Chapter 2

Number of tagged parts over the partitions with designated summands

2.1 Introduction

In this chapter, we prove the congruences (1.3.5)-(1.3.8) and also find several new congruences and infinite families of congruences modulo 2 and 4.

The following theorem states the exact generating functions of $\mathrm{PDO_t}(8n+6)$ and $\mathrm{PDO_t}(8n+7)$ that immediately implies the congruences (1.3.7) and (1.3.8).

Theorem 2.1.1. *For any nonnegative integer n, we have*

$$\sum_{n=0}^{\infty} \mathrm{PDO_t}(8n+6)q^n = 8\left(2\frac{f_2^{16}f_6^{10}}{f_1^{17}f_3^{3}f_{12}^{4}} - q\frac{f_2^{28}f_3f_{12}^{4}}{f_1^{21}f_6^{2}f_4^{8}} - 16q^2\frac{f_2^{4}f_3f_4^{8}f_{12}^{4}}{f_1^{13}f_6^{2}}\right) \tag{2.1.1}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PDO_t}(8n+7)q^n = 8\left(\frac{f_2^{14}f_3f_6^{4}f_8^{2}}{f_1^{14}f_4^{3}f_{12}^{2}} + 2\frac{f_2^{9}f_3^{2}f_4^{5}f_6}{f_1^{13}f_8^{2}} + 4q\frac{f_2^{8}f_3^{3}f_4f_8^{2}f_{12}^{2}}{f_1^{12}f_6^{2}}\right). \tag{2.1.2}$$

In the next theorem and corollary, we present our new congruences and infinite families of congruences modulo 2 and 4 for $\mathrm{PD_t}(n)$.

Theorem 2.1.2. *For any nonnegative integers k, ℓ and n, we have*

$$\mathrm{PD_t}(24n+12) \equiv 0 \pmod{2}, \tag{2.1.3}$$

$$\mathrm{PD_t}(24n+21) \equiv 0 \pmod{2}, \tag{2.1.4}$$

$$\mathrm{PD_t}(48n+30) \equiv 0 \pmod{2}, \tag{2.1.5}$$

$$\mathrm{PD_t}(144n+102) \equiv 0 \pmod{2}, \tag{2.1.6}$$

$$\mathrm{PD_t}(216n+153) \equiv 0 \pmod{2}, \tag{2.1.7}$$

$$\mathrm{PD_t}(36n+21) \equiv 0 \pmod{4}, \tag{2.1.8}$$

$$\mathrm{PD_t}(36n+33) \equiv 0 \pmod{4}, \tag{2.1.9}$$

$$\mathrm{PD_t}(2^{2k} \cdot 12n) \equiv \mathrm{PD_t}(12n) \pmod{4}, \tag{2.1.10}$$

$$\mathrm{PD_t}(3^{\ell} \cdot 2^{2k}(24n+12)) \equiv \mathrm{PD_t}(24n+12) \pmod{4}, \tag{2.1.11}$$

$$\mathrm{PD_t}(96n+60) \equiv 0 \pmod{4}, \tag{2.1.12}$$

$$\mathrm{PD_t}(96n+84) \equiv 0 \pmod{4}, \tag{2.1.13}$$

$$\mathrm{PD_t}(144n+84) \equiv 0 \pmod{4}, \tag{2.1.14}$$

$$\mathrm{PD_t}(144n+120) \equiv 0 \pmod{4}, \tag{2.1.15}$$

$$\mathrm{PD_t}(144n+132) \equiv 0 \pmod{4}, \tag{2.1.16}$$

$$\mathrm{PD_t}(3^{k}(288n+204)) \equiv \mathrm{PD_t}(288n+204) \equiv 0 \pmod{4}, \tag{2.1.17}$$

$$\mathrm{PD_t}(864n+792) \equiv 0 \pmod{4}, \tag{2.1.18}$$

$$\mathrm{PD_t}(1728n+1224) \equiv 0 \pmod{4}, \tag{2.1.19}$$

$$\mathrm{PD_t}(2592n+1080) \equiv 0 \pmod{4}, \tag{2.1.20}$$

$$\mathrm{PD_t}(36n+30) \equiv 0 \pmod{4}, \tag{2.1.21}$$

$$\mathrm{PD_t}(108n+90) \equiv 0 \pmod{4}, \tag{2.1.22}$$

$$\mathrm{PD_t}(3^{2k}(12n+6)) \equiv \mathrm{PD_t}(12n+6) \pmod{4}. \tag{2.1.23}$$

Corollary 2.1.3. *For any positive integers k, ℓ and any nonnegative integer n, we have*

$$\mathrm{PD_t}(3^\ell \cdot 2^{2k}(8n+5)) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(3^\ell \cdot 2^{2k}(8n+7)) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(3^\ell \cdot 2^{2k}(12n+7)) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(3^\ell \cdot 2^{2k}(12n+11)) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(3 \cdot 2^{2k+1}(6n+5)) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(3^{\ell+1} \cdot 2^{2k}(24n+17)) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(3^2 \cdot 2^{2k+1}(12n+11)) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(3^2 \cdot 2^{2k+1}(24n+17)) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(3^3 \cdot 2^{2k+1}(12n+5)) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(2 \cdot 3^k(6n+5)) \equiv 0 \pmod 4.$$

Proof. Congruences (2.1.12)– (2.1.14) and (2.1.16) may be rewritten as

$$\mathrm{PD_t}(24(4n+2)+12) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(24(4n+3)+12) \equiv 0 \pmod 4,$$

$$\mathrm{PD_t}(24(6n+3)+12) \equiv 0 \pmod 4,$$

and

$$\mathrm{PD_t}(24(6n+5)+12) \equiv 0 \pmod 4,$$

respectively. From (2.1.11) and the above congruences, we easily arrive at the first four infinite families of congruences of the corollary. Since the other congruences can also be proved in a similar way, we omit the details. □

In Section 2.2, we present some preliminary results. In Section 2.3, we prove Theorem 2.1.1 whereas Section 2.4 is devoted to proving the congruences (1.3.5) and (1.3.6). In Section 2.5, we prove Theorem 2.1.2.

2.2 Preliminary results

In the following lemma we state some useful 2-dissections.

Lemma 2.2.1. *We have*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.2.1}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{2.2.2}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.2.3}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{2.2.4}$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}, \tag{2.2.5}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{2.2.6}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{2.2.7}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{2.2.8}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.2.9}$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}, \tag{2.2.10}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{2.2.11}$$

Proof. Identities (2.2.1) and (2.2.3) are the 2-dissections of $\varphi(q)$ and $\varphi(q^2)$ (see [27, Eqs. (1.9.4) and (1.10.1)]). Replacing q by $-q$ in (2.2.1) and (2.2.3), and then using

$$(-q;-q)_\infty = \frac{f_2^2}{f_1 f_4}, \tag{2.2.12}$$

we readily arrive at (2.2.2) and (2.2.4), respectively. Identities (2.2.5), (2.2.6), (2.2.7), (2.2.9), (2.2.10), and (2.2.11) are Eqs. (30.12.1), (30.12.3), (22.1.13), (22.1.14), (30.10.2), and (30.10.4), respectively, in [27]. Finally, (2.2.8) follows from (2.2.7) by replacing q by $-q$ and then using (2.2.12). □

The next lemma we present some useful 3-dissections.

Lemma 2.2.2. *We have*

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{2.2.13}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{2.2.14}$$

$$f_1^3 = f_3 a(q^3) - 3q f_9^3, \tag{2.2.15}$$

$$\frac{1}{f_1^3} = a^2(q^3)\frac{f_9^3}{f_3^{10}} + 3qa(q^3)\frac{f_9^6}{f_3^{11}} + 9q^2\frac{f_9^9}{f_3^{12}}, \tag{2.2.16}$$

$$\frac{1}{f_1 f_2} = a(q^6)\frac{f_9^3}{f_3^4 f_6^3} + qa(q^3)\frac{f_{18}^3}{f_3^3 f_6^4} + 3q^2\frac{f_9^3 f_{18}^3}{f_3^4 f_6^4}, \tag{2.2.17}$$

where

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = 1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right).$$

Proof. The first identity is equivalent to the 3-dissection of $\varphi(-q)$ (see [27, Eq. (14.3.2)]). The second can be obtained from the first by replacing q with ωq and $\omega^2 q$ and then multiplying the two results, where ω is a primitive cube root of unity. Identities (2.2.15), (2.2.16) and (2.2.17) are in [27, Eqs. (21.3.1), (39.2.8) and (22.9.4)]. $\square$

We also recall the following useful results from [27, Eqs. (22.1.12), (22.11.8) and (22.11.9)], where the first is a 2-dissection of $a(q)$:

$$a(q) = a(q^4) + 6q\frac{f_4^2 f_{12}^2}{f_2 f_6},$$

$$a(q) + 2a(q^2) = 3\frac{f_2 f_3^6}{f_1^2 f_6^3},$$

$$a(q) + a(q^2) = 2\frac{f_2^6 f_3}{f_1^3 f_6^2}.$$

We end this section by noting the following congruences which can be easily established:

$$a(q) \equiv 1 \pmod 2,$$

$$\begin{aligned}a^2(q)&\equiv 1 \pmod 4,\\ f_1^2&\equiv f_2 \pmod 2,\\ f_1^4&\equiv f_2^2 \pmod 4,\\ f_1^8&\equiv f_2^4 \pmod 8.\end{aligned}$$

We will use the identities and congruences of this section in the subsequent sections without referring to these.

2.3 Proof of Theorem 2.1.1

We have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{PDO_t}(n)q^n &= q\frac{f_2f_{12}^2}{f_6}\cdot\frac{f_3^2}{f_1^2}\\ &= q\frac{f_2f_{12}^2}{f_6}\left(\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}\right),\end{aligned} \tag{2.3.1}$$

from which we extract

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{PDO_t}(2n)q^n &= 2qf_2f_4f_6f_{12}\cdot\frac{f_3}{f_1^3}\\ &= 2qf_2f_4f_6f_{12}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right).\end{aligned}$$

From the above, we extract

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathrm{PDO_t}(4n+2)q^n\\ &= 2\frac{f_2^7}{f_6}\cdot\frac{1}{f_1^8}\cdot f_3^4\\ &= 2\frac{f_2^7}{f_6}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2\left(\frac{f_{12}^{10}}{f_6^2f_{24}^4}-4q^3\frac{f_6^2f_{24}^4}{f_{12}^2}\right)\\ &= 2\frac{f_4^{28}f_{12}^{10}}{f_2^{21}f_6^3f_8^8f_{24}^4}+16q\frac{f_4^{16}f_{12}^{10}}{f_2^{17}f_6^3f_{24}^4}+32q^2\frac{f_4^4f_8^8f_{12}^{10}}{f_2^{13}f_6^3f_{24}^4}\\ &\quad -8q^3\frac{f_4^{28}f_6f_{24}^4}{f_2^{21}f_8^8f_{12}^2}-64q^4\frac{f_4^{16}f_6f_{24}^4}{f_2^{17}f_{12}^2}-128q^5\frac{f_4^4f_6f_8^8f_{24}^4}{f_2^{13}f_{12}^2},\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} \mathrm{PDO_t}(8n+6)q^n = 16\frac{f_2^{16}f_6^{10}}{f_1^{17}f_3^3f_{12}^4} - 8q\frac{f_2^{28}f_3f_{12}^4}{f_1^{21}f_4^8f_6^2} - 128q^2\frac{f_2^4f_3f_4^8f_{12}^4}{f_1^{13}f_6^2},$$

which is (2.1.1).

Next, from (2.3.1) we also extract

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PDO_t}(2n+1)q^n &= \frac{f_2^4f_6^4}{f_4f_{12}} \cdot \frac{1}{f_1^4} \\ &= \frac{f_2^4f_6^4}{f_4f_{12}} \cdot \left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right),\end{aligned}$$

from which we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathrm{PDO_t}(4n+3)q^n \\ &= 4\frac{f_2f_4^4}{f_6} \cdot \frac{1}{f_1^2} \cdot \frac{f_3^4}{f_1^4} \\ &= 4\frac{f_2f_4^4}{f_6}\left(\frac{f_8^5}{f_2^5f_{16}^2} + 2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right)\left(\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}} + 2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}\right)^2 \\ &= 4\frac{f_4^{12}f_6f_8^3f_{12}^4}{f_2^{14}f_{16}^2f_{24}^2} + 16q\frac{f_4^9f_6^2f_8^5f_{12}}{f_2^{13}f_{16}^2} + 8q\frac{f_4^{14}f_6f_{12}^4f_{16}^2}{f_2^{14}f_8^3f_{24}^2} \\ &\quad + 16q^2\frac{f_4^6f_6^3f_8^7f_{24}^2}{f_2^{12}f_{12}^2f_{16}^2} + 32q^2\frac{f_4^{11}f_6^2f_{12}f_{16}^2}{f_2^{13}f_8} + 32q^3\frac{f_4^8f_6^3f_8f_{16}^2f_{24}^2}{f_2^{12}f_{12}^2},\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} \mathrm{PDO_t}(8n+7)q^n = 8\frac{f_2^{14}f_3f_6^4f_8^2}{f_1^{14}f_4^3f_{12}^2} + 16\frac{f_2^9f_3^2f_4^5f_6}{f_1^{13}f_8^2} + 32q\frac{f_2^8f_3^3f_4f_8^2f_{12}^2}{f_1^{12}f_6^2},$$

which is (2.1.2).

2.4 Proofs of (1.3.5) and (1.3.6)

We have

$$\begin{aligned}2\sum_{n=0}^{\infty} \mathrm{PD_t}(n)q^n &= \frac{f_3^5}{f_6^2} \cdot \frac{1}{f_1^3} - \frac{f_6}{f_3} \cdot \frac{1}{f_1f_2} \\ &= \frac{f_3^5}{f_6^2}\left(a^2(q^3)\frac{f_9^3}{f_3^{10}} + 3qa(q^3)\frac{f_9^6}{f_3^{11}} + 9q^2\frac{f_9^9}{f_3^{12}}\right)\end{aligned}$$

$$-\frac{f_6}{f_3}\left(a(q^6)\frac{f_9^3}{f_3^4f_6^3}+qa(q^3)\frac{f_{18}^3}{f_3^3f_6^4}+3q^2\frac{f_9^3f_{18}^3}{f_3^4f_6^4}\right), \tag{2.4.1}$$

from which we extract

$$\begin{aligned}
2\sum_{n=0}^{\infty}\mathrm{PD_t}(3n+1)q^n &= 3a(q)\frac{f_3^6}{f_1^6f_2^2}-a(q)\frac{f_6^3}{f_1^4f_2^3}\\
&=\left(3\frac{f_3^6}{f_1^6f_2^2}-\frac{f_6^3}{f_1^4f_2^3}\right)a(q)\\
&=\frac{f_6^3}{f_1^4f_2^3}\left(3\frac{f_2f_3^6}{f_1^2f_6^3}-1\right)a(q)\\
&=\frac{f_6^3}{f_1^4f_2^3}\big(a(q)+2a(q^2)-1\big)\,a(q)\\
&=\frac{f_6^3}{f_1^4f_2^3}\big(a(q)+a(-q)+2a(q^2)-1-a(-q)\big)\,a(q)\\
&=\frac{f_6^3}{f_1^4f_2^3}\big(2a(q^4)+2a(q^2)-1-a(-q)\big)\,a(q)\\
&=\frac{f_6^3}{f_1^4f_2^3}\left(4\frac{f_4^6f_6}{f_2^3f_{12}^2}-1-a(-q)\right)a(q)\\
&=\frac{f_6^3}{f_1^4f_2^3}\left(\left(4\frac{f_4^6f_6}{f_2^3f_{12}^2}-1\right)\left(a(q^4)+6q\frac{f_4^2f_{12}^2}{f_2f_6}\right)-a(-q)a(q)\right)\\
&=\frac{f_6^3}{f_1^4f_2^3}\Bigg(\left(4\frac{f_4^6f_6}{f_2^3f_{12}^2}-1\right)\left(a(q^4)+6q\frac{f_4^2f_{12}^2}{f_2f_6}\right)\\
&\quad-\left(a^2(q^4)-36q^2\frac{f_4^4f_{12}^4}{f_2^2f_6^2}\right)\Bigg)\\
&=\frac{f_6^3}{f_2^3}\Bigg(\left(4\frac{f_4^6f_6}{f_2^3f_{12}^2}a(q^4)-a(q^4)-a^2(q^4)+36q^2\frac{f_4^4f_{12}^4}{f_2^2f_6^2}\right)\\
&\quad+q\left(24\frac{f_4^8}{f_2^4}-6\frac{f_4^2f_{12}^2}{f_2f_6}\right)\Bigg)\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right).
\end{aligned}$$

Extracting the terms involving q^{2n+1} from both sides of the above and then dividing by 2, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathrm{PD_t}(6n+4)q^n &= \frac{f_3^3}{f_1^3}\Bigg(2\frac{f_2^2f_4^4}{f_1^{10}}\left(4\frac{f_2^6f_3}{f_1^3f_6^2}a(q^2)-a(q^2)-a^2(q^2)+36q\frac{f_2^4f_6^4}{f_1^2f_3^2}\right)\\
&\quad+\frac{f_2^{14}}{f_1^{14}f_4^4}\left(12\frac{f_2^8}{f_1^4}-3\frac{f_2^2f_6^2}{f_1f_3}\right)\Bigg).
\end{aligned}$$

Taking congruences modulo 8, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathrm{PD_t}(6n+4)q^n \\
&\equiv \frac{f_3^3}{f_1^3}\left(6\frac{f_2^2f_4^4}{f_1^{10}}\left(a(q^2)+a^2(q^2)\right)+4\frac{f_2^{22}}{f_1^{18}f_4^4}+5\frac{f_2^{16}f_6^2}{f_1^{15}f_3f_4^4}\right) \\
&\equiv \frac{f_3^3}{f_1^3}\left(6\frac{f_2^2f_4^4}{f_1^{10}}a(q^2)+6\frac{f_2^2f_4^4}{f_1^{10}}+4\frac{f_2^{22}}{f_1^{18}f_4^4}+5\frac{f_2^{16}f_6^2}{f_1^{15}f_3f_4^4}\right) \\
&\equiv \left(6f_4^2f_6^2a(q^2)+10f_4^2f_6^2\right)\cdot\frac{1}{f_1f_3}+5f_6^2\cdot\frac{f_3^2}{f_1^2} \\
&\equiv \left(6f_4^2f_6^2a(q^2)+10f_4^2f_6^2\right)\left(\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}+q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}\right) \\
&\quad+5f_6^2\left(\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}\right) \pmod 8. \qquad (2.4.2)
\end{aligned}$$

We extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathrm{PD_t}(12n+4)q^n \\
&\equiv \left(6f_2^2f_3^2a(q)+10f_2^2f_3^2\right)\frac{f_4^2f_6^5}{f_1^2f_2f_3^4f_{12}^2}+5\frac{f_2^4f_3^3f_6^2}{f_1^5f_4f_{12}} \\
&\equiv 6\frac{f_2f_4^2}{f_6}\cdot a(q)\cdot\frac{f_3^2}{f_1^2}+10f_4^2+5\frac{f_6^2}{f_4f_{12}}\cdot f_1^3f_3^3 \\
&\equiv 6\frac{f_2f_4^2}{f_6}\left(a(q^4)+6q\frac{f_4^2f_{12}^2}{f_2f_6}\right)\left(\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}\right) \\
&\quad+10f_4^2+5\frac{f_6^2}{f_4f_{12}}\left(\frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2}-q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}\right)^3 \\
&\equiv 6\left(a(q^4)\frac{f_4^6f_{12}^2}{f_2^4f_8f_{24}}+12q^2\frac{f_4^5f_8f_{12}f_{24}}{f_2^4}+q\left(2a(q^4)\frac{f_4^3f_6f_8f_{24}}{f_2^3f_{12}}\right.\right. \\
&\quad\left.\left.+6\frac{f_4^8f_{12}^4}{f_2^5f_6f_8f_{24}}\right)\right)+10f_4^2+5\left(\frac{f_2^3f_8^6f_{12}^{11}}{f_4^7f_6f_{24}^6}-3q\frac{f_2f_6f_8^2f_{12}^5}{f_4f_{24}^2}\right. \\
&\quad\left.+3q^2\frac{f_4^5f_6^3f_{24}^2}{f_2f_8^2f_{12}}-q^3\frac{f_4^{11}f_6^5f_{24}^6}{f_2^3f_8^6f_{12}^7}\right) \pmod 8,
\end{aligned}$$

from which we extract

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(24n+4)q^n$$

$$\equiv 6a(q^2)\frac{f_2^6f_6^2}{f_1^4f_4f_{12}} + 10f_2^2 + 5\left(\frac{f_1^3f_4^6f_6^{11}}{f_2^7f_3f_{12}^6} + 3q\frac{f_2^5f_3^3f_{12}^2}{f_1f_4^2f_6}\right)$$
$$\equiv 6a(q^2)\frac{f_4f_6^2}{f_{12}} + 10f_2^2 + 5\left(\frac{f_2f_4^2f_6^3}{f_{12}^3}\cdot\frac{f_1^3}{f_3} + 3q\frac{f_2^5f_{12}^2}{f_4^2f_6}\cdot\frac{f_3^3}{f_1}\right)$$
$$\equiv 6a(q^2)\frac{f_4f_6^2}{f_{12}} + 10f_2^2 + 5\left(\frac{f_2f_4^2f_6^3}{f_{12}^2}\left(\frac{f_4^3}{f_{12}} - 3q\frac{f_2^2f_{12}^3}{f_4f_6^2}\right)\right.$$
$$\left.+ 3q\frac{f_2^5f_{12}^2}{f_4^2f_6}\left(\frac{f_4^3f_6^2}{f_2^2f_{12}} + q\frac{f_{12}^3}{f_4}\right)\right)$$
$$\equiv 6a(q^2)\frac{f_4f_6^2}{f_{12}} + 10f_2^2 + 5\left(\frac{f_2f_4^5f_6^3}{f_{12}^3} + 3q^2\frac{f_2^5f_{12}^5}{f_4^3f_6}\right) \pmod 8.$$

Therefore,

$$\mathrm{PD_t}(48n+28) \equiv 0 \pmod 8.$$

which is (1.3.5).

Now, from (2.4.2), we also extract

$$\sum_{n=0}^{\infty}\mathrm{PD_t}(12n+10)q^n$$
$$\equiv \left(6f_2^2f_3^2a(q) + 10f_2^2f_3^2\right)\frac{f_2^5f_{12}^2}{f_1^4f_3^2f_4^2f_6} + 10\frac{f_2f_3^4f_4f_{12}}{f_1^4f_6}$$
$$\equiv 6\frac{f_2f_{12}^2}{f_6}\cdot a(q) + 10\frac{f_2f_{12}^2}{f_6} + 10\frac{f_2f_6f_{12}}{f_2}$$
$$\equiv 6\frac{f_2f_{12}^2}{f_6}\left(a(q^4) + 6q\frac{f_4^2f_{12}^2}{f_2f_6}\right) + 10\frac{f_2f_{12}^2}{f_6} + 10\frac{f_2f_6f_{12}}{f_2} \pmod 8,$$

from which we extract

$$\sum_{n=0}^{\infty}\mathrm{PD_t}(24n+22)q^n \equiv 36\frac{f_2^2f_6^4}{f_3^2} \equiv 36f_2^2f_6^3 \pmod 8,$$

Thus,

$$\mathrm{PD_t}(48n+46) \equiv 0 \pmod 8,$$

which is (1.3.6).

2.5 Proof of Theorem 2.1.2

From (2.4.1) we extract

$$\begin{aligned}2\sum_{n=0}^{\infty}\mathrm{PD_t}(3n)q^n &= \frac{f_3^3}{f_1^5f_2^2}a^2(q)-\frac{f_3^3}{f_1^5f_2^2}a(q^2)\\ &\equiv \frac{1}{f_4^2}\cdot\frac{f_3^3}{f_1}-\frac{a(q^2)}{f_4^2}\cdot\frac{f_3^3}{f_1}\\ &\equiv \left(\frac{1}{f_4^2}-\frac{a(q^2)}{f_4^2}\right)\left(\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}\right) \pmod 4, \end{aligned} \tag{2.5.1}$$

from which we extract

$$\begin{aligned}&2\sum_{n=0}^{\infty}\mathrm{PD_t}(6n)q^n\\ &\equiv \left(\frac{1}{f_2^2}-\frac{a(q)}{f_2^2}\right)\frac{f_2^3f_3^2}{f_1^2f_6}\\ &\equiv \frac{f_2}{f_6}\cdot\frac{f_3^2}{f_1^2}-\frac{f_2}{f_6}\cdot\frac{f_3^2}{f_1^2}\cdot a(q)\\ &\equiv \frac{f_2}{f_6}\left(\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}\right)\\ &\quad-\frac{f_2}{f_6}\left(\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}\right)\left(a(q^4)+6q\frac{f_4^2f_{12}^2}{f_2f_6}\right) \pmod 4. \end{aligned} \tag{2.5.2}$$

Therefore,

$$\begin{aligned}2\sum_{n=0}^{\infty}\mathrm{PD_t}(12n)q^n &\equiv \frac{f_2^4f_6^2}{f_1^4f_4f_{12}}-\frac{f_1}{f_3}\frac{f_2^4f_3f_6^2}{f_1^5f_4f_{12}}\ a(q^2)\\ &\equiv \frac{f_2^2f_6^2}{f_4f_{12}}-\frac{f_2^2f_6^2}{f_4f_{12}}\ a(q^2) \pmod 4,\end{aligned}$$

from which we readily arrive at

$$\mathrm{PD_t}(24n+12)\equiv 0 \pmod 2,$$

which is (2.1.3).

Next, extracting the terms involving q^{2n+1} from both sides of (2.5.2), and then dividing by 2, we have

$$\sum_{n=0}^{\infty}\mathrm{PD_t}(12n+6)q^n \equiv \frac{f_2f_3f_4f_{12}}{f_1^3f_6}-a(q^2)\frac{f_2f_3f_4f_{12}}{f_1^3f_6}-3\frac{f_2^6f_6^4}{f_1^5f_3f_4f_{12}}$$

$$\equiv \frac{f_2 f_4 f_{12}}{f_6} \cdot \frac{f_3}{f_1^3}$$
$$\equiv \frac{f_2 f_4 f_{12}}{f_6} \left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7} \right) \pmod 2,$$

from which we extract

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(24n+6) q^n \equiv \frac{f_1 f_2 f_6}{f_3} \cdot \frac{f_2^6 f_3^3}{f_1^9 f_6^2} \equiv f_2^3 \pmod 2,$$

from which we further extract

$$\mathrm{PD_t}(48n+30) \equiv 0 \pmod 2,$$

which is (2.1.5), and

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(48n+6) q^n \equiv f_1^3 \equiv f_3 a(q^3) - 3q f_9^3 \pmod 2,$$

which implies

$$\mathrm{PD_t}(144n+102) \equiv 0 \pmod 2,$$

which is (2.1.6).

Now, from (2.5.1) we extract

$$2\sum_{n=0}^{\infty} \mathrm{PD_t}(6n+3) q^n \equiv \left(\frac{1}{f_2^2} - \frac{a(q)}{f_2^2} \right) \frac{f_6^3}{f_2}$$
$$\equiv \frac{f_6^3}{f_2^3} - \frac{f_6^3}{f_2^3} \left(a(q^4) + 6q \frac{f_4^2 f_{12}^2}{f_2 f_6} \right) \pmod 4,$$

from which we extract

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(12n+9) q^n \equiv \frac{f_2^2 f_3^2 f_6^2}{f_1^4} \equiv f_6^3 \pmod 2. \tag{2.5.3}$$

This implies

$$\mathrm{PD_t}(24n+21) \equiv 0 \pmod 2,$$

which is (2.1.4). Furthermore,

$$\mathrm{PD_t}(36n+21) \equiv \mathrm{PD_t}(36n+33) \equiv 0 \pmod 2,$$

which are weaker versions of (2.1.8) and (2.1.9).

From (2.5.3) we also extract

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(72n+9)q^n \equiv f_1^3 = f_3 a(q^3) - 3q f_9^3 \pmod 2, \tag{2.5.4}$$

from which we further extract

$$\mathrm{PD_t}(216n+153) \equiv 0 \pmod 2,$$

which is (2.1.7).

From (2.4.1), we extract

$$\begin{aligned}
2\sum_{n=0}^{\infty} \mathrm{PD_t}(3n)q^n &= \frac{f_3^3}{f_1^5 f_2^2} a^2(q) - \frac{f_3^3}{f_1^5 f_2^2} a(q^2) \\
&= \frac{f_3^3}{f_1^5 f_2^2} \left(2(a(q)+a(q^2)) - (a(q)+2a(q^2))\right)^2 \\
&\quad - \frac{f_3^3}{f_1^5 f_2^2} \left((a(q)+2a(q^2)) - (a(q)+a(q^2))\right) \\
&= \frac{f_3^3}{f_1^5 f_2^2} \left(4\frac{f_2^6 f_3}{f_1^3 f_6^2} - 3\frac{f_2 f_3^6}{f_1^2 f_6^3}\right)^2 - \frac{f_3^3}{f_1^5 f_2^2} \left(3\frac{f_2 f_3^6}{f_1^2 f_6^3} - 2\frac{f_2^6 f_3}{f_1^3 f_6^2}\right).
\end{aligned}$$

Taking congruences modulo 8, we have

$$\begin{aligned}
2\sum_{n=0}^{\infty} \mathrm{PD_t}(3n)q^n &\equiv \frac{f_6^2}{f_2^4} \cdot \frac{1}{f_1 f_3} + 5\frac{f_6}{f_2^5} \cdot f_1 f_3 + 2 \\
&\equiv \frac{f_6^2}{f_2^4} \left(\frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}\right) \\
&\quad + 5\frac{f_6}{f_2^5} \left(\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}\right) + 2 \pmod 8,
\end{aligned} \tag{2.5.5}$$

from which we extract

$$\begin{aligned}
&2\sum_{n=0}^{\infty} \mathrm{PD_t}(6n+3)q^n \\
&\equiv \frac{f_2^5 f_{12}^2}{f_1^8 f_4^2 f_6} + 3\frac{f_2^4 f_3^2 f_{12}^2}{f_1^6 f_4^2 f_6^2} \\
&\equiv \frac{f_2 f_{12}^2}{f_4^2 f_6} + 3\frac{f_{12}^2}{f_4^2 f_6^2} \cdot f_1^2 f_3^2 \\
&\equiv \frac{f_2 f_{12}^2}{f_4^2 f_6} + 3\frac{f_{12}^2}{f_4^2 f_6^2} \cdot \left(\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}\right)^2
\end{aligned}$$

$$\equiv \frac{f_2 f_{12}^2}{f_4^2 f_6} + 3\frac{f_{12}^2}{f_4^2 f_6^2} \cdot \left(\frac{f_2^2 f_8^4 f_{12}^8}{f_4^4 f_6^2 f_{24}^4} - 2q f_4^2 f_{12}^2 + q^2 \frac{f_4^8 f_6^2 f_{24}^4}{f_2^2 f_8^4 f_{12}^4} \right) \pmod{8}.$$

Extracting the terms involving q^{2n+1} from both sides, and then dividing by 2, we have

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(12n+9)q^n \equiv \frac{f_6^4}{f_3^2} \pmod{4},$$

from which we extract

$$\mathrm{PD_t}(36n+21) \equiv 0 \pmod{4}$$

and

$$\mathrm{PD_t}(36n+33) \equiv 0 \pmod{4},$$

which are (2.1.8) and (2.1.9), respectively.

Now, from (2.5.5), we extract

$$\begin{aligned} 2\sum_{n=0}^{\infty} \mathrm{PD_t}(6n)q^n &\equiv \frac{f_4^2 f_6^5}{f_1^6 f_2 f_3^2 f_{12}^2} + 5\frac{f_4^2 f_6^4}{f_1^4 f_2^2 f_{12}^2} + 2 \\ &\equiv \frac{f_4^2 f_6^5}{f_2^5 f_{12}^2} \cdot \frac{f_1^2}{f_3^2} - 3\frac{f_4^2 f_6^4}{f_2^2 f_{12}^2} \cdot \frac{1}{f_1^4} + 2 \\ &\equiv \frac{f_4^2 f_6^5}{f_2^5 f_{12}^2} \left(\frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q \frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4} \right) \\ &\quad - 3\frac{f_4^2 f_6^4}{f_2^2 f_{12}^2} \left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}} \right) + 2 \pmod{8}, \end{aligned} \tag{2.5.6}$$

which yields

$$\begin{aligned} 2\sum_{n=0}^{\infty} \mathrm{PD_t}(12n)q^n &\equiv \frac{f_2^4 f_6^2}{f_4 f_{12}} \cdot \frac{1}{f_1^4} - 3f_6^2 \cdot \frac{1}{f_3^4} + 2 \\ &\equiv \frac{f_2^4 f_6^2}{f_4 f_{12}} \left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}} \right) \\ &\quad - 3f_6^2 \left(\frac{f_{12}^{14}}{f_6^{14} f_{24}^4} + 4q^3 \frac{f_{12}^2 f_{24}^4}{f_6^{10}} \right) + 2 \pmod{8}, \end{aligned} \tag{2.5.7}$$

from which we extract

$$2\sum_{n=0}^{\infty} \mathrm{PD_t}(24n)q^n \equiv \frac{f_2}{f_6} \cdot \frac{f_3^2}{f_1^2} - 3f_6^2 \cdot \frac{1}{f_3^4} + 2$$

$$\equiv \frac{f_2}{f_6}\left(\frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}\right)$$
$$- 3f_6^2\left(\frac{f_{12}^{14}}{f_6^{14} f_{24}^4} + 4q^3\frac{f_{12}^2 f_{24}^4}{f_6^{10}}\right) + 2 \pmod 8. \quad (2.5.8)$$

We extract

$$2\sum_{n=0}^{\infty} \mathrm{PD_t}(48n)q^n \equiv \frac{f_2^4 f_6^2}{f_4 f_{12}} \cdot \frac{1}{f_1^4} - 3\frac{f_6^2}{f_3^4} + 2 \pmod 8. \quad (2.5.9)$$

From (2.5.7) and (2.5.9), we arrive at

$$\mathrm{PD_t}(12n) \equiv \mathrm{PD_t}(48n) \pmod 4,$$

which, by iteration, gives (2.1.10).

We also extract from (2.5.7)

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(24n+12)q^n \equiv 2f_4^3 - 2qf_{12}^3$$
$$\equiv 2f_{12}a(q^{12}) - 6q^4 f_{36}^3 - 2qf_{12}^3 \pmod 4, \quad (2.5.10)$$

from which we extract

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(24(3n+1)+12)q^n \equiv 2f_4^3 - 2qf_{12}^3 \pmod 4.$$

From the above two, we have

$$\mathrm{PD_t}(24(3n+1)+12) = \mathrm{PD_t}(3(24n+12)) \equiv \mathrm{PD_t}(24n+12) \pmod 4.$$

Thus, for any nonnegative integer ℓ,

$$\mathrm{PD_t}(3^\ell(24n+12)) \equiv \mathrm{PD_t}(24n+12) \pmod 4.$$

Combining the above with (2.1.10), we readily arrive at (2.1.11).

Next, from (2.5.8) we also have

$$2\sum_{n=0}^{\infty} \mathrm{PD_t}(24n)q^n \equiv \frac{f_3^2}{f_6} \cdot \frac{f_2}{f_1^2} - 3\frac{f_6^2}{f_3^4} + 2$$
$$\equiv \frac{f_3^2}{f_6}\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right) - 3\frac{f_6^2}{f_3^4} + 2 \pmod 8,$$

from which we extract

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(72n+48)q^n \equiv 2\frac{f_2 f_6^3}{f_1^4}$$
$$\equiv 2\frac{f_6^3}{f_2} \pmod 4,$$

from which we further extract

$$\mathrm{PD_t}(144n+120) \equiv 0 \pmod 4,$$

which is (2.1.15).

From (2.5.10), we extract

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(48n+12)q^n \equiv 2f_2^3 \pmod 4 \tag{2.5.11}$$

and

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(48n+36)q^n \equiv 2f_6^3 \pmod 4, \tag{2.5.12}$$

which readily implies

$$\mathrm{PD_t}(96n+60) \equiv 0 \pmod 4$$

and

$$\mathrm{PD_t}(96n+84) \equiv 0 \pmod 4,$$

which are (2.1.12) and (2.1.13), respectively. Furthermore, equating the coefficients of q^{3n+1} and q^{3n+2} from both sides of (2.5.12), we arrive at

$$\mathrm{PD_t}(144n+84) \equiv 0 \pmod 4$$

and

$$\mathrm{PD_t}(144n+132) \equiv 0 \pmod 4,$$

which are (2.1.14) and (2.1.16), respectively.

From (2.5.11) and (2.5.12), we also have

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(96n+12)q^n \equiv \sum_{n=0}^{\infty} \mathrm{PD_t}(288n+36)q^n \equiv 2f_1^3 \equiv 2\left(f_3 a(q^3) - 3q f_9^3\right) \pmod 4,$$

from which we extract

$$\mathrm{PD_t}(288n+204) \equiv \mathrm{PD_t}(3(288n+204)) \equiv 0 \pmod 4,$$

which, by iteration, yields (2.1.17).

From (2.5.8), we extract

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PD_t}(48n+24)q^n &\equiv \frac{f_2 f_3 f_4 f_{12}}{f_1^3 f_6} - 2q\frac{f_6^2 f_{12}^4}{f_3^8} \\ &\equiv \frac{f_2 f_4 f_{12}}{f_6} \cdot \frac{f_3}{f_1^3} - 2q f_{12}^3 \\ &\equiv \frac{f_2 f_4 f_{12}}{f_6}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right) - 2q f_{12}^3 \pmod 4,\end{aligned}$$

from which we further extract

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PD_t}(96n+24)q^n &\equiv \frac{f_2^7 f_3^2}{f_1^8 f_6} \\ &\equiv \frac{f_3^2}{f_6} \cdot f_2^3 \\ &\equiv \frac{f_3^2}{f_6}\left(f_6 a(q^6) - 3q^2 f_{18}^3\right) \pmod 4 \end{aligned} \tag{2.5.13}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PD_t}(96n+72)q^n &\equiv 3\frac{f_2^3 f_6^3}{f_1^6} - 2f_6^3 \\ &\equiv 2f_6^3 + 3f_2 f_6^3 \cdot \frac{1}{f_1^2} \\ &\equiv 2f_6^3 + 3f_2 f_6^3\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right) \pmod 4. \end{aligned} \tag{2.5.14}$$

From (2.5.13) we extract

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(288n+216)q^n \equiv f_6^3 \cdot \frac{f_1^2}{f_2}$$

$$\equiv f_6^3\left(\frac{f_9^2}{f_{18}} - 2q\frac{f_3f_{18}^2}{f_6f_9}\right) \pmod 4,$$

from which we extract

$$\mathrm{PD_t}(864n+792) \equiv 0 \pmod 4,$$

which is (2.1.18). We further extract

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{PD_t}(864n+216)q^n &\equiv \frac{f_3^2}{f_6}\cdot f_2^3\\ &\equiv \frac{f_3^2}{f_6}\left(f_6a(q^6) - 3q^2f_{18}^3\right) \pmod 4,\end{aligned}$$

from which we have

$$\mathrm{PD_t}(2592n+1080) \equiv 0 \pmod 4,$$

which is (2.1.20).

From (2.5.14) we extract

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathrm{PD_t}(192n+72)q^n\\ &\equiv 2f_3^3 + 3\frac{f_3^3f_4^5}{f_1^4f_8^2}\\ &\equiv 2f_3^3 + 3f_3^3\cdot\frac{f_4}{f_2^2}\\ &\equiv 2f_3^3 + 3f_3^3\left(\frac{f_{12}^4f_{18}^6}{f_6^8f_{36}^3} + 2q^2\frac{f_{12}^3f_{18}^3}{f_6^7} + 4q^4\frac{f_{12}^2f_{36}^3}{f_6^6}\right) \pmod 4,\end{aligned}$$

from which we extract

$$\begin{aligned}\sum_{n=0}^{\infty}\mathrm{PD_t}(576n+72)q^n &\equiv 2f_1^3 + 3\frac{f_1^3f_4^4f_6^6}{f_2^8f_{12}^3}\\ &\equiv \left(2+3\frac{f_6^2}{f_{12}}\right)\cdot f_1^3\\ &\equiv \left(2+3\frac{f_6^2}{f_{12}}\right)\left(f_3a(q^3) - 3qf_9^3\right) \pmod 4,\end{aligned}$$

from which we further extract

$$\mathrm{PD_t}(1728n+1224) \equiv 0 \pmod 4,$$

which is (2.1.19).

From (2.5.6), we extract

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PD_t}(12n+6)q^n &\equiv -\frac{f_2 f_3 f_4 f_{12}}{f_1^3 f_6} - 2\frac{f_2^4 f_3^4 f_4^4}{f_1^{12} f_6^2} \\ &\equiv 2f_4^3 + 3\frac{f_2 f_4 f_{12}}{f_6}\cdot\frac{f_3}{f_1^3} \\ &\equiv 2f_4^3 + 3\frac{f_2 f_4 f_{12}}{f_6}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right) \\ &\equiv \left(2 + 3\frac{f_6^2}{f_{12}}\right)\cdot f_4^3 + qf_{12}^3\cdot\frac{f_2^2}{f_4} \qquad (2.5.15) \\ &\equiv \left(2 + 3\frac{f_6^2}{f_{12}}\right)\left(f_{12}a(q^{12}) - 3q^4 f_{36}^3\right) \\ &\quad + qf_{12}^3\left(\frac{f_{18}^2}{f_{36}} - 2q^2\frac{f_6 f_{36}^2}{f_{12} f_{18}}\right) \pmod 4,\end{aligned}$$

from which we extract

$$\mathrm{PD_t}(36n+30) \equiv 0 \pmod 4,$$

which is (2.1.21), and

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{PD_t}(36n+18)q^n &\equiv 2qf_{12}^3 + 3qf_{12}^3\cdot\frac{f_2^2}{f_4} + \frac{f_6^2}{f_{12}}\cdot f_4^3 \\ &\equiv 2qf_{12}^3 + 3qf_{12}^3\left(\frac{f_{18}^2}{f_{36}} - 2q^2\frac{f_6 f_{36}^2}{f_{12} f_{18}}\right) \\ &\quad + \frac{f_6^2}{f_{12}}\left(f_{12}a(q^{12}) - 3q^4 f_{36}^3\right) \pmod 4.\end{aligned}$$

From the above we extract

$$\mathrm{PD_t}(108n+90) \equiv 0 \pmod 4,$$

which is (2.1.22), and

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(108n+54)q^n \equiv \left(2 + 3\frac{f_6^2}{f_{12}}\right)\cdot f_4^3 + qf_{12}^3\cdot\frac{f_2^2}{f_4} \pmod 4. \qquad (2.5.16)$$

From (2.5.15) and (2.5.16), we have

$$\mathrm{PD_t}(9(12n+6)) \equiv \mathrm{PD_t}(12n+6) \pmod 4,$$

which, upon iteration, yields (2.1.23). This completes the proof.