Chapter 3

Vanishing coefficients in infinite product expansions

3.1 Introduction

In this chapter, we prove the following theorems.

Theorem 3.1.1. *If*

$$\sum_{n=0}^{\infty} a_n q^n := (-q, -q^4; q^5)_\infty (q, q^9; q^{10})_\infty^3 \textit{ and } \sum_{n=0}^{\infty} b_n q^n := (-q^2, -q^3; q^5)_\infty (q^3, q^7; q^{10})_\infty^3,$$

then

$$\sum_{n=0}^{\infty} b(5n) q^n - \sum_{n=0}^{\infty} a(5n-2) q^n = \frac{f_1^4}{f_2^4}, \tag{3.1.1}$$

$$b_{5n+1} = a_{5n-1}, \tag{3.1.2}$$

$$b_{5n+2} = a_{5n}, \tag{3.1.3}$$

$$b_{5n+3} = a_{5n+1}, \tag{3.1.4}$$

$$b_{5n+4} = a_{5n+2}. \tag{3.1.5}$$

Theorem 3.1.2. *If*

$$\sum_{n=0}^{\infty} c_n q^n := (-q, -q^4; q^5)_\infty^3 (q^3, q^7; q^{10})_\infty \textit{ and } \sum_{n=0}^{\infty} d_n q^n := (-q^2, -q^3; q^5)_\infty^3 (q, q^9; q^{10})_\infty,$$

then

$$c_{5n} = d_{5n}, \tag{3.1.6}$$

$$c_{5n+2} = d_{5n+2}, \tag{3.1.7}$$

$$c_{5n+3} = d_{5n+3}, \tag{3.1.8}$$

and

$$c_{5n+4} = d_{5n+4}. \tag{3.1.9}$$

Furthermore,

$$\sum_{n=0}^{\infty} c_{5n+1}q^n - \sum_{n=0}^{\infty} d_{5n+1}q^n = 4\frac{f_2^4}{f_1^4}, \tag{3.1.10}$$

which shows that $c_{5n+1} > d_{5n+1}$.

As mentioned in our introductory chapter, instead of proving both (1.4.1) and (1.4.2) by Hirschhorn [28], it would have been enough to prove only one of (1.4.1) or (1.4.2). Similarly, instead of proving both (1.4.3) and (1.4.4) by Tang [36], it would have been enough to prove only one of (1.4.3) or (1.4.4).

In addition to the above theorems, we also prove the following theorems by proceeding in a similar way as in Hirschhorn [27].

Theorem 3.1.3. *If*

$$(\mp q, \mp q^4; q^5)_\infty (\pm q^4, \pm q^6; q^{10})_\infty^3 = \sum_{n=0}^{\infty} e_n q^n$$

and

$$(\mp q^2, \mp q^3; q^5)_\infty (\pm q^2, \pm q^8; q^{10})_\infty^3 = \sum_{n=0}^{\infty} f_n q^n,$$

where the signs in the products are taken either both upper ones or both lower ones, then

$$e_{5n+3} = f_{5n+4} = 0.$$

Remark 3.1.4. *The results involving the upper ambiguity signs in Theorem 3.1.3 have already been proved by Tang [36]. Since our proof works for both the signs, we felt it necessary to keep it here as well.*

Theorem 3.1.5. *If*

$$(q,q^4;q^5)_\infty(-q,-q^9;q^{10})_\infty^3=\sum_{n=0}^{\infty}g_nq^n$$

and

$$(q^2,q^3;q^5)_\infty(-q^3,-q^7;q^{10})_\infty^3=\sum_{n=0}^{\infty}h_nq^n,$$

then

$$g_{5n+2}=h_{5n+1}=0.$$

Theorem 3.1.6. *If*

$$(q,q^4;q^5)_\infty(q,q^9;q^{10})_\infty^3=\sum_{n=0}^{\infty}k_nq^n$$

and

$$(q^2,q^3;q^5)_\infty(q^3,q^7;q^{10})_\infty^3=\sum_{n=0}^{\infty}\ell_nq^n,$$

then

$$k_{5n+4}=\ell_{5n+4}=0.$$

Theorem 3.1.7. *If*

$$(q,q^4;q^5)_\infty^3(-q^3,-q^7;q^{10})_\infty=\sum_{n=0}^{\infty}s_nq^n$$

and

$$(q^2,q^3;q^5)_\infty^3(-q,-q^9;q^{10})_\infty=\sum_{n=0}^{\infty}t_nq^n,$$

then

$$s_{5n+3}=t_{5n+4}=0.$$

Theorem 3.1.8. *If*

$$(q,q^4;q^5)_\infty^3(q^3,q^7;q^{10})_\infty=\sum_{n=0}^{\infty}u_nq^n$$

and

$$(q^2,q^3;q^5)_\infty^3(q,q^9;q^{10})_\infty=\sum_{n=0}^{\infty}v_nq^n,$$

then

$$u_{5n+4}=v_{5n+3}=0.$$

In this thesis, we also propose the following conjecture on vanishing coefficients for certain q-products with bases q^7 and q^{14}.

Conjecture 3.1.1. *If*

$$\begin{aligned}
(-q^2,-q^5;q^7)_\infty^3(q^5,q^9;q^{14})_\infty&=\sum_{n=0}^{\infty}A_nq^n,\\
(-q^2,-q^5;q^7)_\infty(q^3,q^{11};q^{14})_\infty^3&=\sum_{n=0}^{\infty}B_nq^n,\\
(q,q^6;q^7)_\infty^3(-q,-q^{13};q^{14})_\infty&=\sum_{n=0}^{\infty}C_nq^n,\\
(q^2,q^5;q^7)_\infty^3(-q^5,-q^9;q^{14})_\infty&=\sum_{n=0}^{\infty}D_nq^n,\\
(q^3,q^4;q^7)_\infty^3(-q^3,-q^{11};q^{14})_\infty&=\sum_{n=0}^{\infty}E_nq^n,\\
(q,q^6;q^7)_\infty^3(q,q^{13};q^{14})_\infty&=\sum_{n=0}^{\infty}F_nq^n,\\
(q^2,q^5;q^7)_\infty^3(q^5,q^9;q^{14})_\infty&=\sum_{n=0}^{\infty}G_nq^n,\\
(q^3,q^4;q^7)_\infty^3(q^3,q^{11};q^{14})_\infty&=\sum_{n=0}^{\infty}H_nq^n,\\
(q,q^6;q^7)_\infty(-q^5,-q^9;q^{14})_\infty^3&=\sum_{n=0}^{\infty}J_nq^n,
\end{aligned}$$

$$(q^2,q^5;q^7)_\infty(-q^3,-q^{11};q^{14})^3_\infty=\sum_{n=0}^{\infty}K_nq^n,$$
$$(q^3,q^4;q^7)_\infty(-q,-q^{13};q^{14})^3_\infty=\sum_{n=0}^{\infty}L_nq^n,$$
$$(q,q^6;q^7)_\infty(q^5,q^9;q^{14})^3_\infty=\sum_{n=0}^{\infty}M_nq^n,$$
$$(q^2,q^5;q^7)_\infty(q^3,q^{11};q^{14})^3_\infty=\sum_{n=0}^{\infty}N_nq^n,$$
$$(q^3,q^4;q^7)_\infty(q,q^{13};q^{14})^3_\infty=\sum_{n=0}^{\infty}P_nq^n,$$

then

$$A_{7n+1}=A_{7n+3}=B_{7n+1}=B_{7n+4}=C_{7n+2}=D_{7n+3}=E_{7n+6}=F_{7n+5}$$
$$=G_{7n+1}=H_{7n+5}=J_{7n+3}=K_{7n+4}=L_{7n+3}=M_{7n+4}=N_{7n+1}=P_{7n+5}$$
$$=0.$$

We employ simple q-series manipulations, Jacobi triple product identity, some preliminary identities for Ramanujan's theta functions, and two known identities for a certain quotient of q-products.
In the next section, we state some preliminary identities. In Sections 3.3 and 3.4, we prove Theorems 1.4.6 and 1.4.7, respectively. The remaining sections are devoted to proving our new Theorems 3.1.3 – 3.1.8.

3.2 Preliminary identities

The following preliminary identities easily follow from [21, p. 46, Entry 30].

Lemma 3.2.1. *We have*

$$f(a,ab^2)f(b,a^2b)=\frac{1}{2}f(1,ab)f(a,b),$$
$$f(a,b)f(-a,-b)=f(-ab,-ab)f(-a^2,-b^2),$$

$$f(a,b) = f(a^3b, ab^3) + af\left(\frac{b}{a}, a^5b^3\right),$$
$$f^2(a,b) = f(a^2,b^2)f(ab,ab) + af\left(\frac{b}{a}, a^3b\right)f(1,a^2b^2).$$

From [13, Eqs. (1.19) and (1.20)], we also recall the following two identities which will be used in our next two sections.

Lemma 3.2.2. *Let*

$$R(q) = \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}.$$

We have

$$\frac{1}{R(q)R^2(q^2)} - q^2R(q)R^2(q^2) = \frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{10};q^{10})_\infty^5}$$

and

$$\frac{R(q^2)}{R^2(q)} - \frac{R^2(q)}{R(q^2)} = 4q\frac{(q^{10};q^{10})_\infty^5(q;q)_\infty}{(q^5;q^5)_\infty(q^2;q^2)_\infty}.$$

The Jacobi triple product identity (1.2.4) and the identities in the above lemmas will be used frequently in our proofs without referring.

3.3 Proof of Theorem 3.1.1

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_nq^n &= (-q,-q^4;q^5)_\infty(q,q^9;q^{10})_\infty^3 \\
&= \frac{(q^2,q^8;q^{10})_\infty}{(q,q^4;q^5)_\infty}\cdot\frac{(q,q^4,q^6,q^9;q^{10})_\infty^3}{(q^4,q^6;q^{10})_\infty^3} \\
&= \frac{(q^2,q^8;q^{10})_\infty(q,q^4;q^5)_\infty^2}{(q^4,q^6;q^{10})_\infty^3} \\
&= \frac{(q,q^2,q^3,q^4;q^5)_\infty}{(q^2,q^4,q^6,q^8;q^{10})_\infty}\cdot\frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}\cdot\frac{(q^2,q^8;q^{10})_\infty^2}{(q^4,q^6;q^{10})_\infty^2} \\
&= \frac{(q;q)_\infty(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(q^5;q^5)_\infty}\cdot R(q)R^2(q^2)
\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} b_n q^n &= (-q^2,-q^3;q^5)_\infty (q^3,q^7;q^{10})_\infty^3 \\ &= \frac{(q^4,q^6;q^{10})_\infty}{(q^2,q^3;q^5)_\infty}\cdot\frac{(q^2,q^3,q^7,q^8;q^{10})_\infty^3}{(q^2,q^8;q^{10})_\infty^3} \\ &= \frac{(q^4,q^6;q^{10})_\infty (q^2,q^3;q^5)_\infty^2}{(q^2,q^8;q^{10})_\infty^3} \\ &= \frac{(q,q^2,q^3,q^4;q^5)_\infty}{(q^2,q^4,q^6,q^8;q^{10})_\infty}\cdot\frac{(q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty}\cdot\frac{(q^4,q^6;q^{10})_\infty^2}{(q^2,q^8;q^{10})_\infty^2} \\ &= \frac{(q;q)_\infty (q^{10};q^{10})_\infty}{(q^2;q^2)_\infty (q^5;q^5)_\infty}\cdot\frac{1}{R(q)R^2(q^2)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} b_n q^n - \sum_{n=0}^{\infty} a_n q^{n+2} &= \frac{(q;q)_\infty (q^{10};q^{10})_\infty}{(q^2;q^2)_\infty (q^5;q^5)_\infty}\left(\frac{1}{R(q)R^2(q^2)} - q^2 R(q) R^2(q^2)\right) \\ &= \frac{(q;q)_\infty (q^{10};q^{10})_\infty}{(q^2;q^2)_\infty (q^5;q^5)_\infty}\cdot\frac{(q^2;q^2)_\infty (q^5;q^5)_\infty^5}{(q;q)_\infty (q^{10};q^{10})_\infty^5} \\ &= \frac{(q^5;q^5)_\infty^4}{(q^{10};q^{10})_\infty^4}.\end{aligned}$$

Equating the coefficients of q^{5n+r}, $r = 0,1,2,3,4$ from both sides of the above, we readily arrive at (3.1.1) – (3.1.5) to finish the proof.

3.4 Proof of Theorem 3.1.2

We have

$$\begin{aligned}\sum_{n=0}^{\infty} c_n q^n &= (-q,-q^4;q^5)_\infty^3 (q^3,q^7;q^{10})_\infty \\ &= \frac{(q^2,q^8;q^{10})_\infty^3}{(q,q^4;q^5)_\infty^3}\cdot (q^3,q^7;q^{10})_\infty \\ &= \frac{(q^2,q^8;q^{10})_\infty^2 (q^2,q^3;q^5)_\infty}{(q,q^4;q^5)_\infty^3} \\ &= \frac{(q^2,q^4,q^6,q^8;q^{10})_\infty}{(q,q^2,q^3,q^4)_\infty}\cdot\frac{(q^2,q^3;q^5)_\infty^2}{(q,q^4;q^5)_\infty^2}\cdot\frac{(q^2,q^8;q^{10})_\infty}{(q^4,q^6;q^{10})_\infty} \\ &= \frac{(q^5;q^5)_\infty (q^2;q^2)_\infty}{(q;q)_\infty (q^{10};q^{10})_\infty}\cdot\frac{R(q^2)}{R^2(q)}\end{aligned}$$

and

$$
\begin{aligned}
\sum_{n=0}^{\infty} d_n q^n &= (-q^2, -q^3; q^5)_\infty^3 (q, q^9; q^{10})_\infty \\
&= \frac{(q^4, q^6; q^{10})_\infty^3}{(q^2, q^3; q^5)_\infty^3} \cdot (q, q^9; q^{10})_\infty \\
&= \frac{(q^4, q^6; q^{10})_\infty^2 (q, q^4; q^5)_\infty}{(q^2, q^3; q^5)_\infty^3} \\
&= \frac{(q^2, q^4, q^6, q^8; q^{10})_\infty}{(q, q^2, q^3, q^4)_\infty} \cdot \frac{(q, q^4; q^5)_\infty^2}{(q^2, q^3; q^5)_\infty^2} \cdot \frac{(q^4, q^6; q^{10})_\infty}{(q^2, q^8; q^{10})_\infty} \\
&= \frac{(q^5; q^5)_\infty (q^2; q^2)_\infty}{(q; q)_\infty (q^{10}; q^{10})_\infty} \cdot \frac{R^2(q)}{R(q^2)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{n=0}^{\infty} c_n q^n - \sum_{n=0}^{\infty} d_n q^n &= \frac{(q^5; q^5)_\infty (q^2; q^2)_\infty}{(q; q)_\infty (q^{10}; q^{10})_\infty} \left(\frac{R(q^2)}{R^2(q)} - \frac{R^2(q)}{R(q^2)} \right) \\
&= 4q \frac{(q^5; q^5)_\infty (q^2; q^2)_\infty}{(q; q)_\infty (q^{10}; q^{10})_\infty} \cdot \frac{(q^{10}; q^{10})_\infty^5 (q; q)_\infty}{(q^5; q^5)_\infty (q^2; q^2)_\infty} \\
&= 4q \frac{(q^{10}; q^{10})_\infty^4}{(q^5; q^5)_\infty^4}. \qquad (3.4.1)
\end{aligned}
$$

Equating the coefficients of q^{5n+r}, $r = 0, 2, 3, 4$ from both sides of the above, we have

$$c_{5n} = d_{5n},$$

$$c_{5n+2} = d_{5n+2},$$

$$c_{5n+3} = d_{5n+3}$$

and

$$c_{5n+4} = d_{5n+4},$$

which are (3.1.6), (3.1.8) – (3.1.10). Similarly, extracting the terms involving q^{5n+1} from both sides of (3.4.1), diving by q, and then replacing q^5 by q, we arrive at (3.1.7), to complete the proof.

3.5 Proof of Theorem 3.1.3

Throughout this section, we consider the ambiguity signs in the products to be either all upper ones or all lower ones.

We have

$$\begin{aligned}\sum_{n=0}^{\infty} e_n q^n &= (\mp q, \mp q^4; q^5)_\infty (\pm q^4, \pm q^6; q^{10})_\infty^3 \\ &= (\mp q, \mp q^4, \pm q^4, \pm q^4, \pm q^4, \mp q^6, \pm q^6, \pm q^6, \pm q^6, \mp q^9; q^{10})_\infty \\ &= (\mp q, \pm q^4, \pm q^6, \mp q^9; q^{10})_\infty (q^8, q^{12}; q^{20})_\infty (\pm q^4; \pm q^6; q^{10})_\infty \\ &= U_1(q) U_2 U_3,\end{aligned}$$

where $U_1(q) = (\mp q, \pm q^4, \pm q^6, \mp q^9; q^{10})_\infty$, $U_2 = (q^8, q^{12}; q^{20})_\infty$, $U_3 = (\pm q^4; \pm q^6; q^{10})_\infty$.

Now,

$$\begin{aligned}U_1(-q) &= (\pm q, \pm q^4, \pm q^6, \pm q^9; q^{10})_\infty \\ &= (\pm q, \pm q^4; q^5)_\infty \\ &= \frac{(\pm q, \pm q^4, q^5; q^5)_\infty}{(q^5; q^5)_\infty} \\ &= \frac{1}{(q^5; q^5)_\infty} \sum_{m=-\infty}^{\infty} (\mp 1)^m q^{(5m^2+3m)/2} \\ &= \frac{1}{(q^5; q^5)_\infty} \left(\sum_{m=-\infty}^{\infty} q^{10m^2+3m} \mp q \sum_{m=-\infty}^{\infty} q^{10m^2+7m} \right) \\ &= \frac{1}{(q^5; q^{10})_\infty (q^{10}; q^{10})_\infty} \big((-q^7, -q^{13}, q^{20}; q^{20})_\infty \mp q(-q^3, -q^{17}, q^{20}; q^{20})_\infty \big),\end{aligned}$$

and hence,

$$U_1(q) = \frac{(q^5; q^5)_\infty (q^{20}; q^{20})_\infty}{(q^{10}; q^{10})_\infty^3} \big((q^7, q^{13}, q^{20}; q^{20})_\infty \pm q(q^3, q^{17}, q^{20}; q^{20})_\infty \big)$$

Therefore,

$$\begin{aligned}&U_1(q) U_2 \\ &= \frac{(q^5; q^5)_\infty}{(q^{10}; q^{10})_\infty^3} (q^8, q^{12}, q^{20}; q^{20}) \big((q^7, q^{13}, q^{20}; q^{20})_\infty \pm q(q^3, q^{17}, q^{20}; q^{20})_\infty \big) \\ &= \frac{(q^5; q^5)_\infty}{(q^{10}; q^{10})_\infty^3} \sum_{m=-\infty}^{\infty} (-1)^m q^{10m^2+2m}\end{aligned}$$

$$
\begin{aligned}
&\times\Bigg(\sum_{n=-\infty}^{\infty}(-1)^n q^{10n^2+3n}\pm q\sum_{n=-\infty}^{\infty}(-1)^n q^{10n^2+7n}\Bigg)\\
&=\frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg(\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}q^{10m^2+2m+10n^2+3n}\\
&\quad\pm q\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}q^{10m^2+7m+10n^2+2n}\Bigg)\\
&=\frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg(\Bigg(\sum_{r,s=-\infty}^{\infty}q^{10(r+s)^2+2(r+s)+10(r-s)^2+3(r-s)}\\
&\quad-\sum_{r,s=-\infty}^{\infty}q^{10(r+s-1)^2+2(r+s-1)+10(r-s)^2+3(r-s)}\Bigg)\\
&\quad\pm q\Bigg(\sum_{r,s=-\infty}^{\infty}q^{10(r+s)^2+7(r+s)+10(r-s)^2+2(r-s)}\\
&\quad-\sum_{r,s=-\infty}^{\infty}q^{10(r+s-1)^2+7(r+s-1)+10(r-s)^2+2(r-s)}\Bigg)\Bigg)\\
&=\frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg(\Bigg(\sum_{r,s=-\infty}^{\infty}q^{20r^2+20s^2+5r+s}-q^8\sum_{r,s=-\infty}^{\infty}q^{20r^2+20s^2+15r+21s}\Bigg)\\
&\quad\pm q\Bigg(\sum_{r,s=-\infty}^{\infty}q^{20r^2+20s^2+9r+5s}-q^3\sum_{r,s=-\infty}^{\infty}q^{20r^2+20s^2+11r+15s}\Bigg)\Bigg)\\
&=\frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg((-q^{15},-q^{25},q^{40};q^{40})_\infty\Bigg(\sum_{n=-\infty}^{\infty}q^{20n^2+n}\pm q\sum_{n=-\infty}^{\infty}q^{20n^2+9n}\Bigg)\\
&\quad\mp q^4(-q^5,-q^{35},q^{40};q^{40})_\infty\Bigg(\sum_{n=-\infty}^{\infty}q^{20n^2+11n}\pm q^4\sum_{n=-\infty}^{\infty}q^{20n^2+21n}\Bigg)\Bigg).
\end{aligned}
$$

We also have

$$
\begin{aligned}
U_3&=(\pm q^4,\pm q^6;q^{10})_\infty\\
&=\frac{1}{(q^{10};q^{10})_\infty}(\pm q^4,\pm q^6,q^{10};q^{10})_\infty\\
&=\frac{1}{(q^{10};q^{10})_\infty}\sum_{m=-\infty}^{\infty}(\mp 1)^m q^{5m^2+m}\\
&=\frac{1}{(q^{10};q^{10})_\infty}\Bigg(\sum_{m=-\infty}^{\infty}q^{20m^2+2m}\mp q^4\sum_{m=-\infty}^{\infty}q^{20m^2+18m}\Bigg).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} e_n q^n \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^4}\Bigg((-q^{15},-q^{25},q^{40};q^{40})_\infty \left(\sum_{m=-\infty}^{\infty} q^{20m^2+2m} \mp q^4 \sum_{m=-\infty}^{\infty} q^{20m^2+18m}\right) \\
&\quad \times \left(\sum_{n=-\infty}^{\infty} q^{20n^2+n} \pm q \sum_{n=-\infty}^{\infty} q^{20n^2+9n}\right) \\
&\quad \mp (-q^5,-q^{35},q^{40};q^{40})_\infty \left(\sum_{m=-\infty}^{\infty} q^{20m^2+2m} \mp q^4 \sum_{m=-\infty}^{\infty} q^{20m^2+18m}\right) \\
&\quad \times \left(q^4 \sum_{n=-\infty}^{\infty} q^{20n^2+11n} \pm q^8 \sum_{n=-\infty}^{\infty} q^{20n^2+21n}\right)\Bigg) \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^4}\big((-q^{15},-q^{25},q^{40};q^{40})_\infty \big(S_1 \mp S_2 \pm S_3 - S_4\big) \\
&\quad \mp (-q^5,-q^{35},q^{40};q^{40})_\infty \big(S_5 \mp S_6 \pm S_7 - S_8\big)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
S_1 &= \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+2m+n}, \quad S_2 = q^4 \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+18m+n}, \\
S_3 &= q \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+2m+9n}, \quad S_4 = q^5 \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+18m+9n}, \\
S_5 &= q^4 \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+2m+11n}, \quad S_6 = q^8 \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+18m+11n}, \\
S_7 &= q^8 \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+2m+21n}, \quad S_8 = q^{12} \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+18m+21n}.
\end{aligned}
$$

Proceeding as in the previous theorem, it can be shown that the 3-components of the sums $S_1, S_2, \ldots, S_8$ are, respectively,

$$
\begin{aligned}
&q^{43} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+125r+40s}, \quad q^{23} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+75r+60s}, \\
&q^{23} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+75r+60s}, \quad q^{43} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+125r+40s}, \\
&q^{13} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+25r+60s}, \quad q^{8} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+25r+40s},
\end{aligned}
$$

$$q^8 \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+25r+40s}, \quad q^{13} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+25r+60s}.$$

Since these cancel in pairs, we conclude that $e_{5n+3} = 0$.

Similarly, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n q^n &= (\mp q^2, \mp q^3; q^5)_\infty (\pm q^2, \pm q^8; q^{10})_\infty^3 \\
&= (\mp q^2, \pm q^2, \pm q^2, \pm q^2, \mp q^3 \mp q^7, \mp q^8 \pm q^8, \pm q^8, \pm q^8; q^{10})_\infty \\
&= (\pm q^2, \mp q^3, \mp q^7, \pm q^8; q^{10})_\infty (q^4, q^{16}; q^{20})_\infty (\pm q^2; \pm q^8; q^{10})_\infty \\
&= V_1(q) V_2 V_3,
\end{aligned}$$

where $V_1(q) = (\pm q^2, \mp q^3, \mp q^7, \pm q^8; q^{10})_\infty$, $V_2 = (q^4, q^{16}; q^{20})_\infty$, $V_3 = (\pm q^2; \pm q^8; q^{10})_\infty$.

Now

$$\begin{aligned}
V_1(-q) &= (\pm q^2, \pm q^3, \pm q^7, \pm q^8; q^{10})_\infty \\
&= (\pm q^2, \pm q^3; q^5)_\infty \\
&= \frac{(\pm q^2, \pm q^3, q^5; q^5)_\infty}{(q^5; q^5)_\infty} \\
&= \frac{1}{(q^5; q^5)_\infty} \sum_{m=-\infty}^{\infty} (\mp 1)^m q^{(5m^2+m)/2} \\
&= \frac{1}{(q^5; q^5)_\infty} \left(\sum_{m=-\infty}^{\infty} q^{10m^2+m} \mp q^2 \sum_{m=-\infty}^{\infty} q^{10m^2+9m} \right) \\
&= \frac{1}{(q^5; q^{10})_\infty (q^{10}; q^{10})_\infty} \big((-q^9, -q^{11}, q^{20}; q^{20})_\infty \mp q^2 (-q, -q^{19}, q^{20}; q^{20})_\infty \big).
\end{aligned}$$

Therefore,

$$V_1(q) = \frac{(q^5; q^5)_\infty (q^{20}; q^{20})_\infty}{(q^{10}; q^{10})_\infty^3} \big((q^9, q^{11}, q^{20}; q^{20})_\infty \mp q^2 (q, q^{19}, q^{20}; q^{20})_\infty \big),$$

and hence,

$$\begin{aligned}
&V_1(q) V_2 \\
&= \frac{(q^5; q^5)_\infty}{(q^{10}; q^{10})_\infty^3} (q^4, q^{16}, q^{20}; q^{20}) \big((q^9, q^{11}, q^{20}; q^{20})_\infty \mp q^2 (q, q^{19}, q^{20}; q^{20})_\infty \big) \\
&= \frac{(q^5; q^5)_\infty}{(q^{10}; q^{10})_\infty^3} \sum_{m=-\infty}^{\infty} (-1)^m q^{10m^2+6m} \Bigg(\sum_{n=-\infty}^{\infty} (-1)^n q^{10n^2+n}
\end{aligned}$$

$$
\begin{aligned}
&\mp q^2 \sum_{n=-\infty}^{\infty} (-1)^n q^{10n^2+9n}\Bigg) \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{10m^2+6m+10n^2+n} \\
&\mp q^2 \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{10m^2+9m+10n^2+6n}\Bigg) \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg(\Bigg(\sum_{r,s=-\infty}^{\infty} q^{10(r+s)^2+6(r+s)+10(r-s)^2+(r-s)} \\
&- \sum_{r,s=-\infty}^{\infty} q^{10(r+s-1)^2+6(r+s-1)+10(r-s)^2+(r-s)}\Bigg) \\
&\mp q^2\Bigg(\sum_{r,s=-\infty}^{\infty} q^{10(r+s)^2+9(r+s)+10(r-s)^2+6(r-s)} \\
&- \sum_{r,s=-\infty}^{\infty} q^{10(r+s-1)^2+9(r+s-1)+10(r-s)^2+6(r-s)}\Bigg)\Bigg) \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg(\Bigg(\sum_{r,s=-\infty}^{\infty} q^{20r^2+20s^2+7r+5s} - q^4 \sum_{r,s=-\infty}^{\infty} q^{20r^2+20s^2+13r+15s}\Bigg) \\
&\mp q^2\Bigg(\sum_{r,s=-\infty}^{\infty} q^{20r^2+20s^2+15r+3s} - q \sum_{r,s=-\infty}^{\infty} q^{20r^2+20s^2+5r+17s}\Bigg)\Bigg) \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg((-q^{15},-q^{25},q^{40};q^{40})_\infty\Bigg(\sum_{n=-\infty}^{\infty} q^{20n^2+7n} \pm q^3 \sum_{n=-\infty}^{\infty} q^{20n^2+17n}\Bigg) \\
&\mp (-q^5,-q^{35},q^{40};q^{40})_\infty\Bigg(q^2 \sum_{n=-\infty}^{\infty} q^{20n^2+3n} \pm q^4 \sum_{n=-\infty}^{\infty} q^{20n^2+13n}\Bigg)\Bigg).
\end{aligned}
$$

Also,

$$
\begin{aligned}
V_3(q) &= (\pm q^2, \pm q^8; q^{10})_\infty \\
&= \frac{1}{(q^{10};q^{10})_\infty}(\pm q^2, \pm q^8, q^{10}; q^{10})_\infty \\
&= \frac{1}{(q^{10};q^{10})_\infty}\sum_{m=-\infty}^{\infty} (\mp 1)^m q^{5m^2+3m} \\
&= \frac{1}{(q^{10};q^{10})_\infty}\Bigg(\sum_{m=-\infty}^{\infty} q^{20m^2+6m} \mp q^2 \sum_{m=-\infty}^{\infty} q^{20m^2+14m}\Bigg).
\end{aligned}
$$

It follows that

$$\sum_{n=0}^{\infty} f_n q^n$$
$$= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^4}\Bigg((-q^{15},-q^{25},q^{40};q^{40})_\infty\Bigg(\sum_{m=-\infty}^{\infty} q^{20m^2+6m} \mp q^2 \sum_{m=-\infty}^{\infty} q^{20m^2+14m}\Bigg)$$
$$\times\Bigg(\sum_{n=-\infty}^{\infty} q^{20n^2+7n} \pm q^3 \sum_{n=-\infty}^{\infty} q^{20n^2+17n}\Bigg)$$
$$\mp(-q^5,-q^{35},q^{40};q^{40})_\infty\Bigg(\sum_{m=-\infty}^{\infty} q^{20m^2+6m} \mp q^2 \sum_{m=-\infty}^{\infty} q^{20m^2+14m}\Bigg)$$
$$\times\Bigg(q^2\sum_{n=-\infty}^{\infty} q^{20n^2+3n} \pm q^4 \sum_{n=-\infty}^{\infty} q^{20n^2+13n}\Bigg)\Bigg)$$
$$= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^4}\Big((-q^{15},-q^{25},q^{40};q^{40})_\infty\big(T_1 \mp T_2 \pm T_3 - T_4\big)$$
$$\mp(-q^5,-q^{35},q^{40};q^{40})_\infty\big(T_5 \mp T_6 \pm T_7 - T_8\big)\Big),$$

where

$$T_1 = \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+6m+7n}, \quad T_2 = q^2\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+14m+7n},$$
$$T_3 = q^3\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+6m+17n}, \quad T_4 = q^5\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+14m+17n},$$
$$T_5 = q^2\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+6m+3n}, \quad T_6 = q^4\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+14m+3n},$$
$$T_7 = q^4\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+6m+13n}, \quad T_8 = q^6\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+14m+13n}.$$

It can be shown that, the 4-components of the sums $T_1, T_2, \ldots, T_8$ are, respectively,

$$q^{14}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+20r+75s}, \quad q^{29}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+80r+75s},$$
$$q^{29}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+80r+75s}, \quad q^{14}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+20r+75s},$$
$$q^{19}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+80r+25s}, \quad q^{4}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+20r+25s},$$

$$q^4 \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+20r+25s}, \quad q^{19} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+80r+25s},$$

and these cancel in pairs. Therefore, we arrive at $f_{5n+4} = 0$ to finish the proof.

3.6 Proof of Theorem 3.1.5

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} g_n q^n &= (q, q^4; q^5)_\infty (-q, -q^9; q^{10})_\infty^3 \\
&= (q, q^4, q^6, q^9; q^{10})_\infty (-q, -q^9; q^{10})_\infty^3 \\
&= (q, -q, -q, -q, q^4, q^6, q^9, -q^9, -q^9, -q^9; q^{10})_\infty \\
&= (-q, q^4, q^6, -q^9; q^{10})_\infty (q^2, q^{18}; q^{20})_\infty (-q, -q^9; q^{10})_\infty \\
&= K_1(q) K_2 K_3,
\end{aligned}$$

where

$$K_1(q) = (-q, q^4, q^6, -q^9; q^{10})_\infty, \quad K_2 = (q^2, q^{18}; q^{20})_\infty, \quad K_3 = (-q, -q^9; q^{10})_\infty.$$

Now

$$\begin{aligned}
K_1(-q) &= (q, q^4, q^6, q^9; q^{10})_\infty \\
&= (q, q^4; q^5)_\infty \\
&= \frac{(q, q^4, q^5 : q^5)_\infty}{(q^5; q^5)_\infty} \\
&= \frac{1}{(q^5; q^5)_\infty} \sum_{m=-\infty}^{\infty} (-1)^m q^{(5m^2+3m)/2} \\
&= \frac{1}{(q^5; q^5)_\infty} \left(\sum_{m=-\infty}^{\infty} q^{10m^2+3m} - q \sum_{m=-\infty}^{\infty} q^{10m^2+7m} \right) \\
&= \frac{1}{(q^5; q^{10})_\infty (q^{10}; q^{10})_\infty} \big((-q^7, -q^{13}, q^{20}; q^{20})_\infty - q(-q^3, -q^{17}, q^{20}; q^{20})_\infty \big).
\end{aligned}$$

Therefore,

$$K_1(q) = \frac{(q^5; q^5)_\infty (q^{20}; q^{20})_\infty}{(q^{10}; q^{10})_\infty^3} \big((q^7, q^{13}, q^{20}; q^{20})_\infty + q(q^3, q^{17}, q^{20}; q^{20})_\infty \big)$$

and hence,

$$
\begin{aligned}
&K_1(q)K_2 \\
&= \frac{(q^5;q^5)_\infty(q^{20};q^{20})_\infty}{(q^{10};q^{10})_\infty^3}(q^2,q^{18};q^{20})_\infty\big((q^7,q^{13},q^{20};q^{20})_\infty + q(q^3,q^{17},q^{20};q^{20})_\infty\big) \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg(\sum_{m=-\infty}^{\infty}(-1)^m q^{10m^2+8m}\Bigg) \\
&\quad \times \Bigg(\sum_{n=-\infty}^{\infty}(-1)^n q^{10n^2+3n} + q\sum_{n=-\infty}^{\infty}(-1)^n q^{10n^2+7n}\Bigg) \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg(\sum_{m,n=-\infty}^{\infty}(-1)^{m+n} q^{10m^2+8m+10n^2+3n} \\
&\quad + q\sum_{m,n=-\infty}^{\infty}(-1)^{m+n} q^{10m^2+8m+10n^2+7n}\Bigg) \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg(\Bigg(\sum_{r,s=-\infty}^{\infty} q^{10(r+s)^2+8(r+s)+10(r-s)^2+3(r-s)} \\
&\quad - \sum_{r,s=-\infty}^{\infty} q^{10(r+s-1)^2+8(r+s-1)+10(r-s)^2+3(r-s)}\Bigg) \\
&\quad + q\Bigg(\sum_{r,s=-\infty}^{\infty} q^{10(r+s)^2+8(r+s)+10(r-s)^2+7(r-s)} \\
&\quad - \sum_{r,s=-\infty}^{\infty} q^{10(r+s-1)^2+8(r+s-1)+10(r-s)^2+7(r-s)}\Bigg)\Bigg) \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg(\sum_{r,s=-\infty}^{\infty} q^{20r^2+20s^2+11r+5s} - q^2\sum_{r,s=-\infty}^{\infty} q^{20r^2+20s^2-9r-15s} \\
&\quad + q\sum_{r,s=-\infty}^{\infty} q^{20r^2+20s^2+15r+s} - q^3\sum_{r,s=-\infty}^{\infty} q^{20r^2+20s^2-5r-19s})\Bigg) \\
&= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^3}\Bigg((-q^{15},-q^{25},q^{40};q^{40})_\infty\Bigg(\sum_{n=-\infty}^{\infty} q^{20n^2+11n} - q^3\sum_{n=-\infty}^{\infty} q^{20n^2+19n}\Bigg) \\
&\quad + q(-q^5,-q^{35},q^{40};q^{40})_\infty\Bigg(\sum_{n=-\infty}^{\infty} q^{20n^2+n} - q\sum_{n=-\infty}^{\infty} q^{20n^2+9n}\Bigg)\Bigg).
\end{aligned}
$$

Also,

$$K_3 = \frac{1}{(q^{10};q^{10})_\infty}(-q,-q^9,q^{10};q^{10})_\infty$$

$$= \frac{1}{(q^{10};q^{10})_\infty} \sum_{m=-\infty}^{\infty} q^{5m^2+4m}$$
$$= \frac{1}{(q^{10};q^{10})_\infty} \biggl(\sum_{m=-\infty}^{\infty} q^{20m^2+8m} + q \sum_{m=-\infty}^{\infty} q^{20m^2+12m} \biggr).$$

It follows that

$$\sum_{n=0}^{\infty} g_n q^n$$
$$= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^4} \biggl((-q^{15},-q^{25},q^{40};q^{40})_\infty \biggl(\sum_{m=-\infty}^{\infty} q^{20m^2+8m} + q \sum_{m=-\infty}^{\infty} q^{20m^2+12m} \biggr)$$
$$\times \biggl(\sum_{n=-\infty}^{\infty} q^{20n^2+11n} - q^3 \sum_{n=-\infty}^{\infty} q^{20n^2+19n} \biggr)$$
$$+ q(-q^5,-q^{35},q^{40};q^{40})_\infty \biggl(\sum_{m=-\infty}^{\infty} q^{20m^2+8m} + q \sum_{m=-\infty}^{\infty} q^{20m^2+12m} \biggr)$$
$$\times \biggl(\sum_{n=-\infty}^{\infty} q^{20n^2+n} - q \sum_{n=-\infty}^{\infty} q^{20n^2+9n} \biggr) \biggr)$$
$$= \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^4} \bigl(-q^{15},-q^{25},q^{40};q^{40})_\infty (S_1+S_2-S_3-S_4)$$
$$+ q(-q^5,-q^{35},q^{40};q^{40})_\infty (S_5+S_6-S_7-S_8),$$

where

$$S_1 = \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+8m+11n}, \quad S_2 = q \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+12m+11n},$$
$$S_3 = q^3 \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+8m+19n}, \quad S_4 = q^4 \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+12m+19n},$$
$$S_5 = q \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+8m+n}, \quad S_6 = q^2 \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+12m+n},$$
$$S_7 = q^2 \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+8m+9n}, \quad S_8 = q^3 \sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+12m+9n}.$$

It can be shown that the 2-components of each of these eight sums $S_1, S_2, \ldots, S_8$ are, respectively,

$$q^{12} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+75r+10s}, \quad q^{32} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+75r+90s},$$

$$q^{32}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+75r+90s}, \quad q^{12}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+75r+10s},$$
$$q^{22}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+25r+90s}, \quad q^{2}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+25r+10s},$$
$$q^{2}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+25r+10s}, \quad q^{22}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+25r+90s}.$$

Since these cancel in pairs, we conclude that $g_{5n+2}=0$.

Proceeding similarly, we obtain

$$\sum_{n=0}^{\infty} h_n q^n = \frac{(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty^4}\big((-q^{15},-q^{25},q^{40};q^{40})_\infty(T_1+T_2-T_3-T_4) + q^3(-q^5,-q^{35},q^{40};q^{40})_\infty(T_5+T_6-T_7-T_8),$$

where

$$T_1=\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+4m+3n}, \quad T_2=q^3\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+16m+3n},$$
$$T_3=q^2\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+4m+13n}, \quad T_4=q^5\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+16m+13n},$$
$$T_5=q^3\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+4m+7n}, \quad T_6=q^6\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+16m+7n},$$
$$T_7=q^6\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+4m+17n}, \quad T_8=q^9\sum_{m,n=-\infty}^{\infty} q^{20m^2+20n^2+16m+17n}.$$

The 1-components of $T_1, T_2, \ldots, T_8$ are, respectively,

$$q^{16}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+75r+30s}, \quad q^{26}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+75r+70s},$$
$$q^{26}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+75r+70s}, \quad q^{16}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+75r+30s},$$
$$q^{16}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+70r+25s}, \quad q^{6}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+30r+25s},$$
$$q^{6}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+30r+25s}, \quad q^{16}\sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+70r+25s}.$$

Since these cancel in pairs, we arrive at $h_{5n+1}=0$ to finish the proof.

3.7 Proof of Theorem 3.1.6

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} k_n q^n &= (q,q^4;q^5)_\infty (q,q^9;q^{10})_\infty^3 \\
&= \frac{1}{(q^5;q^5)_\infty (q^{10};q^{10})_\infty^3} f(-q,-q^4) f^3(-q,-q^9) \\
&= \frac{1}{(q^5;q^5)_\infty (q^{10};q^{10})_\infty^3} f(-q,-q^4) f(-q,-q^9) \\
&\times \big(f(q^2,q^{18}) f(q^{10},q^{10}) - q f(q^8,q^{12}) f(1,q^{20}) \big) \\
&= \frac{(q^{20};q^{20})_\infty^5}{(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty^3 (q^{40};q^{40})_\infty^2}\ P_1P_2P_3 \\
&\quad - 2q \frac{(q^{40};q^{40})_\infty^2}{(q^5;q^5)_\infty (q^{10};q^{10})_\infty^3 (q^{20};q^{20})_\infty}\ P_1P_4P_3,
\end{aligned}$$

where $P_1 = f(-q,-q^4)$, $\quad P_2 = f(q^2,q^{18})$, $\quad P_3 = f(-q,-q^9)$, $\quad P_4 = (q^8,q^{12})$.

Now

$$\begin{aligned}
P_1P_2 &= f(-q,-q^4) f(q^2,q^{18}) \\
&= \sum_{m=-\infty}^{\infty} (-1)^m q^{(5m^2+3m)/2} \sum_{n=-\infty}^{\infty} q^{10n^2+8n} \\
&= \left(\sum_{m=-\infty}^{\infty} q^{10m^2+3m} - q \sum_{m=-\infty}^{\infty} q^{10m^2+7m} \right) \sum_{n=-\infty}^{\infty} q^{10n^2+8n} \\
&= \sum_{m,n=-\infty}^{\infty} q^{10m^2+8m+10n^2+3n} - q \sum_{m,n=-\infty}^{\infty} q^{10m^2+8m+10n^2+7n} \\
&= \left(\sum_{r,s=-\infty}^{\infty} q^{20r^2+11r+20s^2+5s} + q^2 \sum_{r,s=-\infty}^{\infty} q^{20r^2+9r+20s^2+15s} \right) \\
&\quad - q \left(\sum_{r,s=-\infty}^{\infty} q^{20r^2+15r+20s^2+s} + q^2 \sum_{r,s=-\infty}^{\infty} q^{20r^2+5r+20s^2+19s} \right) \\
&= (-q^{15},-q^{25},q^{40};q^{40})_\infty \left(\sum_{n=-\infty}^{\infty} q^{20n^2+11n} - q^3 \sum_{n=-\infty}^{\infty} q^{20n^2+19n} \right) \\
&\quad - q(-q^5,-q^{35},q^{40};q^{40})_\infty \left(\sum_{n=-\infty}^{\infty} q^{20n^2+n} - q \sum_{n=-\infty}^{\infty} q^{20n^2+9n} \right)
\end{aligned}$$

and

$$P_3 = f(-q,-q^9)$$

$$= \sum_{m=-\infty}^{\infty} (-1)^m q^{5m^2+4m}$$
$$= \sum_{m=-\infty}^{\infty} q^{20m^2+8m} - q \sum_{m=-\infty}^{\infty} q^{20m^2+12m}.$$

Therefore,

$$\begin{aligned} P_1P_2P_3 &= (-q^{15}, -q^{25}, q^{40}; q^{40})_\infty \Big(\sum_{m=-\infty}^{\infty} q^{20m^2+8m} - q \sum_{m=-\infty}^{\infty} q^{20m^2+12m} \Big) \\ &\quad \times \Big(\sum_{n=-\infty}^{\infty} q^{20n^2+11n} - q^3 \sum_{n=-\infty}^{\infty} q^{20n^2+19n} \Big) \\ &\quad - q(-q^5, -q^{35}, q^{40}; q^{40})_\infty \Big(\sum_{m=-\infty}^{\infty} q^{20m^2+8m} - q \sum_{m=-\infty}^{\infty} q^{20m^2+12m} \Big) \\ &\quad \times \Big(\sum_{n=-\infty}^{\infty} q^{20n^2+n} - q \sum_{n=-\infty}^{\infty} q^{20n^2+9n} \Big) \\ &= (-q^{15}, -q^{25}, q^{40}; q^{40})_\infty \big(S_1 - S_2 - S_3 + S_4 \big) \\ &\quad - (-q^5, -q^{35}, q^{40}; q^{40})_\infty \big(S_5 - S_6 - S_7 + S_8 \big), \end{aligned}$$

where

$$S_1 = \sum_{m,n=-\infty}^{\infty} q^{20m^2+8m+20n^2+11n}, \quad S_2 = q \sum_{m,n=-\infty}^{\infty} q^{20m^2+12m+20n^2+11n},$$
$$S_3 = q^3 \sum_{m,n=-\infty}^{\infty} q^{20m^2+8m+20n^2+19n}, \quad S_4 = q^4 \sum_{m,n=-\infty}^{\infty} q^{20m^2+12m+20n^2+19n},$$
$$S_5 = q \sum_{m,n=-\infty}^{\infty} q^{20m^2+8m+20n^2+n}, \quad S_6 = q^2 \sum_{m,n=-\infty}^{\infty} q^{20m^2+12m+20n^2+n},$$
$$S_7 = q^2 \sum_{m,n=-\infty}^{\infty} q^{20m^2+8m+20n^2+9n}, \quad S_8 = q^3 \sum_{m,n=-\infty}^{\infty} q^{20m^2+12m+20n^2+9n}.$$

Next,

$$\begin{aligned} qP_1P_4 &= f(-q, -q^4) f(q^8, q^{12}) \\ &= q \sum_{m=-\infty}^{\infty} (-1)^m q^{(5m^2+3m)/2} \sum_{n=-\infty}^{\infty} q^{10n^2+2n} \\ &= q \Big(\sum_{m=-\infty}^{\infty} q^{10m^2+3m} - q \sum_{m=-\infty}^{\infty} q^{10m^2+7m} \Big) \sum_{n=-\infty}^{\infty} q^{10n^2+2n} \end{aligned}$$

$$
\begin{aligned}
&= q\sum_{m,n=-\infty}^{\infty} q^{10m^2+3m+10n^2+2n} - q^2\sum_{m,n=-\infty}^{\infty} q^{10m^2+7m+10n^2+2n}\\
&= q\Big(\sum_{r,s=-\infty}^{\infty} q^{20r^2+5r+20s^2+s} + q^7\sum_{r,s=-\infty}^{\infty} q^{20r^2+15r+20s^2+19s}\Big)\\
&\quad - q^2\Big(\sum_{r,s=-\infty}^{\infty} q^{20r^2+9r+20s^2+5s} + q^3\sum_{r,s=-\infty}^{\infty} q^{20r^2+11r+20s^2+15s}\Big)\\
&= (-q^{15},-q^{25},q^{40};q^{40})_\infty\Big(q\sum_{n=-\infty}^{\infty} q^{20n^2+n} - q^2\sum_{n=-\infty}^{\infty} q^{20n^2+9n}\Big)\\
&\quad - (-q^{5},-q^{35},q^{40};q^{40})_\infty\Big(q^5\sum_{n=-\infty}^{\infty} q^{20n^2+11n} - q^8\sum_{n=-\infty}^{\infty} q^{20n^2+19n}\Big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
qP_1P_4P_3 &= (-q^{15},-q^{25},q^{40};q^{40})_\infty\Big(\sum_{m=-\infty}^{\infty} q^{20m^2+8m} - q\sum_{m=-\infty}^{\infty} q^{20m^2+12m}\Big)\\
&\qquad \times\Big(q\sum_{n=-\infty}^{\infty} q^{20n^2+n} - q^2\sum_{n=-\infty}^{\infty} q^{20n^2+9n}\Big)\\
&\qquad - (-q^{5},-q^{35},q^{40};q^{40})_\infty\Big(\sum_{m=-\infty}^{\infty} q^{20m^2+8m} - q\sum_{m=-\infty}^{\infty} q^{20m^2+12m}\Big)\\
&\qquad \times\Big(q^5\sum_{n=-\infty}^{\infty} q^{20n^2+11n} - q^8\sum_{n=-\infty}^{\infty} q^{20n^2+19n}\Big)\\
&= (-q^{15},-q^{25},q^{40};q^{40})_\infty\big(S_9 - S_{10} - S_{11} + S_{12}\big)\\
&\qquad - (-q^{5},-q^{35},q^{40};q^{40})_\infty\big(S_{13} - S_{14} - S_{15} + S_{16}\big),
\end{aligned}
$$

where

$$
\begin{aligned}
S_9 &= q\sum_{m,n=-\infty}^{\infty} q^{20m^2+8m+20n^2+n}, & S_{10} &= q^2\sum_{m,n=-\infty}^{\infty} q^{20m^2+12m+20n^2+n},\\
S_{11} &= q^2\sum_{m,n=-\infty}^{\infty} q^{20m^2+8m+20n^2+9n}, & S_{12} &= q^3\sum_{m,n=-\infty}^{\infty} q^{20m^2+12m+20n^2+9n},\\
S_{13} &= q^5\sum_{m,n=-\infty}^{\infty} q^{20m^2+8m+20n^2+11n}, & S_{14} &= q^6\sum_{m,n=-\infty}^{\infty} q^{20m^2+12m+20n^2+11n},\\
S_{15} &= q^8\sum_{m,n=-\infty}^{\infty} q^{20m^2+8m+20n^2+19n}, & S_{16} &= q^9\sum_{m,n=-\infty}^{\infty} q^{20m^2+12m+20n^2+19n}.
\end{aligned}
$$

The 4-components of these sixteen sums $S_1, S_2, \ldots, S_{16}$, are, respectively,

$$q^9 \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+50r+45s}, \quad q^9 \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+50r+45s},$$
$$q^4 \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+50r+5s}, \quad q^4 \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+50r+5s},$$
$$q^{29} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+95r+50s}, \quad q^{29} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+95r+50s},$$
$$q^{14} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+55r+50s}, \quad q^{14} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+55r+50s},$$
$$q^{29} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+95r+50s}, \quad q^{29} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+95r+50s},$$
$$q^{14} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+55r+50s}, \quad q^{14} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+55r+50s},$$
$$q^{14} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+45r+50s}, \quad q^{14} \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+45r+50s},$$
$$q^9 \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+5r+50s}, \quad q^9 \sum_{r,s=-\infty}^{\infty} q^{100r^2+100s^2+5r+50s}.$$

Since these cancel in pairs, we arrive at $k_{5n+4} = 0$.

In a same fashion, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \ell_n q^n \\
&= (q^2, q^3; q^5)_\infty (q^3, q^7; q^{10})_\infty^3 \\
&= \frac{1}{(q^5; q^5)_\infty (q^{10}; q^{10})_\infty^3} f(-q^2, -q^3) f^3(-q^3, -q^7) \\
&= \frac{1}{(q^5; q^5)_\infty (q^{10}; q^{10})_\infty^3} f(-q^2, -q^3) f(-q^3, -q^7) \\
&\times \big(f(q^6, q^{14}) f(q^{10}, q^{10}) - q^3 f(q^4, q^{16}) f(1, q^{20}) \big) \\
&= \frac{(q^{20}; q^{20})_\infty^5}{(q^5; q^5)_\infty^3 (q^{10}; q^{10})_\infty^3 (q^{40}; q^{40})_\infty^2} \cdot f(-q^2, -q^3) f(q^6, q^{14}) f(-q^3, -q^7) \\
&\quad - 2q^3 \frac{(q^{40}; q^{40})_\infty^2}{(q^5; q^5)_\infty (q^{10}; q^{10})_\infty^3 (q^{20}; q^{20})_\infty} \cdot f(-q^2, -q^3), f(-q^3, -q^7)(q^4, q^{16}) \\
&= \frac{(q^{20}; q^{20})_\infty^5}{(q^5; q^5)_\infty^3 (q^{10}; q^{10})_\infty^3 (q^{40}; q^{40})_\infty^2} \Big(f(q^{15}, q^{25}) \big(T_1 - T_2 - T_3 + T_4 \big)
\end{aligned}$$

$$
\begin{aligned}
&- f(q^5,q^{35})\big(T_5-T_6-T_7+T_8\big)\Big)\\
&-2\frac{(q^{40};q^{40})_\infty^2}{(q^5;q^5)_\infty(q^{10};q^{10})_\infty^3(q^{20};q^{20})_\infty}\Big(f(q^{15},q^{25})\big(T_9-T_{10}-T_{11}+T_{12}\big)\\
&- f(q^5,q^{35})\big(T_{13}-T_{14}-T_{15}+T_{16}\big),\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 &= \sum_{m,n=-\infty}^{\infty} q^{20m^2+4m+20n^2+3n}, \quad T_2 = q^3\sum_{m,n=-\infty}^{\infty} q^{20m^2+16m+20n^2+3n},\\
T_3 &= q^2\sum_{m,n=-\infty}^{\infty} q^{20m^2+4m+20n^2+13n}, \quad T_4 = q^5\sum_{m,n=-\infty}^{\infty} q^{20m^2+16m+20n^2+13n},\\
T_5 &= q^3\sum_{m,n=-\infty}^{\infty} q^{20m^2+4m+20n^2+7n}, \quad T_6 = q^6\sum_{m,n=-\infty}^{\infty} q^{20m^2+16m+20n^2+7n},\\
T_7 &= q^6\sum_{m,n=-\infty}^{\infty} q^{20m^2+4m+20n^2+17n}, \quad T_8 = q^9\sum_{m,n=-\infty}^{\infty} q^{20m^2+16m+20n^2+17n},\\
T_9 &= q^3\sum_{m,n=-\infty}^{\infty} q^{20m^2+4m+20n^2+7n}, \quad T_{10} = q^6\sum_{m,n=-\infty}^{\infty} q^{20m^2+16m+20n^2+7n},\\
T_{11} &= q^6\sum_{m,n=-\infty}^{\infty} q^{20m^2+4m+20n^2+17n}, \quad T_{12} = q^9\sum_{m,n=-\infty}^{\infty} q^{20m^2+16m+20n^2+17n},\\
T_{13} &= q^5\sum_{m,n=-\infty}^{\infty} q^{20m^2+4m+20n^2+3n}, \quad T_{14} = q^8\sum_{m,n=-\infty}^{\infty} q^{20m^2+16m+20n^2+3n},\\
T_{15} &= q^7\sum_{m,n=-\infty}^{\infty} q^{20m^2+4m+20n^2+13n}, \quad T_{16} = q^{10}\sum_{m,n=-\infty}^{\infty} q^{20m^2+16m+20n^2+13n}.
\end{aligned}
$$

It can be shown that the 4-components of these sixteen sums $T_1, T_2, \ldots, T_{16}$, are, respectively,

$$
\begin{aligned}
&q^{24}\sum_{r,s=-\infty}^{\infty} q^{100r^2+85r+100s^2+50s}, \quad q^{24}\sum_{r,s=-\infty}^{\infty} q^{100r^2+85r+100s^2+50s},\\
&q^{9}\sum_{r,s=-\infty}^{\infty} q^{100r^2+50r+100s^2+35s}, \quad q^{9}\sum_{r,s=-\infty}^{\infty} q^{100r^2+50r+100s^2+35s},\\
&q^{19}\sum_{r,s=-\infty}^{\infty} q^{100r^2+65r+100s^2+50s}, \quad q^{19}\sum_{r,s=-\infty}^{\infty} q^{100r^2+65r+100s^2+50s},\\
&q^{9}\sum_{r,s=-\infty}^{\infty} q^{100r^2+50r+100s^2+15s}, \quad q^{9}\sum_{r,s=-\infty}^{\infty} q^{100r^2+50r+100s^2+15s},
\end{aligned}
$$

$$q^{19}\sum_{r,s=-\infty}^{\infty} q^{100r^2+65r+100s^2+50s}, \quad q^{19}\sum_{r,s=-\infty}^{\infty} q^{100r^2+65r+100s^2+50s},$$
$$q^{9}\sum_{r,s=-\infty}^{\infty} q^{100r^2+50r+100s^2+15s}, \quad q^{9}\sum_{r,s=-\infty}^{\infty} q^{100r^2+50r+100s^2+15s},$$
$$q^{29}\sum_{r,s=-\infty}^{\infty} q^{100r^2+50r+100s^2+85s}, \quad q^{29}\sum_{r,s=-\infty}^{\infty} q^{100r^2+50r+100s^2+85s},$$
$$q^{14}\sum_{r,s=-\infty}^{\infty} q^{100r^2+50r+100s^2+35s}, \quad q^{14}\sum_{r,s=-\infty}^{\infty} q^{100r^2+50r+100s^2+35s}.$$

As these cancel in pairs, we arrive at $\ell_{5n+4} = 0$ to complete the proof.

3.8 Proof of Theorem 3.1.7

We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} s_n q^n \\
&= (q, q^4; q^5)_\infty^3 (-q^3, -q^7; q^{10})_\infty \\
&= \frac{1}{(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty} f^3(-q,-q^4) f(q^3,q^7) \\
&= \frac{1}{(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty} f(-q,-q^4) f(q^3,q^7) \\
&\quad \times \big(f(q^2,q^8)f(q^5,q^5) - qf(q^3,q^7)f(1,q^{10})\big) \\
&= \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^6 (q^{20};q^{20})_\infty^2} f(-q,-q^4) f(q^2,q^3) \\
&\quad - 2q \frac{(q^{20};q^{20})_\infty^2}{(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty^2} f(-q,-q^4) f^2(q^3,q^7) \\
&= \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^6 (q^{20};q^{20})_\infty^2} \big(f(q^7,q^{13}) - qf(q^3,q^{17})\big)\big(f(q^9,q^{11}) + q^2 f(q,q^{19})\big) \\
&\quad - 2q \frac{(q^{20};q^{20})_\infty^2}{(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty^2} f(-q,-q^4) \\
&\qquad \times \big(f(q^6,q^{14})f(q^{10},q^{10}) + q^3 f(q^4,q^{16})f(q^{20},q^{20})\big) \\
&= \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^6 (q^{20};q^{20})_\infty^2} \big(f(q^7,q^{13})f(q^9,q^{11}) - qf(q^3,q^{17})f(q^9,q^{11}) \\
&\quad + q^2 f(q^7,q^{13})f(q,q^{19}) - q^3 f(q^3,q^{17})f(q,q^{19})\big)
\end{aligned}$$

$$
\begin{aligned}
&- 2q\frac{(q^{20};q^{20})_\infty^7}{(q^5;q^5)_\infty^3(q^{10};q^{10})_\infty^4(q^{40};q^{40})_\infty^2}\big(f(q^7,q^{13})-qf(q^3,q^{17})\big)f(q^6,q^{14})\\
&- 4q^4\frac{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty^2}{(q^5;q^5)_\infty^3(q^{10};q^{10})_\infty^2}\big(f(q^7,q^{13})-qf(q^3,q^{17})\big)f(q^4,q^{16})\\
=&\frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^6(q^{20};q^{20})_\infty^2}\big(S_1-S_2+S_3-S_4\big)\\
&- 2\frac{(q^{20};q^{20})_\infty^7}{(q^5;q^5)_\infty^3(q^{10};q^{10})_\infty^4(q^{40};q^{40})_\infty^2}\big(S_5-S_6\big)\\
&- 4\frac{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty^2}{(q^5;q^5)_\infty^3(q^{10};q^{10})_\infty^2}\big(S_7-S_8\big),
\end{aligned}
$$

where

$$
\begin{aligned}
S_1 &= \sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+3m+n}, \quad S_2 = q\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+7m+n},\\
S_3 &= q^2\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+3m+9n}, \quad S_4 = q^3\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+7m+9n},\\
S_5 &= q\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+3m+4n}, \quad S_6 = q^2\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+7m+4n},\\
S_7 &= q^4\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+3m+6n}, \quad S_8 = q^5\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+7m+6n}.
\end{aligned}
$$

The 3-components of these eight sums $S_1, S_2, \ldots, S_8$, are, respectively,

$$
\begin{aligned}
&q^{13}\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+45r+25s}, \quad q^{13}\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+45r+25s},\\
&q^{3}\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+5r+25s}, \quad q^{3}\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+5r+25s},\\
&q^{8}\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+30r+25s}, \quad q^{8}\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+30r+25s},\\
&q^{8}\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+20r+25s}, \quad q^{8}\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+20r+25s}.,
\end{aligned}
$$

and these cancel in pairs. Therefore, $s_{5n+3} = 0$.

Similarly, it can be shown that

$$
\sum_{n=0}^{\infty} t_n q^n = (q^2,q^3;q^5)_\infty^3(-q,-q^9;q^{10})_\infty
$$

$$
\begin{aligned}
&= \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^6(q^{20};q^{20})_\infty^2}\big(T_1 - T_2 + T_3 - T_4\big)\\
&\quad - 2\frac{(q^{20};q^{20})_\infty^7}{(q^5;q^5)_\infty^3(q^{10};q^{10})_\infty^4(q^{40};q^{40})_\infty^2}\big(T_5 - T_6\big)\\
&\quad - 4\frac{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty^2}{(q^5;q^5)_\infty^3(q^{10};q^{10})_\infty^2}\big(T_7 - T_8\big),
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 &= \sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+m+3n}, \quad T_2 = q^2\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+9m+3n},\\
T_3 &= q\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+m+7n}, \quad T_4 = q^3\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+9m+7n},\\
T_5 &= q^2\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+m+8n}, \quad T_6 = q^4\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+9m+8n},\\
T_7 &= q^3\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+m+2n}, \quad T_8 = q^5\sum_{m,n=-\infty}^{\infty} q^{10m^2+10n^2+9m+2n}.
\end{aligned}
$$

The 4-components of $T_1, T_2, \ldots, T_8$, are, respectively,

$$
\begin{aligned}
&q^9\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+25r+35s}, \quad q^9\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+25r+35s},\\
&q^4\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+25r+15s}, \quad q^4\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+25r+15s},\\
&q^4\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+25r+10s}, \quad q^4\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+25r+10s},\\
&q^{14}\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+25r+40s}, \quad q^{14}\sum_{r,s=-\infty}^{\infty} q^{50r^2+50s^2+25r+40s}.
\end{aligned}
$$

Since these cancel in pairs, we arrive at $t(5n+4) = 0$ to complete the proof.

3.9 Proof of Theorem 3.1.8

Omitting some intermediate steps, we have

$$\sum_{n=0}^{\infty} u_n q^n \tag{3.9.1}$$

$$
\begin{aligned}
&= (q,q^4;q^5)_\infty^3 (q^3,q^7;q^{10})_\infty \\
&= \frac{1}{(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty} f^3(-q,-q^4) f(-q^3,-q^7) \\
&= \frac{(q^{10};q^{10})_\infty^3}{(q^5;q^5)_\infty^4 (q^{20};q^{20})_\infty} \big(f(q^7,q^{13}) f(-q^9,-q^{11}) - q f(q^3,q^{17}) f(-q^9,-q^{11}) \\
&\quad + q^2 f(q^7,q^{13}) f(-q,-q^{19}) - q^3 f(q^3,q^{17}) f(-q,-q^{19}) \big) \\
&\quad - 2q \frac{(q^{20};q^{20})_\infty}{(q^5;q^5)_\infty^3} \big(f(q^7,q^{13}) f(-q^6,-q^{14}) - q f(q^3,q^{17}) f(-q^6,-q^{14}) \big) \\
&= \frac{(q^{10};q^{10})_\infty^3}{(q^5;q^5)_\infty^4 (q^{20};q^{20})_\infty} \big(P_1P_3 - qP_2P_3 + q^2 P_1P_4 - q^3 P_2 P_4 \big) \\
&\quad - 2 \frac{(q^{20};q^{20})_\infty}{(q^5;q^5)_\infty^3} \big(qP_1P_5 - q^2 P_2 P_5 \big),
\end{aligned}
\tag{3.9.2}
$$

where

$$
P_1 = f(q^7,q^{13}),\ P_2 = f(q^3,q^{17}),\ P_3 = f(-q^9,-q^{11}),\ P_4 = f(-q,-q^{19}),
$$

and

$$
P_5 = f(-q^6,-q^{14}).
$$

Now,

$$
\begin{aligned}
&P_1P_3 \\
&= f(q^7,q^{13}) f(-q^9,-q^{11}) \\
&= \sum_{m=-\infty}^{\infty} q^{10m^2+3m} \sum_{n=-\infty}^{\infty} (-1)^n q^{10n^2+n} \\
&= \left(\sum_{m=-\infty}^{\infty} q^{40m^2+6m} + q^7 \sum_{m=-\infty}^{\infty} q^{40m^2+34m} \right) \\
&\quad \times \left(\sum_{n=-\infty}^{\infty} q^{40n^2+2n} - q^9 \sum_{n=-\infty}^{\infty} q^{40n^2+38n} \right) \\
&= S_1 + S_2 - S_3 - S_4,
\end{aligned}
$$

where

$$
S_1 = \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+6m+2n}, \quad S_2 = q^7 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+34m+2n},
$$

$$S_3 = q^9 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+6m+38n}, \quad S_4 = q^{16} \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+34m+38n}.$$

Similarly,

$$\begin{aligned} qP_2P_3 &= S_5 + S_6 - S_7 - S_8, \\ q^2P_1P_4 &= S_9 + S_{10} - S_{11} - S_{12}, \\ q^3P_2P_4 &= S_{13} + S_{14} - S_{15} - S_{16}, \\ qP_1P_5 &= S_{17} + S_{18} - S_{19} - S_{20}, \\ q^2P_2P_5 &= S_{21} + S_{22} - S_{23} - S_{24}, \end{aligned}$$

where

$$S_5 = q \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+14m+2n}, \quad S_6 = q^4 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+26m+2n},$$

$$S_7 = q^{10} \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+14m+38n}, \quad S_8 = q^{13} \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+26m+38n},$$

$$S_9 = q^2 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+6m+18n}, \quad S_{10} = q^9 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+34m+18n},$$

$$S_{11} = q^3 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+6m+22n}, \quad S_{12} = q^{10} \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+34m+22n},$$

$$S_{13} = q^3 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+14m+18n}, \quad S_{14} = q^6 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+26m+18n},$$

$$S_{15} = q^4 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+14m+22n}, \quad S_{16} = q^7 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+26m+22n},$$

$$S_{17} = q \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+6m+8n}, \quad S_{18} = q^8 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+34m+8n},$$

$$S_{19} = q^7 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+6m+32n}, \quad S_{20} = q^{14} \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+34m+32n},$$

$$S_{21} = q^2 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+14m+8n}, \quad S_{22} = q^5 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+26m+8n},$$

$$S_{23} = q^8 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+14m+32n}, \quad S_{24} = q^{11} \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+26m+32n}.$$

The 4-components of each of S_1, S_2, ..., S_{24}, are, respectively,

$$q^{84}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+70r+250s}, \quad q^{49}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+130r+150s},$$
$$q^{9}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+70r+50s}, \quad q^{94}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+270r+50s},$$
$$q^{69}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+230r+50s}, \quad q^{4}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+30r+50s},$$
$$q^{64}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+170r+150s}, \quad q^{79}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+30r+250s},$$
$$q^{94}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+270r+50s}, \quad q^{9}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+70r+50s},$$
$$q^{49}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+130r+150s}, \quad q^{84}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+70r+250s},$$
$$q^{79}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+30r+250s}, \quad q^{64}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+170r+150s},$$
$$q^{4}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+30r+50s}, \quad q^{69}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+230r+50s},$$
$$q^{79}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+230r+100s}, \quad q^{14}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+30r+100s},$$
$$q^{79}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+230r+100s}, \quad q^{14}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+30r+100s},$$
$$q^{34}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+130r+100s}, \quad q^{19}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+70r+100s},$$
$$q^{34}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+130r+100s}, \quad q^{19}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+70r+100s}.$$

These components cancel in pairs. Therefore, from (3.9.1), we conclude that $u_{5n+4} = 0$.

In a similar way, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} v_n q^n &= (q^2, q^3; q^5)_\infty^3 (q, q^9; q^{10})_\infty \\ &= \frac{1}{(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty} f^3(-q^2,-q^3) f(-q,-q^9)\end{aligned}$$

$$= \frac{(q^{10};q^{10})_\infty^3}{(q^5;q^5)_\infty^4(q^{20};q^{20})_\infty}\big(Q_1Q_3 - q^2Q_2Q_3 - qQ_1Q_4 + q^3Q_2Q_4\big)$$
$$- 2\frac{(q^{20};q^{20})_\infty}{(q^5;q^5)_\infty^3}\big(q^2Q_1Q_5 - q^4Q_2Q_5\big), \tag{3.9.3}$$

where

$$Q_1 = f(q^9,q^{11}),\ Q_2 = f(q,q^{19}),\ Q_3 = f(-q^7,-q^{13}),\ Q_4 = f(-q^3,-q^{17})$$

and

$$Q_5 = f(-q^2,-q^{18}).$$

We have

$$Q_1Q_3 = T_1 + T_2 - T_3 - T_4,$$
$$q^2Q_2Q_3 = T_5 + T_6 - T_7 - T_8,$$
$$qQ_1Q_4 = T_9 + T_{10} - T_{11} - T_{12},$$
$$q^3Q_2Q_4 = T_{13} + T_{14} - T_{15} - T_{16},$$
$$q^4Q_2Q_5 = T_{21} + T_{22} - T_{23} - T_{24},$$
$$q^2Q_1Q_5 = T_{17} + T_{18} - T_{19} - T_{20},$$

where

$$T_1 = \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+2m+6n}, \quad T_2 = q^9\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+38m+6n},$$
$$T_3 = q^7\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+2m+34n}, \quad T_4 = q^{16}\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+38m+34n},$$
$$T_5 = q^2\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+18m+6n}, \quad T_6 = q^3\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+22m+6n},$$
$$T_7 = q^9\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+18m+34n}, \quad T_8 = q^{10}\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+22m+34n},$$
$$T_9 = q\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+2m+14n}, \quad T_{10} = q^{10}\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+38m+14n},$$
$$T_{11} = q^4\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+2m+26n}, \quad T_{12} = q^{13}\sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+38m+26n},$$

$$T_{13} = q^3 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+18m+14n}, \quad T_{14} = q^4 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+22m+14n},$$

$$T_{15} = q^6 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+18m+26n}, \quad T_{16} = q^7 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+22m+26n},$$

$$T_{17} = q^2 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+2m+16n}, \quad T_{18} = q^{11} \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+38m+16n},$$

$$T_{19} = q^4 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+2m+24n}, \quad T_{16} = q^{13} \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+38m+24n},$$

$$T_{21} = q^4 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+18m+16n}, \quad T_{22} = q^5 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+22m+16n},$$

$$T_{23} = q^6 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+18m+24n}, \quad T_{24} = q^7 \sum_{m,n=-\infty}^{\infty} q^{40m^2+40n^2+22m+24n}.$$

The respective 3-components of T_1, T_2, $\ldots$, T_{24} are

$$q^{38} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+150r+90s}, \quad q^{43} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+150r+110s},$$

$$q^{13} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+50r+90s}, \quad q^{18} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+50r+110s},$$

$$q^{48} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+50r+190s}, \quad q^{3} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+50r+10s},$$

$$q^{73} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+150r+190s}, \quad q^{28} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+150r+10s},$$

$$q^{43} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+150r+110s}, \quad q^{38} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+150r+90s},$$

$$q^{18} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+50r+110s}, \quad q^{13} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+50r+90s},$$

$$q^{3} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+50r+10s}, \quad q^{48} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+50r+190s},$$

$$q^{28} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+150r+10s}, \quad q^{73} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+150r+190s},$$

$$q^{58} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+100r+190s}, \quad q^{13} \sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+100r+10s},$$

$$q^{58}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+100r+190s}, \quad q^{13}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+100r+10s},$$
$$q^{28}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+100r+110s}, \quad q^{23}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+100r+90s},$$
$$q^{28}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+100r+110s}, \quad q^{23}\sum_{r,s=-\infty}^{\infty} q^{200r^2+200s^2+100r+90s}.$$

These cancel in pairs. Therefore, from (3.9.3), we arrive at $v(5n+3) = 0$ to finish the proof.