Chapter 4

Discrete Quasi-Lindley Distribution

4.1 Introduction

Shanker and Mishra [44] introduced Quasi Linley distribution, with parameter $\alpha\ and\ \theta$ which is given by

$$f(x{:}\,\alpha,\theta) = \frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}. \quad x \geq 0, \theta > 0,\ \alpha > 0. \tag{4.1.1}$$

4.2 Derivation of Discrete Quasi-Lindley (DQL) distribution

Let us consider the survival function of Quasi Lindley distribution

$$S(x) = \frac{1+\alpha+\theta x}{\alpha+1}e^{-\theta x}, \qquad . \tag{4.2.1}$$

where $\lambda = e^{-\theta}$ and $log\lambda = -\theta$.

4.2.1 Probability Mass Function

The probability mass function (pmf) of two parameter discrete quasi Lindley (DQL) distribution may be obtained by using the survival function of quasi Lindley distribution

$$p_x = Pr(X = x) = S(x) - S(x+1)$$

$$= \frac{\lambda^x}{(\alpha+1)}\{(\alpha+1)(1-\lambda) + \{(\lambda-1)\,x + \lambda\}\log\lambda\},\ x = 0, 1, 2, 3, \ldots \tag{4.2.2}$$

It is reduced to the pmf of discrete Lindley (DL) distribution as

$$p_x = \frac{\lambda^x}{1+\alpha}[\lambda log\lambda + (1-\lambda)(1-log\lambda^{x+1})] \text{ , x=0,1,...} \quad (4.2.3)$$

whrere $\lambda = e^{-\theta}$.

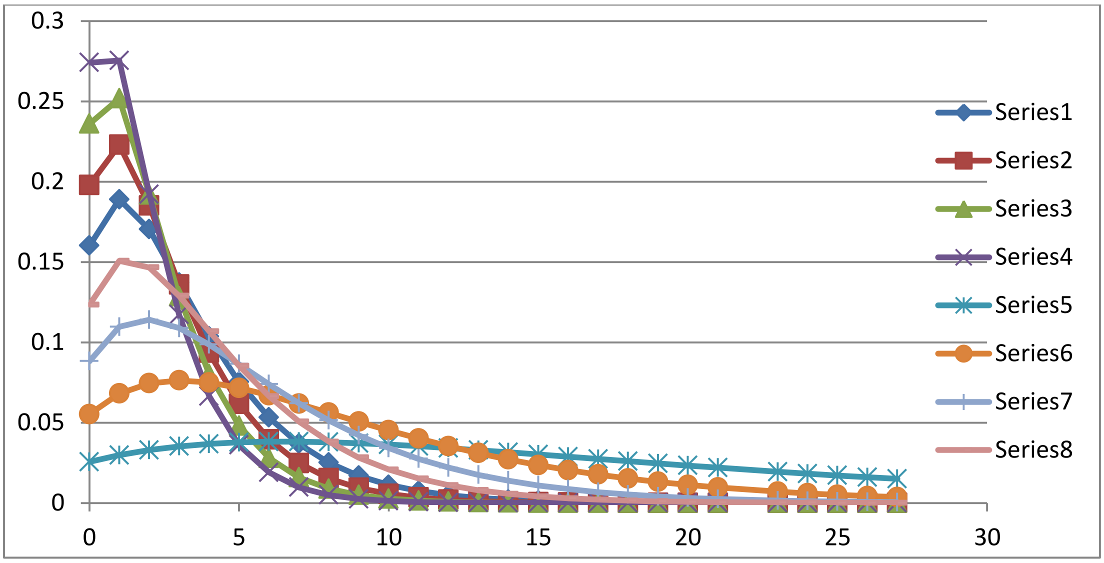

Figure 5: Probability graph for Discrete Quasi Lindley distribution $\alpha = 2, \theta = 0.1\ (series1) \alpha = 2, \theta = 0.3\ (series3) \alpha = 2, \theta = 0.4 (series4) \alpha = 2, \theta = 0.4 (series4) \alpha = 2, \theta = 0.5\ (series5) \alpha = 2, \theta = 0.6 (series6)$

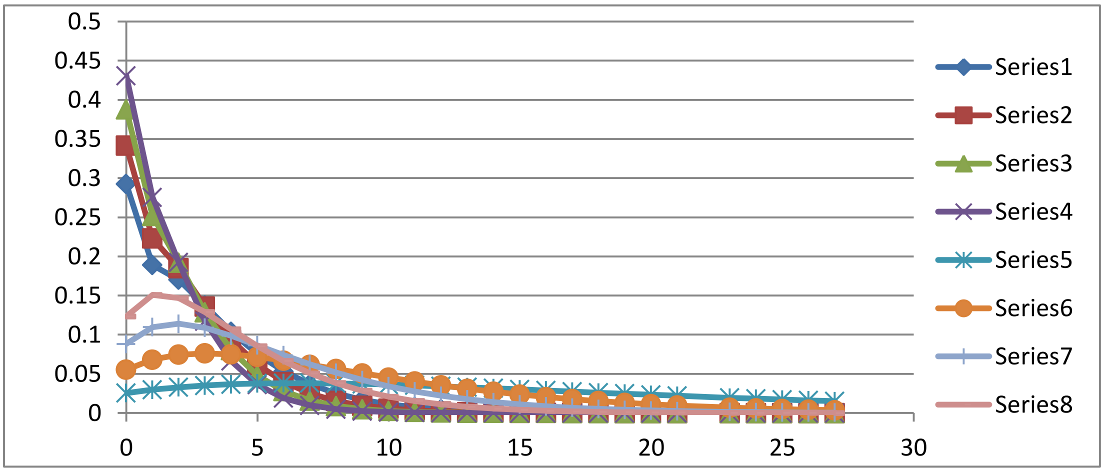

Figure 6: Probability graph for Discrete Quasi Lindley distribution $\theta = 1.5, \alpha = 2\ (series1) \theta = 1.5, \alpha = 3\ \ (series3) \theta = 1.5, \alpha = 4\ (series4) \theta = 1.5, \alpha = 2\ (series4) \theta = 1.5, \alpha = 5\ \ (series5) \theta = 1.5, \alpha = 6 (series6)$

4.2.2 Probability Generating Function

The probability generating function (pgf) of a discrete random variable following the DQL distribution is given by

$$G(t) = \frac{[(1-\lambda)(\alpha+1)+\lambda log\lambda](1-\lambda t)-(1-\lambda)\lambda t log\lambda}{(\alpha+1)(1-\lambda t)^2}\ . \qquad (4.2.4)$$

4.2.3 Recurrence Relation for Probabilities

The probability recurrence relation for two parameter DQL distribution can also be obtained as

$$p_{r+2} = \lambda(2p_{r+1} - \lambda p_r)\ , r \geq 0 \qquad (4.2.5)$$

where,

$$p_0 = \frac{(\alpha+1)(1-\lambda)+\lambda\ log\lambda}{\alpha+1}, \qquad (4.2.6)$$

$$p_1 = \lambda\frac{(\alpha+1)(1-\lambda)+(2\lambda-1)log\lambda}{\alpha+1}. \qquad (4.2.7)$$

4.2.4 Cumulative Distribution Function

The cumulative distribution function (cdf) of a discrete random variable following the is given by

$$F(x) = \frac{1}{\alpha+1}[1 - \lambda^{x+1} + \alpha(1 - \lambda^{x+1}) + (x + 1)\lambda^{x+1}log\lambda]. \qquad (4.2.8)$$

4.2.5 Survival Function

The survival function of DQL distribution can be obtained from the distribution function as

$$S_{DQL}(x) = 1 - F(x),$$

$$= \frac{\lambda^{x+1}\{1+\alpha-(x+1)log\lambda\}}{\alpha+1}\ . \qquad (4.2.9)$$

4.2.6 Failure or Hazard Rate Function

The failure rate function of DQL distribution can be obtained from the distribution function as

$$r(x) = \frac{P(X=x)}{P(X>x-1)}$$

$$= \frac{(\alpha+1)(1-\lambda)+\{(\lambda-1)x+\lambda\}log\lambda}{1+\alpha-xlog\lambda}. \quad (4.2.10)$$

4.2.7 Reversed Failure Rate Function

The failure rate function of DQL distribution can be obtained from the distribution function as

$$r^*(x) = \frac{P(X=x)}{P(X\leq x)}$$

$$= \frac{\lambda^x[((\alpha+1)(1-\lambda)+\{(\lambda-1)x+\lambda\}log\lambda]}{\{1-\lambda^{x+1}+\alpha(1-\lambda^{x+1})+(x+1)\lambda^{x+1}log\lambda\}}. \quad (4.2.11)$$

4.2.8 Second Rate of Failure Function

The second rate of failure function of DQL distribution can be obtained from the distribution function as

$$r^{**}(x) = log\left[\frac{S(x)}{S(x+1)}\right]$$

$$= log\left[\frac{\{(x+2)log\lambda-1\}}{\lambda\{(x+3)log\lambda-1\}}\right] = log\left[\frac{1}{\lambda}\left\{\frac{(x+2)log\lambda-1}{(x+3)log\lambda-1}\right\}\right]. \quad (4.2.12)$$

4.2.9 Proportion of Probabilities

The portion of probabilities of DQL distribution may be given as

$$\frac{P_{x+1}}{P_x} = \lambda\left[1+\frac{(2\lambda-1)log\lambda}{(\alpha+1)(1-\lambda)+\{(\lambda-1)x+\lambda\}log\lambda}\right], \quad (4.2.13)$$

4.2.10 Factorial Moment Generating Function

The factorial moment generating function $M_x(t)$ and r^{th} ordered factorial moment $\mu_{[r]}$ for two parameter DQL distribution can be written as

$$M_x(t) = \frac{[(\alpha+1)(1-\lambda)+\lambda log\lambda](1-\lambda-\lambda t)-(1-\lambda)\lambda(1+t)\log\lambda)}{(1-\lambda-\lambda t)^2\,(\alpha+1\,)}, \quad \text{and} \quad (4.2.14)$$

$$\mu_{[r]} = \frac{r!\lambda^r[(\alpha+1)(1-\lambda)-rlog\lambda]}{(1-\lambda)^{r+1}(\alpha+1)} \quad r = 1, 2, \ldots \text{ respectively.} \quad (4.2.15)$$

4.2.11 Factorial Moment Recurrence Relation

The recurrence relation for factorial moment can be obtained as

$$\mu_{[r+2]} = \frac{\lambda(r+2)}{(1-\lambda)^2}\left[2(1-\lambda)\mu_{[r+1]} - \lambda(r+1)\mu_{[r]}\right], \tag{4.2.16}$$

where

$$\mu_{[1]} = \frac{\lambda[(\alpha+1)(1-\lambda)-log\lambda]}{(1-\lambda)^2(\alpha+1)}, \tag{4.2.17}$$

$$\mu_{[2]} = \frac{2\lambda^2[(\alpha+1)(1-\lambda)-2log\lambda]}{(1-\lambda)^3(\alpha+1)}, \tag{4.2.18}$$

$$\mu_{[3]} = \frac{6\lambda^3[(\alpha+1)(1-\lambda)-3log\lambda]}{(1-\lambda)^4(\alpha+1)}, \text{ etc.} \tag{4.2.19}$$

From the above factorial moments, the mean μ and variance σ^2 can be derived as

$$\mu = \frac{\lambda[(\alpha+1)(1-\lambda)-log\lambda]}{(1-\lambda)^2(\alpha+1)}. \tag{4.2.20}$$

$$\sigma^2 = \frac{\lambda[(\alpha+1)^2(1-\lambda)^2-(\alpha+1)(1-\lambda)(1+\lambda)log\lambda-\lambda(log\lambda)^2]}{(1-\lambda)^4(\alpha+1)^2}. \tag{4.2.21}$$

4.3 Zero Truncated of ZTDQL Distribution

The pmf of Zero-truncated discrete Quasi Lindley (ZTDQL) $P_z(x)$distribution has been derived as

$$P_z(x) = \frac{P_x}{1-P_0}, \tag{4.3.1}$$

$where\ P_x$ denotes the pmf of discrete Quasi-Lindley distribution.

$$Hence,\quad P_z(x) = \lambda^{x-1}\frac{[(\alpha+1)(1-\lambda)+\{(\lambda-1)\,x+\lambda\}\log\lambda]}{(\alpha+1)-\log\lambda}, \quad x = 1,\ 2, \ldots \tag{4.3.2}$$

4.3.1 Probability Generating Function of ZTDQL Distribution

Probability generating function $G_z(t)$ of ZTDQL distribution may be obtained as

$$G_z(t) = \sum_{x=1}^{\infty} t^x P_z(x)$$

$$= \frac{t[\{(1-\lambda)(\alpha+1)+\lambda\log\lambda\}(1-\lambda t)-(1-\lambda)\log\lambda]}{[(\alpha+1)-\log\lambda](1-\lambda t)^2}. \qquad (4.3.3)$$

4.3.2 Probability Recurrence Relation of ZTDQL Distribution

Probability recurrence relation ZTDQL distribution may obtained as

$$P_r = e^{-\theta}\left[2P_{r-1} - e^{-\theta}P_{r-2}\right], \quad r > 2. \qquad (4.3.4)$$

Where

$$P_1 = \frac{[(\alpha+1)(1-\lambda)+\{(\lambda-1)+\lambda\}\log\lambda]}{(\alpha+1)-\log\lambda} \qquad (4.3.5)$$

$$P_2 = \lambda\frac{[(\alpha+1)(1-\lambda)+\{2(\lambda-1)+\lambda\}\log\lambda]}{(\alpha+1)-\log\lambda}, \qquad (4.3.6)$$

4.3.3 Cumulative Distribution of ZTDQL Distribution

The cumulative distribution of ZTDQL Lindley distribution is given by

$$F_z(x) = \frac{(\alpha+1)-\log\lambda-[(\alpha+1)-(1+x)\log\lambda]\lambda^x}{(\alpha+1)-\log\lambda}. \qquad (4.3.7)$$

4.3.4 Survival Function of ZTDQL Distribution

The survival function of Zero truncated of ZTDQL distribution is given by

$$S_z(x) = \frac{[(\alpha+1)-(1+x)\log\lambda]\lambda^x}{(\alpha+1)-\log\lambda}. \qquad (4.3.8)$$

4.3.5 Failure Hazard Rate Function of ZTDQL Distribution

The failure hazard rate function of Zero truncated of a new discrete Quasi Lindley Distribution is given by

$$r_z(x) = \frac{P(X=x)}{P(X\geq x-1)}$$

$$= \frac{(\alpha+1)(1-\lambda)+\{(\lambda-1)x+\lambda\}\log\lambda}{(\alpha+1)-x\log\lambda}. \qquad (4.3.9)$$

4.3.6 Reversed Failure Rate of ZTDQL Distribution

The reversed failure rate function of Zero truncated of a new discrete Quasi Lindley Distribution is given by

$$r_z^*(x) = \frac{P(X=x)}{P(X\le x)}$$

$$= \frac{[(\alpha+1)(1-\lambda)+\{(\lambda-1)\ x+\lambda\}\log\lambda]\lambda^{x-1}}{(\alpha+1)-\log\lambda-[(\alpha+1)-(1+x)\log\lambda]\lambda^{x}}\ . \qquad (4.3.10)$$

4.3.7 Second Rate of Failure of ZTDQL Distribution

The second rate failure rate function of Zero truncated of a new discrete Quasi Lindley is given by

$$r_z^{**}(x) = log\left[\frac{s(x)}{s(x+1)}\right]$$

$$= log\left[\frac{(\alpha+1)-(1+x)\log\lambda}{\lambda\{(\alpha+1)-(2+x)\log\lambda\}}\right]. \qquad (4.3.11)$$

4.3.8 Proportions of Probabilities of ZTDQL Distribution

The proportions of probabilities of Zero truncated of a new discrete Quasi Lindley Distribution is given by

$$\frac{P_z(x+1)}{P_z(x)} = \lambda\left[1+\frac{(2\lambda-1)\log\lambda}{(\alpha+1)(1-\lambda)+\{(\lambda-1)\ x+\lambda\}\log\lambda}\right]. \qquad (4.3.12)$$

4.3.9 Factorial Moment Generating Function of ZTNDQL Distribution

Factorial moment generating function $M_z(t)$ of ZTNDQL distribution may be obtained as

$$M_z(t) = \frac{(1+t)[\{(1-\lambda)(\alpha+1)+\lambda log\lambda\}(1-\lambda-\lambda t)-\lambda(1-\lambda)log\lambda]}{[(\alpha+1)-\log\lambda](1-\lambda-\lambda t)^2}. \qquad (4.3.13)$$

Factorial moment recurrence relation of zero-truncatedof ZTNDQL distribution may be obtained as

$$\mu'_{[r]} = \frac{e^{-\theta}}{(1-e^{-\theta})^2}\left[2(1-e^{-\theta})r - e^{-\theta}\mu'_{[r-1]} - r(r-1)e^{-\theta}\mu'_{[r-2]}\right],\quad r>2 \ (4.3.14)$$

where,

$$\mu'_{[1]} = \frac{[(1-\lambda)(\alpha+1)-\log\lambda)]}{(\alpha+1-\log\lambda)(1-\lambda)^2}, \qquad (4.3.15)$$

$$\mu'_{[2]} = \frac{2\,\lambda^1[(1-\lambda)(\alpha+1)-2\log\lambda)]}{(\alpha+1-\log\lambda)(1-\lambda)^3}, \tag{4.3.16}$$

$$\mu'_{[3]} = \frac{6\,\lambda^2[(1-\lambda)(\alpha+1)-3\log\lambda)]}{(\alpha+1-\log\lambda)(1-\lambda)^4}, \tag{4.3.17}$$

The mean μ and the variance σ^2 of the distribution may be obtained as

$$\mu \ = \frac{[(1-\lambda)(\alpha+1)-\log\lambda)]}{(\alpha+1-\log\lambda)(1-\lambda)^2}. \tag{4.3.18}$$

$$\sigma^2 =. \mu'_{[2]}+\mu'_{[1]}-\mu'^2_{[1]} \ . \tag{4.3.19}$$

The general form of factorial moments of may also be written as

$$\mu'_{[r]} = \frac{r!\,\lambda^{r-1}[(1-\lambda)(\alpha+1)-r\log\lambda)]}{(\alpha+1-\log\lambda)(1-\lambda)^{r+1}}. \qquad r = 1, 2, \ldots \tag{4.3.20}$$

4.4 Size–Biazed Discrete Quasi Lindley (SBDQL) Distribution

If the random variable X has pmf $f(x;\theta)$, with unknown parameter θ, then the corresponding weighted distribution is of the form $f^w(x;\theta) = \frac{w(x)f(x;\theta)}{E[w(x)]}$, where $w(x)$ is a non-negative weight function such that $E[w(x)]$.

$$f_s(x,\alpha) = \frac{xp_x}{\mu} = x\lambda^{x-1}\frac{[(\alpha+1)(1-\lambda)+\{(\lambda-1)\,x+\lambda\}\log\lambda](1-\lambda)^2}{(\alpha+1)(1-\lambda)-\log\lambda} \quad \text{x=1, 2, ...} \tag{4.4.1}$$

A special case of interest arise when the weight function $w(x) = x^\alpha$. Such distributions are known as sized biased distributions of order α. The most common case of size-biased distribution occur when $\alpha = 1$and $\alpha = 2$, these special cases may be termed as length (size) and area biased respectively. If a random variable X having pmf $f(x;\theta)$, hence the pmf of size-biased QL distribution may obtained as

$$f_s(x,\alpha) = \frac{xp_x}{\mu}, \mu \text{ denotes the mean of the DQL distribution}$$

$$= x\lambda^{x-1}\frac{[(\alpha+1)(1-\lambda)+\{(\lambda-1)\,x+\lambda\}\log\lambda](1-\lambda)^2}{(\alpha+1)(1-\lambda)-\log\lambda}\ .\, x = 1, 2, 3, \ldots \tag{4.4.2}$$

4.4.1 Probability Generating Function of SBDQL Distribution

The probability generating function of a discrete random variable following the SBDQL distribution is given by

$$G_s(t) = \frac{t[(1-\lambda)(\alpha+1)+\lambda log\lambda](1-\lambda)^2(1-\lambda t)-t(1-\lambda)^3(1+\lambda t)log\lambda}{[(\alpha+1)(1-\lambda)-\log\lambda](1-\lambda t)^3} \quad (4.3.3)$$

4.4.2 Probability Recurrence Relation of SBDQL Distribution

Probability recurrence relation of the DQL distribution is obtained as

$$p_r = 3\lambda P_{r-1} - 3\lambda^2 p_{r-2} + \lambda^3 p_{r-3} \text{for } r > 2, \text{ and} \quad (4.4.4)$$

where,

$$p_1 = \frac{[(\alpha+1)(1-\lambda)+(2\lambda-1)\log\lambda](1-\lambda)^2}{(\alpha+1)(1-\lambda)-\log\lambda}, \quad (4.4.5)$$

$$p_2 = 2\lambda \frac{[(\alpha+1)(1-\lambda)+(3\lambda-2)\log\lambda](1-\lambda)^2}{(\alpha+1)(1-\lambda)-\log\lambda} \quad (4.4.6)$$

$$p_3 = 3\lambda^2 \frac{[(\alpha+1)(1-\lambda)+(4\lambda-3)\log\lambda](1-\lambda)^2}{(\alpha+1)(1-\lambda)-\log\lambda} \;. \quad (4.4.7)$$

4.4.3 Factorial Moments Generating Function (SBDQL) Distribution

The factorial moments of size biased discrete quasi Lindley (SBDQL) distribution may be obtained as

$$\mu'_{[r]} = r!\,\lambda^{r-1} \frac{[(\alpha+1)(1-\lambda)(r+\lambda)-\{r^2+(2r+1)\lambda\}\log\lambda]}{[(\alpha+1)(1-\lambda)-\log\lambda](1-\lambda)^r} r = 1, 2, ... \quad (4.4.8)$$

from its factorial moment generating function (fmgf)

$$M_s(t) = \frac{(1+t)[(1-\lambda)(\alpha+1)+\lambda log\lambda](1-\lambda)^2(1-\lambda-\lambda t)-(1-\lambda)^3(1+\lambda+\lambda t)(1+t)log\lambda}{[(\alpha+1)(1-\lambda)-\log\lambda](1-\lambda-\lambda t)^3} \quad (4.4.9)$$

$$\mu'_{[1]} = \frac{[(\alpha+1)(1-\lambda)(1+\lambda)-\{1+3)\lambda\}\log\lambda]}{[(\alpha+1)(1-\lambda)-\log\lambda](1-\lambda)} \;. \quad (4.4.10)$$

$$\mu'_{[2]} = 2\lambda \frac{[(\alpha+1)(1-\lambda)(2+\lambda)-\{4+5\lambda\}\log\lambda]}{[(\alpha+1)(1-\lambda)-\log\lambda](1-\lambda)^2} \;. \quad (4.4.11)$$

$$\mu'_{[3]} = 6\lambda^2 \frac{[(\alpha+1)(1-\lambda)(3+\lambda)-\{9+7\lambda\}\log\lambda]}{[(\alpha+1)(1-\lambda)-\log\lambda](1-\lambda)^3} \;. \quad (4.4.12)$$

4.4.4 Factorial Recurrence Relation of SBDQL Distribution

Factorial recurrence relation may also be obtained as

$$\mu_{[r]} = \frac{1}{(1-\lambda)^3}[3(1-\lambda)^2\lambda r\mu_{[r-1]} - 3(1-\lambda)\lambda^2 r(r-1)\mu_{[r-2]} + \lambda^3 r(\mathrm{r}-1)(\mathrm{r} - 2)\mu_{[r-3]}]$$

for $r > 2$

(4.4.13)

where

$$\mu_{[1]} = \frac{[(\alpha+1)(1-\lambda)(1+\lambda)-(7+3\lambda)\log\lambda]}{[(\alpha+1)(1-\lambda)-\log\lambda](1-\lambda)} \quad (4.4.14)$$

$$\mu_{[2]} = 2\lambda\frac{[(\alpha+1)(1-\lambda)(2+\lambda)-(4+5\lambda)\log\lambda]}{[(\alpha+1)(1-\lambda)-\log\lambda](1-\lambda)^2} \quad (4.4.15)$$

$$\mu_{[3]} = 6\lambda^2\frac{[(\alpha+1)(1-\lambda)(3+\lambda)-(9+7\lambda)\log\lambda]}{[(\alpha+1)(1-\lambda)-\log\lambda](1-\lambda)^3} \quad (4.4.16)$$

4.5 Zero-Modified Discrete Quasi Lindley (ZMDQL) Distribution

In recent years there has been considerable and growing interest in modeling zero-modified count data. Zero-modified DQL model address the problem, that the data display a higher fraction of zeros, or non-occurrences, than can be possibly explained through any fitted standard count model. The zero-modified distributions are appropriate alternatives for modeling clustered samples when the population consists of two sub-populations, one containing only zeros, while in the other, counts from a discrete distribution are observed.

$$P_z[X=0] = \omega + (1-\omega)P_0$$

$$= \omega + (1-\omega)\left[\frac{(\alpha+1)(1-\lambda)+\lambda log\lambda}{\alpha+1}\right] \quad (4.5.1)$$

$$P_z[X=x] = (1-\omega)\lambda^x\frac{[(\alpha+1)(1-\lambda)+\{(\lambda-1)\,x+\lambda\}\log\lambda]}{(\alpha+1)} \qquad \text{x=1, 2, ...} \quad (4.5.2)$$

$$\alpha \geq 0, \qquad 0 < \lambda < 1, \qquad \omega \geq \frac{-P_0}{1-P_0}$$

where $P_z[X=x]$denotes the probability of ZMDQL distribution**.**

4.6 Estimation of Parameter of DQL Distribution

4.6.1. Estimation of parameters in terms of mean and variance of DQL distribution

From $\mu = \frac{\lambda[(\alpha+1)(1-\lambda)-log\lambda]}{(1-\lambda)^2(\alpha+1)}$ the mean of DQL distribution, the value of $\lambda log\lambda$ may be expressed as $(1-\lambda)(\alpha+1)-(\lambda-(1-\lambda)\mu)$. Now putting the value of $\lambda log\lambda$ in $\sigma^2 = \frac{\lambda[(\alpha+1)^2(1-\lambda)^2-(\alpha+1)(1-\lambda)(1+\lambda)log\lambda-\lambda(log\lambda)^2]}{(1-\lambda)^4(\alpha+1)^2}$ the variance of DQL distribution, the quadratic equation in λ may be obtained as

$$\lambda^2 A - 2\lambda B + C = 0 \tag{4.6.1}$$

Given a random sample $x_1, x_2, \ldots x_n$ of size *n* from the DQL distribution with the pmf (4.2.3), the moment estimate $\hat{\lambda}$ of DQL distribution may be obtained by the quadratic equation (4.6.1) as

$$\hat{\lambda} = \frac{B \pm \sqrt{B^2 - AC}}{A}, \tag{4.6.2}$$

where $A = \sigma^2 + \mu^2 + 3\mu + 2$, $B = \sigma^2 + \mu^2 + \mu$ and $C = \sigma^2 + \mu^2 - \mu$.
There are two values of λ had solving equation (4.6.2). We choose that the value $\hat{\lambda}$ had which minimizes the value of χ^2 static in table 2.1- 2.3, column 5. Now putting the value of $\hat{\lambda}$ from (4.6.2) and putting in mean we have α as following

$$\alpha = \frac{-\lambda log\lambda}{\mu(1-\lambda)^2 - \lambda(1-\lambda)} - 1 \tag{4.6.3}$$

4.6.2 Maximum Likelihood Estimates

The likelihood function, L of the two parameter discrete quasi Lindley distribution (4.2.3) is given by

$$L = \prod_{x=1}^{k} P_x^{\,f_x} \tag{4.6.4}$$

$$L = \frac{e^{-\theta n\bar{x}}}{(\alpha+1)^n} \prod_{x=1}^{k} \left[(\alpha+1)\left(1-e^{-\theta}\right) - \theta\left\{\left(e^{-\theta}-1\right)x + e^{-\theta}\right\}\right]^{f_x}. \tag{4.6.5}$$

The log likelihood function is obtained as

$$logL = -\theta n\bar{x} - nlog(\alpha+1) + G \tag{4.6.6}$$

Where,

$$G = \sum_{x=1}^{k} f_x log\left[(\alpha+1)\left(1-e^{-\theta}\right) - \theta\left\{\left(e^{-\theta}-1\right)x + e^{-\theta}\right\}\right]$$

The derivative of log likelihood equations are thus obtained as

$$\frac{\partial logL}{\partial \theta} = -n\bar{x} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[(\alpha+1)\left(1-e^{-\theta}\right)-\theta\left\{\left(e^{-\theta}-1\right)x+e^{-\theta}\right\}\right]}{\partial\theta}}{\left[(\alpha+1)\left(1-e^{-\theta}\right)-\theta\left\{\left(e^{-\theta}-1\right)x+e^{-\theta}\right\}\right]} = 0. \tag{4.6.7}$$

$$\frac{\partial logL}{\partial \alpha} = -\frac{n}{(\alpha+1)} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[(\alpha+1)\left(1-e^{-\theta}\right)-\theta\left\{\left(e^{-\theta}-1\right)x+e^{-\theta}\right\}\right]}{\partial \alpha}}{\left[(\alpha+1)\left(1-e^{-\theta}\right)-\theta\left\{\left(e^{-\theta}-1\right)x+e^{-\theta}\right\}\right]} = 0. \quad (4.6.8)$$

The two equations (4.6.7) and (4.6.8) cannot be solved directly. However the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 logL}{\partial \theta^2} = \frac{\partial}{\partial \theta} \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[(\alpha+1)\left(1-e^{-\theta}\right)-\theta\left\{\left(e^{-\theta}-1\right)x+e^{-\theta}\right\}\right]}{\partial \theta}}{\left[(\alpha+1)\left(1-e^{-\theta}\right)-\theta\left\{\left(e^{-\theta}-1\right)x+e^{-\theta}\right\}\right]}. \quad (4.6.9)$$

$$\frac{\partial^2 logL}{\partial \theta \partial \alpha} = \frac{\partial}{\partial \alpha} \sum_{x=1}^{k} f_x \frac{\frac{\partial l\left[(\alpha+1)\left(1-e^{-\theta}\right)-\theta\left\{\left(e^{-\theta}-1\right)x+e^{-\theta}\right\}\right]}{\partial \theta}}{\left[(\alpha+1)\left(1-e^{-\theta}\right)-\theta\left\{\left(e^{-\theta}-1\right)x+e^{-\theta}\right\}\right]}. \quad (4.6.10)$$

$$\frac{\partial^2 logL}{\partial \alpha^2} = \frac{n}{(\alpha+1)^2} + \frac{\partial}{\partial \alpha} \sum_{x=1}^{k} f_x \frac{\frac{\partial l\left[(\alpha+1)\left(1-e^{-\theta}\right)-\theta\left\{\left(e^{-\theta}-1\right)x+e^{-\theta}\right\}\right]}{\partial \alpha}}{\left[(\alpha+1)\left(1-e^{-\theta}\right)-\theta\left\{\left(e^{-\theta}-1\right)x+e^{-\theta}\right\}\right]}. \quad (4.6.11)$$

The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved

$$\begin{bmatrix} \frac{\partial^2 logL}{\partial \theta^2} & \frac{\partial^2 logL}{\partial \theta \partial \alpha} \\ \frac{\partial^2 logL}{\partial \alpha \partial \theta} & \frac{\partial^2 logL}{\partial \alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial logL}{\partial \theta} \\ \frac{\partial logL}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}, \quad (4.6.12)$$

where θ_0 and $\alpha_0$are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently close estimates of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

4.7 Goodness of Fit

The fittings of the two-parameter DQL distribution based on three data-sets have been presented in the following tables. The expected frequencies according to the one parameter Poisson- Lindley with parameter θ in Table 4.1 presented by Sankaran [40], two parameter Poisson- Lindley distributions with parameter θ and α in Table 4.2 presented by Shanker et al. [45] have also been given for ready comparison with DQL distribution. The estimates of the parameters have been obtained by the method of moments.

Table 4.1 Observed and expected frequencies for distribution of Pyrausta nublilalis in 1937.

No. of accidents	Observed frequencies	Expected frequencies		
		Poisson-Lindley (θ)	Poisson-Lindley (θ, α)	DQL (α, θ)
0	33	31.49	31.9	28.16
1	12	14.16	13.8	16.65
2	6	6.09	5.9	7.21
3	3	2.54	2.5	2.72
4	1	1.04	1.1	0.94
≥5	1	0.42	0.8	0.32
	56	56	56	56
		$\hat{\theta} = 1.8082$	$\hat{\alpha} = 0.2573$	$\hat{\alpha} = 0.637080$
			$\hat{\theta} = 0.39249$	$\hat{\theta} = 1.295001$
	χ^2	4.82	0.36	2.134
	$P\ value$	0.1855	0.8353	0.3441

Table 4.2 Distribution of number of epileptic seizure counts

European red mites	Observed frequencies	Expected frequencies		
		GPL (α, θ)	NBD (r,p)	TPDL (θ, β)
0	126	121.51	91.0	125.33
1	80	95.81	86.60	89.20
2	59	59.81	63.37	57.42
3	42	34.49	42.57	34.85
4	24	19.24	27.60	20.55
5	8	10.59	17.60	11.36
6	5	5.81	10.50	6.32
7	4	3.18	6.52	3.66
8	3	3.88	5.00	1.90
	351	351 $\hat{\alpha} = 1.139$ $\hat{\theta} = 1.292$ $\chi^2 = 5.94$	351 $\hat{r} = 1.757$ $\hat{p} = .463$ $\chi^2 = 22.53$	351 $\hat{\alpha} = 1.260164$ $\hat{\theta} = 0.717397$ $\chi^2 = 4.677$

4.8 Conclusion

Two-parameter DQL distribution has been introduced. Several properties of the two-parameter DQL distribution haven discussed. Estimation of parameters by the method of maximum likelihood and the method of moments have been discussed. The properties of size- biased and Zero- truncated version of DQL distribution have also been investigated. Finally, the proposed distribution has been fitted to a number of data sets. It is observed that two-parameter DQL provides better fits.