujan functions $G(q)$ and $H(q)$ are defined for $|q| < 1$ by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}.$$

We now recall the following 2-dissections of $G(q)$ and $H(q)$ from [22, Lemma 4.1, p. 19]:

$$G(q) = \frac{f_8}{f_2}\left(G(q^{16}) + qH(-q^4)\right)$$

and

$$H(q) = \frac{f_8}{f_2}\left(q^3 H(q^{16}) + G(-q^4)\right).$$

With the aid of the above two identities, (5.3.8) reduces to

$$\frac{f_3^2}{f_1 f_9} = \frac{f_8 f_{72}}{f_2 f_{18}}\big[G(q^{16})G(q^{144}) + qH(-q^4)G(q^{144}) + q^2 G(-q^4)G(-q^{36})$$

$$+ q^5 H(q^{16})G(-q^{36}) + q^9 G(q^{16})H(-q^{36}) + q^{10} H(-q^4)H(-q^{36})$$
$$+ q^{29} G(-q^4)H(q^{144}) + q^{32} H(q^{16})H(q^{144})\big].$$

Employing the above in (5.3.7), we extract

$$\sum_{n=0}^{\infty} N(1,1,27;48n+20)q^n - \sum_{n=0}^{\infty} N(1,1,27;12n+5)q^n$$
$$= 16q f_4 f_{36} \times \frac{f_9}{f_3} f(q,q^5) \times \big[H(-q^2)G(q^{72}) + q^{14}G(-q^2)H(q^{72})$$
$$+ q^2 H(q^8)G(-q^{18}) + q^4 G(q^8)H(-q^{18})\big]. \tag{5.3.9}$$

Now, two more identities among the Ramanujan's forty identities are [22, Entries 3.13 and 3.14, p. 8]

$$H(q^4)G(q^9) - qG(q^4)H((q^9)) = \frac{\chi(-q)\chi(q^3)}{\chi(-q^{18})}$$

and

$$H(q)G(q^{36}) - q^7 G(q)H(q^{36}) = \frac{\chi(q^3)\chi(-q^9)}{\chi(-q^2)},$$

where we also used the trivial identity $\chi(q)\chi(-q) = \chi(-q^2)$, with $\chi(q) = (-q;q^2)_\infty$. Replacing q by $-q^2$ in the above and then using the resulting identities in (5.3.9), we find that

$$\sum_{n=0}^{\infty} N(1,1,27;48n+20)q^n - \sum_{n=0}^{\infty} N(1,1,27;12n+5)q^n$$
$$= 16q f_4 f_{36} \times \frac{f_9}{f_3} f(q,q^5)\left(L(q^2) + q^2 M(q^2)\right), \tag{5.3.10}$$

where $L(q) := \dfrac{\chi(-q^3)\chi(q^9)}{\chi(-q^2)}$ and $M(q) := \dfrac{\chi(q)\chi(-q^3)}{\chi(-q^{18})}$.

Now, Yao and Xia [47] proved that

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}.$$

Replacing q by q^3, we have the following 2-dissection of f_9/f_3:

$$\frac{f_9}{f_3} = A(q^2) + q^3 B(q^2), \tag{5.3.11}$$

where $A(q)=\dfrac{f_6f_9f_{24}f_{36}^2}{f_3^2f_{12}f_{18}f_{72}}$ and $B(q)=\dfrac{f_9f_{12}^2f_{72}}{f_3^2f_{24}f_{36}}$.

We also have

$$f(q,q^5)=C(q^2)+qD(q^2), \tag{5.3.12}$$

where $C(q)=f(q^4,q^8)$ and $D(q)=f(q^2,q^{10})$.

Employing (5.3.11) and (5.3.12), we can rewrite (5.3.10) as

$$\begin{aligned}&\sum_{n=0}^{\infty}N(1,1,27;48n+20)q^n-\sum_{n=0}^{\infty}N(1,1,27;12n+5)q^n\\&=16qf_4f_{36}\left(L(q^2)+q^2M(q^2)\right)\left(A(q^2)+q^3B(q^2)\right)\left(C(q^2)+qD(q^2)\right),\end{aligned}$$

which implies that

$$\begin{aligned}&\sum_{n=0}^{\infty}N(1,1,27;96n+116)q^n-\sum_{n=0}^{\infty}N(1,1,27;24n+29)q^n\\&=16f_2f_{18}\left(L(q)+qM(q)\right)\left(A(q)D(q)+qB(q)C(q)\right).\end{aligned} \tag{5.3.13}$$

We now simplify the right side.

By (1.2.4) and simple q-product manipulations, we find that

$$\begin{aligned}&A(q)D(q)+qB(q)C(q)\\&=\frac{f_6f_9f_{24}f_{36}^2}{f_3^2f_{12}f_{18}f_{72}}f(q^2,q^{10})+q\frac{f_9f_{12}^2f_{72}}{f_3^2f_{24}f_{36}}f(q^4,q^8)\\&=\frac{\chi(-q^9)}{\varphi(-q^3)}\left(f(q^2,q^{10})f(q^{12},f^{24})+qf(q^4,q^8)f(q^6,q^{30})\right).\end{aligned} \tag{5.3.14}$$

Now, from (5.2.1) and (5.2.2), we have

$$f(q^8,q^{16})=\frac{1}{2}\left(f(q,q^5)+f(-q,-q^5)\right)$$

and

$$f(q^4,q^{20})=\frac{1}{2}\left(f(q,q^5)-f(-q,-q^5)\right).$$

Therefore,

$$f(q^4,q^{20})f(q^{24},q^{48})+q^2f(q^8,q^{16})f(q^{12},q^{60})$$

$$= \frac{1}{2q}\left(f(q,q^5)f(q^3,q^{15}) - f(-q,-q^5)f(-q^3,-q^{15})\right). \quad (5.3.15)$$

Recall from Cao [23, Corollary 2.2] that, if $|ab| < 1$ and $cd = (ab)^{k_1k_2}$, where both k_1 and k_2 are positive integers, then

$$f(a,b)f(c,d) = \sum_{r=0}^{k_1+k_2-1} a^{\frac{r^2+r}{2}} b^{\frac{r^2-r}{2}} \left(f(a^{\frac{k_1^2+k_1}{2}+k_1r} b^{\frac{k_1^2-k_1}{2}+k_1r} d, a^{\frac{k_1^2-k_1}{2}-k_1r} b^{\frac{k_1^2+k_1}{2}-k_1r} c\right)$$
$$\times \left(f(a^{\frac{k_2^2+k_2}{2}+k_2r} b^{\frac{k_2^2-k_2}{2}+k_2r} c, a^{\frac{k_2^2-k_2}{2}-k_2r} b^{\frac{k_2^2+k_2}{2}-k_2r} d\right).$$

Setting $a = q$, $b = q^5$, $c = q^3$, $d = q^{15}$, $k_1 = 1$ and $k_2 = 3$ in the above, and then simplifying, we find that

$$f(q,q^5)f(q^3,q^{15}) = qf(q^{18},q^{54})f(q^2,q^4) + \varphi(q^{36})f(q^4,q^{20}) + 2q^8\psi(q^{72})f(q^8,q^{16}).$$

Therefore,

$$f(q,q^5)f(q^3,q^{15}) - f(-q,-q^5)f(-q^3,-q^{15}) = 2qf(q^{18},q^{54})f(q^2,q^4).$$

Employing the above in (5.3.15), we have

$$f(q^4,q^{20})f(q^{24},q^{48}) + q^2f(q^8,q^{16})f(q^{12},q^{60}) = f(q^2,q^4)\psi(q^{18}).$$

Replacing q^2 by q in the above, and then using it in (5.3.14), we obtain

$$A(q)D(q) + qB(q)C(q) = \frac{\chi(-q^9)}{\phi(-q^3)} f(q,q^2)\psi(q^9).$$

Plugging the above in (5.3.13), we have

$$\sum_{n=0}^{\infty} N(1,1,27;96n+116)q^n - \sum_{n=0}^{\infty} N(1,1,27;24n+29)q^n$$
$$= 16f_2f_{18}\frac{\chi(-q^9)}{\phi(-q^3)} f(q,q^2)\psi(q^9)\left(L(q) + qM(q)\right)$$
$$= 16f_2f_{18}\frac{\chi(-q^9)}{\phi(-q^3)} f(q,q^2)\psi(q^9)\left(\frac{\chi(-q^3)\chi(q^9)}{\chi(-q^2)} + q\frac{\chi(q)\chi(-q^3)}{\chi(-q^{18})}\right).$$

Elementary q-product manipulations can be applied in the above to arrive at

$$\sum_{n=0}^{\infty} N(1,1,27;96n+116)q^n - \sum_{n=0}^{\infty} N(1,1,27;24n+29)q^n$$

$$= 16\frac{f_2^2}{f_3}\left(\frac{\phi(-q^{18})}{\phi(-q^2)} + q\frac{\psi(-q^9)}{\psi(-q)}\right) f(q,q^2)\psi(q^9). \tag{5.3.16}$$

Now, recall from Berndt [21, Entry 4(i), p. 358] that

$$\frac{\phi(-q^{18})}{\phi(-q^2)} + q\left(\frac{\psi(q^9)}{\psi(q)} - \frac{\psi(-q^9)}{\psi(-q)}\right) = 1.$$

With the help of the above, (5.3.16) reduces to

$$\sum_{n=0}^{\infty} N(1,1,27;96n+116)q^n - \sum_{n=0}^{\infty} N(1,1,27;24n+29)q^n$$
$$= 16\frac{f_2^2}{f_3}\left(1 + 2q\frac{\psi(-q^9)}{\psi(-q)} - q\frac{\psi(q^9)}{\psi(q)}\right) f(q,q^2)\psi(q^9). \tag{5.3.17}$$

Also recall from Berndt [21, Entry 2(ii), p. 349] that

$$\psi(q) - 3q\psi(q^9) = \frac{\phi(-q)}{\chi(-q^3)},$$

which readily gives

$$q\frac{\psi(q^9)}{\psi(q)} = \frac{1}{3}\left(1 - \frac{\chi^3(-q)}{\chi(-q^3)}\right)$$

and

$$q\frac{\psi(-q^9)}{\psi(-q)} = \frac{1}{3}\left(\frac{\chi^3(q)}{\chi(q^3)} - 1\right).$$

Employing the above two identities in (5.3.17), we have

$$\sum_{n=0}^{\infty} N(1,1,27;96n+116)q^n - \sum_{n=0}^{\infty} N(1,1,27;24n+29)q^n$$
$$= \frac{16f_2^2}{3f_3}\left(2\frac{\chi^3(q)}{\chi(q^3)} + \frac{\chi^3(-q)}{\chi(-q^3)}\right) f(q,q^2)\psi(q^9). \tag{5.3.18}$$

Now, from [14, eq. (6.4)], we have

$$\frac{\chi^3(q)}{\chi(q^3)} + \frac{\chi^3(-q)}{\chi(-q^3)} = 2\,\frac{\chi(-q^6)}{\chi^3(-q^2)},$$

which reduces (5.3.18) to

$$\sum_{n=0}^{\infty} N(1,1,27;96n+116)q^n - \sum_{n=0}^{\infty} N(1,1,27;24n+29)q^n$$

$$= \frac{16f_2^2}{3f_3}\left(\frac{\chi^3(q)}{\chi(q^3)} + 2\frac{\chi(-q^6)}{\chi^3(-q^2)}\right) f(q,q^2)\psi(q^9)$$

$$= \frac{16f_6}{3f_2f_3}\left(\frac{f^3(q)}{f(q^3)} + 2\frac{f_4^3}{f_{12}}\right) f(q,q^2)\psi(q^9), \tag{5.3.19}$$

where we have used the fact that $f(q) = (-q;-q)_\infty = \dfrac{f_2^3}{f_1f_4}$.

Now, Hirschhorn, Garvan, Borwein [29] proved that

$$\frac{f_3}{f_1^3} = \frac{f_4^6f_6^3}{f_2^9f_{12}^2} + 3q\frac{f_4^2f_6f_{12}^2}{f_2^7} \tag{5.3.20}$$

and

$$\frac{f_3^3}{f_1} = \frac{f_4^3f_6^2}{f_2^2f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{5.3.21}$$

Replacing q by $-q$ in (5.3.20) we find that

$$\frac{f^3(q)}{f(q^3)} = \frac{f_4^3}{f_{12}} + 3q\frac{f_2^2f_{12^3}}{f_4f_6^2}.$$

Employing the above identity in (5.3.19), we obtain

$$\sum_{n=0}^{\infty} N(1,1,27;96n+116)q^n - \sum_{n=0}^{\infty} N(1,1,27;24n+29)q^n$$
$$= 16\left(\frac{f_4^3f_6}{f_2f_3f_{12}} + q\frac{f_2f_{12}^3}{f_3f_4f_6}\right) f(q,q^2)\psi(q^9),$$

which, by (5.3.21), reduces to

$$\sum_{n=0}^{\infty} N(1,1,27;96n+116)q^n - \sum_{n=0}^{\infty} N(1,1,27;24n+29)q^n$$
$$= 16\frac{f_2f_3^2}{f_1f_6} f(q,q^2)\psi(q^9)$$
$$= 16f(q,q^2)^2\psi(q^9),$$

where, we used $f(q,q^2) = \dfrac{f_2f_3^2}{f_1f_6}$.

From the above and (5.3.6), we arrive at

$$\sum_{n=0}^{\infty} N(1,1,27;96n+116)q^n - \sum_{n=0}^{\infty} N(1,1,27;24n+29)q^n = 16\sum_{n=0}^{\infty} T(1,1,27;3n)q^n,$$

from which (5.3.5) readily implies. Thus, we complete the proof of Theorem 5.1.1 for the case $n \equiv 0 \pmod 3$.

5.4 Proof of Theorem 5.1.2

We present the proof of the case $(a, b, c, d) = (1, 1, 6, 9)$ only. The proofs of the remaining cases can be accomplished similarly.

We have

$$\sum_{n=0}^{\infty} N(1,1,6,9;n)q^n = \phi(q)^2\phi(q^6)\phi(q^9).$$

Employing the 2-dissections of $\phi(q)$ and $\phi(q)^2$ in the above and then proceeding as in the previous section, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,6,9;8n+4)q^n - \sum_{n=0}^{\infty} N(1,1,6,9;2n+1)q^n \\
&= 16q^5\phi(q)\psi(q^2)\psi(q^6)\psi(q^{36}) + 32q^2\psi(q)\psi(q^4)\psi(q^6)\psi(q^9) \\
&= 16q^5(\phi(q^4) + 2q\psi(q^8))\psi(q^2)\psi(q^6)\psi(q^{36}) \\
&\quad + 32q^2\psi(q^4)\psi(q^6)(f(q^6,q^{10}) + qf(q^2,q^{14}))(f(q^{54},q^{90}) + q^9 f(q^{18},q^{126})),
\end{aligned}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,6,9;16n+4)q^n - \sum_{n=0}^{\infty} N(1,1,6,9;4n+1)q^n \\
&= 32q^3\psi(q)\psi(q^3)\psi(q^4)\psi(q^{18}) \\
&\quad + 32q\psi(q^2)\psi(q^3)\left(f(q^3,q^5)f(q^{27},q^{45}) + q^5 f(q,q^7)f(q^9,q^{63})\right) \\
&= 32q^3\psi(q)\psi(q^3)\psi(q^4)\psi(q^{18}) \\
&\quad + 16q\psi(q^2)\psi(q^3)\left(\psi\left(\sqrt{q}\right)\psi\left(\sqrt{q^9}\right) + \psi\left(-\sqrt{q}\right)\psi\left(-\sqrt{q^9}\right)\right),
\end{aligned}$$

where we used (4.2.4).

Now, replacing q^2 by q in (5.2.13), and then using it in the previous identity, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,6,9;16n+4)q^n - \sum_{n=0}^{\infty} N(1,1,6,9;4n+1)q^n \\
&= 32q^3\psi(q)\psi(q^3)\psi(q^4)\psi(q^{18}) + 64q^6\psi(q^2)\psi(q^3)\psi(q^9)\psi(q^{36}) \\
&\quad + 32q\psi(q)\psi(q^2)\psi(q^3)f(q^{12},q^{24}) - 32q^2\psi(q^2)\psi(q^3)\psi(q^9)f(q^{12},q^{24}),
\end{aligned}$$

from which, after employing (4.2.9), we extract

$$\sum_{n=0}^{\infty} N(1,1,6,9;32n+4)q^n - \sum_{n=0}^{\infty} N(1,1,6,9;8n+1)q^n$$
$$= 32q^2\phi(q)\psi(q^2)\psi(q^6)\psi(q^9) + 64q^3\psi(q)\psi(q^6)\phi(q^9)\psi(q^{18})$$
$$+ 32q\psi(q)\psi(q^6)f(q^6,q^{12}))\big(\phi(q)-\phi(q^9)\big)$$
$$= 32q^2\phi(q)\psi(q^2)\psi(q^6)\psi(q^9) + 64q^3\psi(q)\psi(q^6)\phi(q^9)\psi(q^{18})$$
$$+ 64q^2\psi(q)\psi(q^6)f(q^6,q^{12})f(q^3,q^{15}). \tag{5.4.1}$$

Now, (1.2.4) and simple q-product manipulations give

$$f(q^6,q^{12})f(q^3,q^{15}) = \frac{\psi(q^9)\phi(-q^9)}{\chi(-q^3)}.$$

Employing the above in (5.4.1), we find that

$$\sum_{n=0}^{\infty} N(1,1,6,9;32n+4)q^n - \sum_{n=0}^{\infty} N(1,1,6,9;8n+1)q^n$$
$$= 32q^2\psi(q)^2\psi(q^6)\psi(q^9) + 64q^3\psi(q)\psi(q^6)\psi(q^9)^2$$
$$+ 64q^2\psi(q)\psi(q^6)\frac{\psi(q^9)\phi(-q^9)}{\chi(-q^3)} \tag{5.4.2}$$

Now, we have

$$64q^3\psi(q)\psi(q^6)\psi(q^9)^2 = 64q^2\psi(q)\psi(q^6)\psi(q^9)\big(\psi(q)-f(q^3,q^6)\big)$$
$$= 64q^2\psi(q)^2\psi(q^6)\psi(q^9) - 64q^2\psi(q)\psi(q^6)\psi(q^9)\frac{\phi(-q^9)}{\chi(-q^3)},$$

where we used the identity $f(q,q^2) = \dfrac{\phi(-q^3)}{\chi(-q)}$.

Plugging the above into (5.4.2), we arrive at

$$\sum_{n=0}^{\infty} N(1,1,6,9;32n+4)q^n - \sum_{n=0}^{\infty} N(1,1,6,9;8n+1)q^n$$
$$= 96q^2\psi(q)^2\psi(q^6)\psi(q^9)$$
$$= 96\sum_{n=0}^{\infty} T(1,1,6,9;n)q^{n+2},$$

which readily implies that

$$N(1,1,6,9;32n+68) - N(1,1,6,9;8n+17) = 96T(1,1,6,9;n),$$

which is clearly equivalent to (5.1.2) with $(a,b,c,d)=(1,1,6,9)$. Thus, we complete the proof of Theorem 5.1.2 for the proffered case.

5.5 Proof of Theorem 5.1.3

We only sketch the proof of (5.1.4). Identities (5.1.3), (5.1.5) and (5.1.6) can be proved in a similar way.

We have

$$\sum_{n=0}^{\infty} N(1,1,1,7;n)q^n = \phi(q)^3\phi(q^7).$$

Successively employing the two dissections of $\phi(q)$ and $\phi(q)^2$ in the above, we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,1,1,7;8n)q^n - 2\sum_{n=0}^{\infty} N(1,1,1,7;2n)q^n\\&= 12q\phi(q)^2\psi(q)\psi(q^7) - \phi(q^2)^3\phi(q^{14}) - 12q\phi(q^2)\psi^2(q^4)\phi(q^{14})\\&\quad - 12q^4\phi(q^2)^2\psi(q^4)\psi(q^{28}) - 16q^5\psi(q^4)^3\psi(q^{28}),\end{aligned}$$

from which, after using the 2-dissections of $\phi(q)$ and $\psi(q)\psi(q^7)$, we extract

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,1,1,7;16n+8)q^n - 2\sum_{n=0}^{\infty} N(1,1,1,7;4n+2)q^n\\&= 12\phi(q)^2\psi(q^4)\phi(q^{14}) + 12q^3\phi(q)^2\phi(q^2)\psi(q^{28}) + 48q\psi(q)\psi(q^2)^2\psi(q^7)\\&\quad - 12\phi(q)\psi(q^2)^2\phi(q^7) - 16q^2\psi(q^2)^3\psi(q^{14}).\end{aligned}$$

We further extract

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,1,1,7;32n+8)q^n - 2\sum_{n=0}^{\infty} N(1,1,1,7;8n+2)q^n\\&= 12\psi(q)^2\left(\phi(q)\phi(q^7) + 4q^2\psi(q^2)\psi(q^{14})) - (\phi(q^2)\phi(q^{14}) + 4q^4\psi(q^4)\psi(q^{28}))\right)\\&\quad + 32q\psi(q)^3\psi(q^7)\\&= 12\psi(q)^2\left(4q^2\psi(q^2)\psi(q^{14}) + 2q\psi(-q)\psi(-q^7)\right) + 32q\psi(q)^3\psi(q^7), \qquad (5.5.1)\end{aligned}$$

where we used (5.2.12).

Now, from (4.2.10), we have

$$\psi(q)\psi(q^7) - \psi(-q)\psi(-q^7) = 2q\psi(q^2)\psi(q^{14}).$$

Employing the above in (5.5.1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,1,7;32n+8)q^n - 2\sum_{n=0}^{\infty} N(1,1,1,7;8n+2)q^n \\ &= 56q\psi(q)^3\psi(q^7) \\ &= 56\sum_{n=0}^{\infty} T(1,1,1,7;n)q^{n+1},\end{aligned}$$

which readily gives (5.1.4) to complete the proof.

5.6 Proof of Theorem 5.1.4

One way of the theorem has already been proved by Sun [35], namely, if $n = \frac{x(x-1)}{2} + \frac{y(y-1)}{2} + 6\frac{z(z-1)}{2}$, then $n \not\equiv 2\cdot 3^{2r-1} - 1 \pmod{3^{2r}}$ for $r = 1, 2, 3, \ldots$. Therefore, we need to show only the other way, that is, if $n \equiv 2\cdot 3^{2r-1} - 1 \pmod{3^{2r}}$, then $T(1,1,6;n) = 0$. This is equivalent to showing that if $n = 3^{2r}k + 2\cdot 3^{2r-1} - 1$, for some integer k and $r = 1, 2, 3, \ldots$, then

$$T(1,1,6;n) = 0. \tag{5.6.1}$$

Note that, when $r = 1$, then $n = 9k + 5 = 3(3k+1) + 2$.

We have

$$\begin{aligned}\sum_{n=0}^{\infty} T(1,1,6;n)q^n &= \psi(q)^2\psi(q^6) \\ &= (f(q^3,q^6) + q\psi(q^9))^2\psi(q^6),\end{aligned}$$

from which we extract

$$\begin{aligned}\sum_{n=0}^{\infty} T(1,1,6;3n+2)q^n &= \psi(q^2)\psi(q^3)^2 \\ &= (f(q^6,q^{12}) + q^2\psi(q^{18}))\psi(q^3)^2,\end{aligned} \tag{5.6.2}$$

and so,

$$T(1,1,6;3(3n+1)+2)=0. \tag{5.6.3}$$

which is the case $r=1$ of (5.6.1).

From (5.6.2), we also extract

$$\begin{aligned}\sum_{n=0}^{\infty} T(1,1,6;3(3n+2)+2)q^n &= \psi(q)^2\psi(q^6)\\ &= \sum_{n=0}^{\infty} T(1,1,6;n)q^n,\end{aligned}$$

from which we readily arrive at

$$T(1,1,6;3^2n+2.3+2)=T(1,1,6;n),$$

that is,

$$T(1,1,6;3^2n+3.3-1)=T(1,1,6;n).$$

Replacing n by $3(3n+1)+2$ in the above and then using (5.6.3), we have

$$T(1,1,6;3^2(3(3n+1)+2)+3.2-1)=T(1,1,6;3(3n+1)+2)=0$$

that is,

$$T(1,1,6;3^4n+2.3^3-1)=0,$$

which is the case $r=2$ of (5.6.1).

Now, (5.6.1) can be easily proved for $r=1,2,3,\ldots$, by mathematical induction.

From Baruah, Cooper and Hirschhorn [15, Theorem 1.4], we note that

$$N(1,1,6;8n+8)-N(1,1,6;2n+2)=8T(1,1,6,n).$$

From Theorem 5.1.4 and the above we have the following interesting result.

Corollary 5.6.1. *If* $n\equiv 2\cdot 3^{2r-1}-1 \pmod{3^{2r}}$ *for* $r=1,2,3...,$ *then*

$$N(1,1,6;8n+8)=N(1,1,6;2n+2).$$

Remark 5.6.2. *The following general result can be proved by proceeding in a similar way as in the above proof of Theorem 5.1.4.*

If $n \equiv 2 \cdot 3^{2r-1} - 1 \pmod{3^{2r}}$, *then for any nonnegative integer* k, *we have* $T(1, 1, 9k + 6; n) = 0$.

5.7 Proof of Theorem 5.1.5

We have

$$\sum_{n=0}^{\infty} N(1, 1, 8\ell + 4, 4s + 1; n)q^n = \phi(q)^2\phi(q^{8\ell+4})\phi(q^{4s+1})$$
$$= (\phi(q^4) + 2q\psi(q^8))(\phi(q^{16s+4}) + 2q^{4s+1}\psi(q^{32s+8})),$$

from which we extract

$$\sum_{n=0}^{\infty} N(1, 1, 8\ell + 4, 4s + 1; 2n + 1)q^n$$
$$= 4\psi(q^2)^2\phi(q^{4\ell+2})\phi(q^{8s+2}) + 2q^{2s}\psi(q^{16s+4})\phi(q)^2\phi(q^{4\ell+2})$$
$$= 4\psi(q^2)^2\phi(q^{4\ell+2})\phi(q^{8s+2}) + 2q^{2s}\psi(q^{16s+4})(\phi(q^2)^2 + 4q\psi(q^4)^2)\phi(q^{4\ell+2}).$$

We further extract

$$\sum_{n=0}^{\infty} N(1, 1, 8\ell + 4, 4s + 1; 4n + 3)q^n = 8q^s\psi(q^{8s+2})\psi(q^2)^2\phi(q^{2\ell+1})$$
$$= 8q^s\psi(q^{8s+2})\psi(q^2)^2 \left(\phi(q^{8\ell+4}) + 2q^{2\ell+1}\psi(q^{16\ell+8})\right).$$

Now, assuming s to be even (say, $s = 2k$) and odd (say, $s = 2k+1$) respectively, we extract from above that

$$\sum_{n=0}^{\infty} N(1, 1, 8\ell + 4, 8k + 1; 8n + 7)q^n = 16q^{\ell+k}\psi(q)^2\psi(q^{8\ell+4})\psi(q^{8k+1})$$

and

$$\sum_{n=0}^{\infty} N(1, 1, 8\ell + 4, 8k + 5; 8n + 3)q^n = 16q^{\ell+k+1}\psi(q)^2\psi(q^{8\ell+4})\psi(q^{8k+5}).$$

Comparing the terms involving $q^{n+\ell+k}$ and $q^{n+\ell+k+1}$, from both sides of the above two equations, we arrive at

$$N(1,1,8\ell+4,8k+1;8(n+\ell+k)+7)=16T(1,1,8\ell+4,8k+1;n)$$

and

$$N(1,1,8\ell+4,8k+5;8(n+\ell+k)+11)=16T(1,1,8\ell+4,8k+5;n),$$

which are equivalent to (5.1.7) and (5.1.8), respectively. Thus, we finish the proof of Theorem 5.1.5.

5.8 Proof of Theorem 5.1.6

We have

$$\begin{aligned}&\sum_{n=0}^{\infty}N(1,3,10,14;n)q^n\\&=\phi(q)\phi(q^3)\phi(q^{10})\phi(q^{14})\\&=\phi(q^{10})\phi(q^{14})\left(\phi(q^4)\phi(q^{12})+2q\psi(q^2)\psi(q^6)+4q^4\psi(q^8)\psi(q^{24})\right),\end{aligned}$$

where, we used (4.2.8).
We extract from here that

$$\sum_{n=0}^{\infty}N(1,3,10,14;2n)q^n=\phi(q^5)\phi(q^7)\big(\phi(q^2)\phi(q^6)+4q^2\psi(q^4)\psi(q^{12})\big)$$

and

$$\sum_{n=0}^{\infty}N(1,3,10,14;2n+1)q^n=2\psi(q)\psi(q^3)\phi(q^5)\phi(q^7),$$

from which we further extract

$$\begin{aligned}&\sum_{n=0}^{\infty}N(1,3,10,14;4n)q^n\\&\quad=\big(\phi(q^{10})\phi(q^{14})+4q^6\psi(q^{20})\psi(q^{28})\big)\left(\phi(q)\phi(q^3)+4q\psi(q^2)\psi(q^6)\right)\end{aligned}$$

$$= \left(\phi(q^{10})\phi(q^{14}) + 4q^6\psi(q^{20})\psi(q^{28})\right)$$
$$\times \left(\phi(q^4)\phi(q^{12}) + 6q\psi(q^2)\psi(q^6) + 4q^4\psi(q^8)\psi(q^{24})\right).$$

We again extract

$$\sum_{n=0}^{\infty} N(1,3,10,14;8n+4)q^n = 6\psi(q)\psi(q^3)\phi(q^5)\phi(q^7)$$
$$+ 24q^3\psi(q)\psi(q^3)\psi(q^{10})\psi(q^{14}).$$

Therefore,

$$\sum_{n=0}^{\infty} \left(N(1,3,10,14;8n+4) - 3N(1,3,10,14;2n+1)\right) q^n$$
$$= 24\sum_{n=0}^{\infty} T(1,3,10,14;n)q^{n+3},$$

from which we readily arrive at (5.1.9) to finish the proof.

5.9 Proofs of Theorems 5.1.7–5.1.9

The proof is similar to that of Theorem 5.1.6 given in the previous section. We apply (4.2.8) to the generating function

$$\sum_{n=0}^{\infty} N(1,2,3,6;n)q^n = \phi(q)\phi(q^2)\phi(q^3)\phi(q^6),$$

and then extract the even and/or odd powers of q, to find that

$$\sum_{n=0}^{\infty} N(1,2,3,6;2n)q^n = \phi(q)\phi(q^3)\left(\phi(q^2)\phi(q^6) + 4q^2\psi(q^4)\psi(q^{12})\right),$$
$$\sum_{n=0}^{\infty} N(1,3,10,14;2n+1)q^n = 2\psi(q)\psi(q^3)\phi(q)\phi(q^3),$$
$$\sum_{n=0}^{\infty} N(1,2,3,6;4n)q^n = \left(\phi(q^2)\phi(q^6) + 4q^2\psi(q^4)\psi(q^{12})\right)$$
$$\times \left(\phi(q)\phi(q^3) + 4q\psi(q^2)\psi(q^6)\right),$$
$$\sum_{n=0}^{\infty} N(1,2,3,6;8n+4)q^n = 6\psi(q)\psi(q^3)\left(\phi(q)\phi(q^3) + 4q\psi(q^2)\psi(q^6)\right).$$

Thus,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(N(1,2,3,6;8n+4) - 3N(1,2,3,6;2n+1)\right) q^n \\
&= 24q\psi(q)\psi(q^2)\psi(q^3)\psi(q^6) \\
&= 24\sum_{n=0}^{\infty} T(1,2,3,6;n)q^{n+1},
\end{aligned}
$$

which readily gives (5.1.10) to complete the proof.

Since the proofs of Theorem 5.1.8 and Theorem 5.1.9 are similar, so we omit them.