Chapter 1

Introduction

In this thesis, we find arithmetic properties of a partition function, several results on vanishing coefficients in infinite series expansions and various relations between sums of squares and sums of triangular numbers. We use t-dissections of q-products and Ramanujan's theta functions. The thesis comprised of five chapters, including this introductory chapter. In this chapter, we present some basic material on q-products, partitions and other relevant topics. An outline of the work done in the subsequent chapters are also presented here.

1.1 The partition function and q-products

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer n is a finite sequence of non-increasing positive integer *parts* λ_i such that $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$.

The partition function $p(n)$ is the number of partitions of a non-negative integer n, with the convention that $p(0) = 1$. For example, we have $p(4) = 5$, as there are five partitions of 4, namely,

$$(4),\ (3,1),\ (2,2),\ (2,1,1) \text{ and } (1,1,1,1).$$

The generating function for $p(n)$, due to Euler, is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, for any complex number a and q, with $|q| < 1$, we define

$$(a;q)_0 := 1,$$
$$(a;q)_n := \prod_{m=0}^{n-1}(1 - aq^m), \quad n \geq 1,$$
$$(a;q)_\infty := \prod_{m=0}^{\infty}(1 - aq^m).$$

Throghout the thesis, for convenience, we will use $f_k = (q^k;q^k)_\infty$.

1.2 Ramanujan's theta functions and t-dissections

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1.$$

Three special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.2.1}$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{f_2^2}{f_1}, \tag{1.2.2}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2} = f_1, \tag{1.2.3}$$

where the product representations arise from Jacobi's famous triple product identity [21, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{1.2.4}$$

If $P(q)$ denotes a power series in q, then a t-dissection of $P(q)$ is given by

$$P(q) = \sum_{k=0}^{t-1} q^k P_k(q^t),$$

where P_k are power series in q^t.

For example, the 3-dissections of $\phi(-q)$ and $\psi(q)$ are given by [27, p. 132, Eqs. (14.3.2) and (14.3.3)]

$$\phi(-q) = \frac{(q^9;q^9)_\infty^2}{(q^{18};q^{18})_\infty} - 2q\frac{(q^3;q^3)_\infty(q^{18};q^{18})_\infty^2}{(q^6;q^6)_\infty(q^9;q^9)_\infty}$$

and

$$\psi(q) = \frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty} + q\frac{(q^{18};q^{18})_\infty^2}{(q^9;q^9)_\infty}.$$

In the remaining sections of this chapter, we review the literature and briefly outline the work done in the subsequent chapters of the thesis.

1.3 Partition functions $\mathbf{PD_t}(n)$ and $\mathbf{PDO_t}(n)$

In [12], Andrews, Lewis and Lovejoy introduced and studied a new class of partitions, partitions with designated summands. Partitions with designated summands are constructed by taking ordinary partitions and tagging exactly one of each part size. For example, there are 10 partitions of 4 with designated summands, namely,

$$4', \quad 3'+1', \quad 2'+2, \quad 2+2', \quad 2'+1'+1, \quad 2'+1+1',$$

$$1'+1+1+1, \quad 1+1'+1+1, \quad 1+1+1'+1, \quad 1+1+1+1'.$$

The total number of partitions of n with designated summands is denoted by $\mathrm{PD}(n)$. Hence, $\mathrm{PD}(4) = 10$. Andrews, Lewis and Lovejoy [12] also studied $\mathrm{PDO}(n)$, the number of partitions of n with designated summands in which all parts are odd. From the above example, $\mathrm{PDO}(4) = 5$. Further studies on $\mathrm{PD}(n)$ and $\mathrm{PDO}(n)$ were carried out by Chen, Ji, Jin, and Shen [24], Baruah and Ojah [20], and Xia [42].

Recently, Lin [31] introduced two new partition functions $\mathrm{PD_t}(n)$ and $\mathrm{PDO_t}(n)$, which count the total number of tagged parts over all partitions of n with designated summands and the total number of tagged parts over all partitions of n

with designated summands in which all parts are odd, respectively. From the partitions of 4 with designated summands given above, we note that $\mathrm{PD_t}(4) = 13$ and $\mathrm{PDO_t}(4) = 6$. Lin [31] proved that the generating functions of $\mathrm{PD_t}(n)$ and $\mathrm{PDO_t}(n)$ are

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(n) q^n = \frac{1}{2}\left(\frac{f_3^5}{f_1^3 f_6^2} - \frac{f_6}{f_1 f_2 f_3}\right) \tag{1.3.1}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO_t}(n) q^n = \frac{q f_2 f_3^2 f_{12}^2}{f_1^2 f_6}. \tag{1.3.2}$$

Lin [31] also derived several congruences modulo small powers of 3 for $\mathrm{PD_t}(n)$ and $\mathrm{PDO_t}(n)$. For example, for any nonnegative integers n and k,

$$\begin{aligned}
\mathrm{PD_t}(3n) &\equiv 0 \pmod 3, \\
\mathrm{PD_t}(3n+2) &\equiv 0 \pmod 3, \\
\mathrm{PD_t}(36n+21) &\equiv 0 \pmod 9, && (1.3.3) \\
\mathrm{PD_t}(36n+33) &\equiv 0 \pmod 9, && (1.3.4) \\
\mathrm{PD_t}(48n+20) &\equiv 0 \pmod 9, \\
\mathrm{PD_t}(48n+36) &\equiv 0 \pmod 9, \\
\mathrm{PD_t}(72n+42) &\equiv 0 \pmod 9, \\
\mathrm{PD_t}(72n+66) &\equiv 0 \pmod 9, \\
\mathrm{PDO_t}(8n) &\equiv 0 \pmod 9, \\
\mathrm{PDO_t}(24n) &\equiv 0 \pmod{27}, \\
\mathrm{PDO_t}(36n) &\equiv 0 \pmod{27}, \\
\mathrm{PDO_t}(36n+24) &\equiv 0 \pmod{27}, \\
\mathrm{PDO_t}(8 \cdot 5^{2k+1}(30n+6a+5)) &\equiv 0 \pmod{27},
\end{aligned}$$

where $a = 1, 2, 3, 4$.

Very recently, Adansie, Chern and Xia [1] found the following two infinite families of congruences modulo 9.

For any nonnegative integers n and k,

$$\mathrm{PD_t}(3^{2k+1}(9n+2)) \equiv 0 \pmod 9$$

and

$$\mathrm{PD_t}((3^{2k+1}(9n+7)) \equiv 0 \pmod 9.$$

By analyzing a large number of values of $\mathrm{PD_t}(n)$ and $\mathrm{PDO_t}(n)$ via MAPLE, Lin [31] further speculated the existence of congruences modulo small powers of 2. For example, he conjectured that, for any nonnegative integer n,

$$\mathrm{PD_t}(48n+28) \equiv 0 \pmod 8, \tag{1.3.5}$$

$$\mathrm{PD_t}(48n+46) \equiv 0 \pmod 8, \tag{1.3.6}$$

$$\mathrm{PDO_t}(8n+6) \equiv 0 \pmod 8, \tag{1.3.7}$$

and

$$\mathrm{PDO_t}(8n+7) \equiv 0 \pmod 8. \tag{1.3.8}$$

In Chapter 2 of this thesis, we prove the above congruences. In fact, we find the exact generating functions of $\mathrm{PDO_t}(8n+6)$ and $\mathrm{PDO_t}(8n+7)$ that immediately imply (1.3.7) and (1.3.8), respectively. We also find many new congruences and infinite families of congruences for $\mathrm{PD_t}(n)$ modulo 2 and 4.

1.4 Infinite series expansions with vanishing coefficients

In 1978, Richmond and Szekeres [33] proved that if

$$\sum_{n=0}^{\infty} \alpha_n q^n := \frac{(q^3, q^5; q^8)_\infty}{(q, q^7; q^8)_\infty} \quad \text{and} \quad \sum_{n=0}^{\infty} \beta_n q^n := \frac{(q, q^7; q^8)_\infty}{(q^3, q^5; q^8)_\infty},$$

then the coefficients α_{4n+3} and β_{4n+2} always vanish. They also conjectured that if

$$\sum_{n=0}^{\infty} \gamma_n q^n := \frac{(q^5, q^7; q^{12})_\infty}{(q, q^{11}; q^{12})_\infty} \quad \text{and} \quad \sum_{n=0}^{\infty} \delta_n q^n := \frac{(q, q^{11}; q^{12})_\infty}{(q^5, q^7; q^{12})_\infty},$$

then γ_{6n+5} and δ_{6n+3} vanish.

In [11], Andrews and Bressoud proved the following general theorem, which contains the results of Richmond and Szekeres as special cases.

Theorem 1.4.1. (Andrews and Bressoud) *If $1 \leq r < k$ are relatively prime integers of opposite parity and*

$$\frac{(q^r, q^{2k-r}; q^{2k})_\infty}{(q^{k-r}, q^{k+r}; q^{2k})_\infty} =: \sum_{n=0}^{\infty} \phi_n q^n,$$

then $\phi_{kn+r(k-r+1)/2}$ is always zero.

In [10], Alladi and Gordon generalized the above theorem as follows:

Theorem 1.4.2. (Alladi and Gordon) *Let $1 < m < k$ and let $(s, km) = 1$ with $1 \leq s < mk$. Let $r^* = (k-1)s$ and $r \equiv r^* \bmod mk$ with $1 \leq r < mk$. Put $r' = \lceil r^*/mk \rceil \bmod k$ with $1 \leq r' < k$. Write*

$$\frac{(q^r, q^{mk-r}; q^{mk})_\infty}{(q^s, q^{mk-s}; q^{mk})_\infty} =: \sum_{n=0}^{\infty} \mu_n q^n.$$

Then $\mu_n = 0$ for $n \equiv rr' \bmod k$.

They also proved the following companion result to Theorem 1.4.2.

Theorem 1.4.3. (Alladi and Gordon) *Let m, k, s, r^*, r and r' be defined as in Theorem 1.4.2 with k odd. Write*

$$\frac{(q^r, q^{mk-r}; q^{mk})_\infty}{(-q^s, -q^{mk-s}; q^{mk})_\infty} =: \sum_{n=0}^{\infty} \mu'_n q^n.$$

Then $\mu'_n = 0$ for $n \equiv rr' \bmod k$.

The result of Alladi and Gordon in Theorem 1.4.2 does not provide any information about vanishing coefficients in the cases where $k < m$ or $k = m$. In [32], Mc Laughlin proved the following theorem which covers the cases $k \leq m$ as well.

Theorem 1.4.4. (Mc Laughlin) *Let $k > 1$, $m > 1$ be positive integers. Let $r = sm + t$, for some integers s and t, where $0 \leq s < k$, $1 \leq t < m$ and r and k are relatively prime. Let*

$$\frac{(q^{r-tk}, q^{mk-(r-tk)}; q^{mk})_\infty}{(q^r, q^{mk-r}; q^{mk})_\infty} =: \sum_{n=0}^{\infty} \nu_n q^n;$$

then ν_{kn-rs} is always zero.

He also found the following companion result to Theorem 1.4.4.

Theorem 1.4.5. (Mc Laughlin) *Let $k > 1$, $m > 1$ be positive integers with k odd. Let $r = sm + t$, for some integers s and t, where $0 \leq s < k$, $1 \leq t < m$ and r and k are relatively prime. Let*

$$\frac{(q^{r-tk}, q^{mk-(r-tk)}; q^{mk})_\infty}{(-q^r, -q^{mk-r}; q^{mk})_\infty} =: \sum_{n=0}^{\infty} \nu'_n q^n;$$

then ν'_{kn-rs} is always zero.

All the proofs of the above theorems use Ramanujan's well-known ${}_1\psi_1$ summation formula. Very recently, Hirschhorn [28] proved the following interesting result by using only Jacobi triple product identity and elementary q-series manipulations.

Theorem 1.4.6. (Hirschhorn) *If*

$$\sum_{n=0}^{\infty} a_n q^n := (-q, -q^4; q^5)_\infty (q, q^9; q^{10})^3_\infty \text{ and } \sum_{n=0}^{\infty} b_n q^n := (-q^2, -q^3; q^5)_\infty (q^3, q^7; q^{10})^3_\infty,$$

then

$$a_{5n+2} = a_{5n+4} = 0 \tag{1.4.1}$$

and

$$b_{5n+1} = b_{5n+4} = 0. \tag{1.4.2}$$

Note that the forms of the q-products in Theorem 1.4.6 are quite different from those in Theorems 1.4.1–1.4.5.

Motivated by the work of Hirschhorn [28], Tang [36] found more results on vanishing coefficients in some other comparable q-series expansions. In particular, Tang [36] proved the following theorem.

Theorem 1.4.7. (Tang) *If*

$$\sum_{n=0}^{\infty} c_n q^n := (-q,-q^4;q^5)_\infty^3 (q^3,q^7;q^{10})_\infty \text{ and } \sum_{n=0}^{\infty} d_n q^n := (-q^2,-q^3;q^5)_\infty^3 (q,q^9;q^{10})_\infty,$$

then

$$c_{5n+3} = c_{5n+4} = 0 \tag{1.4.3}$$

and

$$d_{5n+3} = d_{5n+4} = 0. \tag{1.4.4}$$

In Chapter 3 of our thesis, we prove the following results.

If a_n, b_n, c_n and d_n are as defined in the previous two theorems, then

$$\sum_{n=0}^{\infty} b(5n)q^n - \sum_{n=0}^{\infty} a(5n-2)q^n = \frac{f_1^4}{f_2^4},$$

$$b_{5n+1} = a_{5n-1},$$

$$b_{5n+2} = a_{5n},$$

$$b_{5n+3} = a_{5n+1},$$

$$b_{5n+4} = a_{5n+2},$$

$$c_{5n} = d_{5n},$$

$$c_{5n+2} = d_{5n+2},$$

$$c_{5n+3} = d_{5n+3},$$

$$c_{5n+4} = d_{5n+4},$$

and

$$\sum_{n=0}^{\infty} c_{5n+1}q^n - \sum_{n=0}^{\infty} d_{5n+1}q^n = 4\frac{f_2^4}{f_1^4}.$$

Furthermore, we notice from the above that instead of proving both (1.4.1) and (1.4.2) by Hirschhorn [28], it would have been enough to prove only one of (1.4.1) or (1.4.2). Similarly, instead of proving both (1.4.3) and (1.4.4) by Tang [36], it would have been enough to prove only one of (1.4.3) or (1.4.4). It also follows from the last identity that $c_{5n+1} > d_{5n+1}$.

In addition to the above results, we prove several new results in that chapter.

1.5 Relations between sums of squares and sums of triangular numbers

Let $\mathbb{N}^+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of positive integers, the set of nonnegative integers, and the set of integers, respectively. For an integer $\ell \geq 2$, let $\mathbb{N}^\ell = \mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}(\ell$ times) and $\mathbb{Z}^\ell = \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}(\ell$ times). For $a_1, a_2, \ldots, a_\ell \in \mathbb{N}^+$ and $n \in \mathbb{N}$, define

$$N(a_1, a_2, \ldots, a_\ell; n) := \left|\left\{(x_1, x_2, \ldots, x_\ell) \in \mathbb{Z}^\ell : a_1x_1^2 + a_2x_2^2 + \cdots + a_\ell x_\ell^2 = n\right\}\right|$$

and

$$\begin{aligned}T(a_1, a_2, \ldots, a_\ell; n) := \Bigg|\Bigg\{&(x_1, x_2, \ldots, x_\ell) \in \mathbb{N}^\ell : a_1\frac{x_1(x_1+1)}{2} + a_2\frac{x_2(x_2+1)}{2} \\ &+ \cdots + x_\ell\frac{x_\ell(x_\ell+1)}{2} = n\Bigg\}\Bigg|,\end{aligned}$$

where we take $N(a_1, a_2, \ldots, a_\ell; 0) = T(a_1, a_2, \ldots, a_\ell; 0) = 1$.

From the above definitions and the definitions of ϕ and ψ in (1.2.1) and (1.2.2), it is clear that

$$\sum_{n=0}^{\infty} N(a_1, a_2, \ldots, a_\ell; n)q^n = \phi(q^{a_1})\phi(q^{a_2})\cdots\phi(q^{a_\ell})$$

and

$$\sum_{n=0}^{\infty} T(a_1, a_2, \ldots, a_\ell; n) q^n = \psi(q^{a_1})\psi(q^{a_2}) \cdots \psi(q^{a_\ell}).$$

Jacobi and Legendre proved that

$$N(1, 1, 1, 1; n) = 8 \sum_{d|n, 4\nmid d} d$$

and

$$T(1, 1, 1, 1; n) = \sigma(2n + 1),$$

respectively, where $\sigma(n) = \sum_{d|n} d$.

For further formulas for $N(a, b, c, d; n)$ and $T(a, b, c, d; n)$ for certain values of $a, b, c, d \in \mathbb{N}^+$, we refer to Dickson's historical comments [11], Cooper's papers [25, 26], Alaca's papers [3, 4], papers [5] – [9] by Alaca, Alaca, Lemire and Williams, Williams' papers [39, 40] and book [41], and papers [37, 38] by Wang and Sun.

Finding relations between $N(a, b, c, d; n)$ and $T(a, b, c, d; n)$ is another interesting area of research. For $a, b, c, d \in \mathbb{N}^+$ with $5 \le a + b + c + d \le 8$, let

$$C(a, b, c, d) = 16 + 4i_1(i_1 - 1)i_2 + 8i_1 i_3,$$

where i_j is the number of elements in $\{a, b, c, d\}$ which are equal to j. When $5 \le a + b + c + d \le 7$, Adiga, Cooper and Han [2] proved that

$$C(a, b, c, d)T(a, b, c, d; n) = N(a, b, c, d; 8n + a + b + c + d).$$

When $a + b + c + d = 8$, Baruah, Cooper and Hirschhorn [15] proved that

$$C(a, b, c, d)T(a, b, c, d; n) = N(a, b, c, d; 8n + 8) - N(a, b, c, d; 2n + 2).$$

Wang and Sun [37, 38] and Sun [34] discovered several new relations between $N(a, b, c, d; n)$ and $T(a, b, c, d; n)$. In particular, in [34], Sun posed 23 conjectures (Conjecture 2.1 – Conjecture 2.23) stating some relations between $N(a, b, c, d; n)$

and $T(a,b,c,d;n)$. Five of the conjectures (Conjectures 2.2, 2.3, 2.4, 2.7, and 2.11) were proved by Yao [45] by utilizing (p,k)-parametrization of theta functions. In Section 8 of [45], Yao also remarked that Conjectures 2.1, 2.14, 2.15, 2.19, 2.20, and 2.21 can also be proved in a similar way. In fact, in another recent paper, Yao [46] proved some general relations from which Conjectures 2.1, 2.15, 2.19, 2.20, and 2.21 follow as special cases. Recently, Sun [35] himself confirmed Conjectures 2.2 and 2.6 – 2.8 by proving the following general result.

Let $m \equiv 1 \pmod 4$ or $m \equiv 4 \pmod 8$. Suppose that there is an odd prime divisor p of m such that $\left(\frac{4n+5}{p}\right) = -1$, where $\left(\frac{a}{p}\right)$ is the Legendre symbol. Then

$$32T(1,1,8,m;n) = N(1,1,8,m;8n+10+m).$$

Most recently, Xia and Zhong [44] proved Conjectures 2.18, 2.22, and 2.23 by using theta function identities.

In Chapter 4 of the thesis, we prove the remaining seven conjectures, namely, Conjectures 2.5, 2.9, 2.10, 2.12, 2.13, 2.16, and 2.17, of Sun [34].

In another paper [35], Sun also found several more relations between N and T and posed seven more open conjectures (Conjecture 6.1 – Conjecture 6.7). Five are on ternary quadratic forms and two are on quaternary quadratic forms. The five conjectures on the ternary case are proved in [30] by an elementary method. Xia and Yan [43] proved Conjecture 6.1 – Conjecture 6.6.

In the final chapter of our thesis, we give alternative proofs of three of the conjectures of Sun [35] that were proved by Xia and Yan [43] and also prove the remaining conjecture, i.e., Conjecture 6.7, in [35]. Furthermore, we prove some new relations between $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$.