Chapter 5

A New Discrete Quasi-Lindley Distribution

5.1 Introduction

Two parameter continuous New Quasi Lindley (NQL) distribution introduced by Shanker and Amannuel [42] with parameter $\theta\ and\ \alpha$ is defined by its probability density function (pdf)

$$f(x;\ \theta,\alpha) = \frac{\theta^2(\theta+\alpha x)e^{-\theta x}}{\theta^2+\alpha}.\quad x \geq 0, \theta > 0,\ \alpha > 0 \qquad (5.1.1)$$

5.2. Discretization of a New Quasi Lindley Distribution

In this paper, our objective is to derive a new discrete distribution and to study some of their properties, which may be called New discrete Quasi–Lindley (NDQL) distribution based on the survival function of the continuous NQL distribution. The survival function may be obtained as

$$S(x) = \int_x^\infty f(x;\ \theta,\alpha)dx$$

$$= \frac{[(\theta^2+\alpha(\theta X+1)]}{\theta^2+\alpha}\ e^{-\theta x}. \qquad (5.2.1)$$

Hence,

$$S(x+1) = \frac{[(\theta^2+\alpha+\alpha\theta(X+1)]}{\theta^2+\alpha}\ e^{-\theta(x+1)}. \qquad (5.2.2)$$

5.2.1 Probability Mass Function (pmf)

The probability mass function (pmf) of NDQL distribution may be obtained as

$$P(X = x) = S(x) - S(x+1)$$

$$= \frac{(\theta^2+\alpha+\alpha\theta x)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}{\theta^2+\alpha}\, e^{-\theta x}, \qquad x = 0,\ 1, 2,\ 3\ \ldots \qquad (5.2.3)$$

Where α and θ denote its parameter.

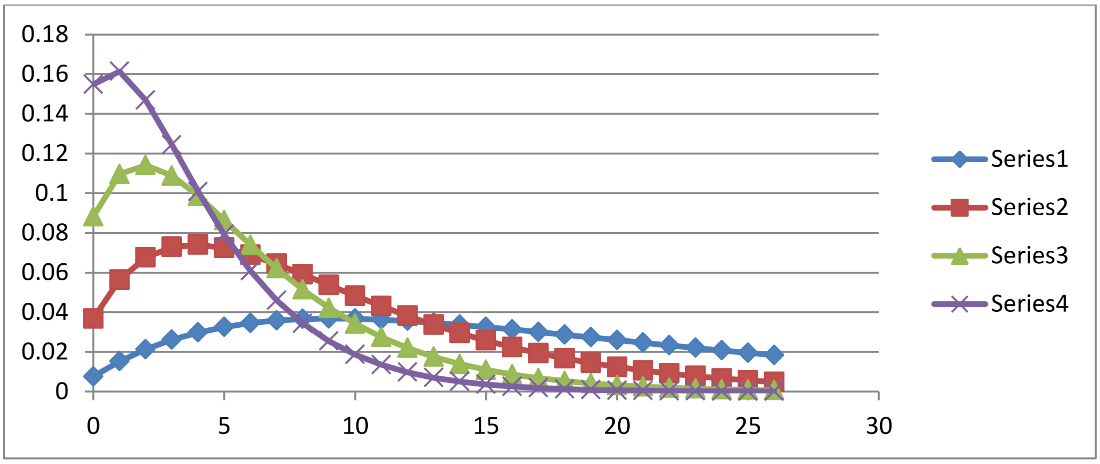

Figure 7: Probability graph for New discrete quasi Lindley distribution $\alpha = 0.3, \theta = 0.1\ (series1) \alpha = 0.3, \theta = 0.2\ (series2)\ \ \alpha = 0.3, \theta = 0.3\ \ (series3).\ \alpha = 0.3, \theta = 0.4 (series4)$

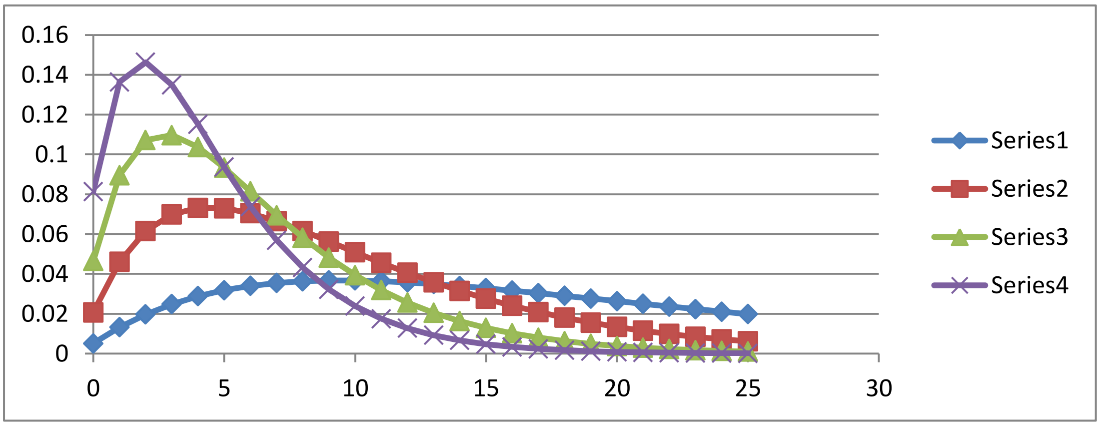

Figure 8: Probability graph for New discrete quasi Lindley distribution $\alpha = 2, \theta = 0.1\ (series1)\ \ \alpha = 2, \theta = 0.2\ (series2)\ \ \alpha = 2, \theta = 0.3\ \ (series3).\ \alpha = 2, \theta = 0.4 (series4)$

5.2.2 Probability Generating Function (pgf)

The probability generating function (pgf) of NDQL distribution may be given as

$$G(t) = \frac{[(\theta^2+\alpha)(1-e^{-\theta})-\alpha\theta e^{-\theta}](1-e^{-\theta}t)+\alpha\theta\, e^{-\theta}t\,(1-e^{-\theta})}{(\theta^2+\alpha)(1-e^{-\theta}t)^2}. \tag{5.2.4}$$

5.2.3 Cumulative Distribution Function

The cumulative distribution of NDQL distribution may be writen as

$$F(x) = \frac{\theta^2+\alpha-[\theta^2+\alpha+\theta\alpha(x+1)]e^{-\theta(x+1)}}{(\theta^2+\alpha)}. \tag{5.2.5}$$

5.2.4 Survival Function

The survival function of NDQL distribution has been obtained as

$$S(x) = \frac{[\theta^2+\alpha+\theta\alpha(x+1)]e^{-\theta(x+1)}}{(\theta^2+\alpha)}. \tag{5.2.6}$$

5.2.5 Failure Rate Function

The failure hazard rate function of NDQL distribution has been obtained as

$$r(x) = \frac{P(X=x)}{P(X\geq x-1)}$$

$$= \frac{(\theta^2+\alpha+\alpha\theta x)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}{\theta^2+\alpha+\theta\alpha x}. \tag{5.2.7}$$

5.2.6 Reversed Failure Rate Function

The reversed failure rate function of NDQL distribution has been obtained as

$$r^*(x) = \frac{P(X=x)}{P(X\leq x)}$$

$$= \frac{[(\theta^2+\alpha+\alpha\theta x)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}]be^{-\theta x}}{\theta^2+\alpha-[\theta^2+\alpha+\theta\alpha(x+1)]e^{-\theta(x+1)}}. \tag{5.2.8}$$

5.2.7 Second Rate of Failure

The second rate of failure rate function of NDQL distribution has been obtained as

$$r^*(x) = log\left[\frac{s(x)}{s(x+1)}\right]$$

$$= log\left[\frac{[\theta^2+\alpha+\theta\alpha(x+1)]}{[\theta^2+\alpha+\theta\alpha(x+2)]e^{-\theta}}\right]. \quad (5.2.9)$$

5.2.8 Proportions of Probabilities

The second rate of failure rate function of NDQL distribution has been obtained as

$$\frac{P(x+1)}{P(x)} = e^{-\theta}\left[1+\frac{\alpha\theta(1-e^{-\theta})}{(\theta^2+\alpha+\alpha\theta x)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}\right] \quad (5.2.10)$$

5.2.9 Probability Recurrence Relation

Probability recurrence relation of NDQL distribution may be obtained as

$$P_{r+2} = e^{-\theta}(2P_{r+1} - e^{-\theta}P_r),\ r \geq 2. \quad (5.2.11)$$

where,

$$P_0 = \frac{(\theta^2+\alpha)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}{\theta^2+\alpha}\ ,\ \text{and} \quad (5.2.12)$$

$$P_1 = \frac{(\theta^2+\alpha+\alpha\theta)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}{\theta^2+\alpha}\ e^{-\theta}. \quad (5.2.13)$$

5.2.10 Factorial Moment Generating Function Relation

Factorial moment generating function (fmgf) may be obtained as

$$M(t) = G(1+t)$$

$$= \frac{[(\theta^2+\alpha)(1-e^{-\theta})-\alpha\theta e^{-\theta}](1-e^{-\theta}-e^{-\theta}t)+\alpha\theta\,e^{-\theta}(1+t)\,(1-e^{-\theta})}{(\theta^2+\alpha)(1-e^{-\theta}-e^{-\theta}t)^2}. \quad (5.2.14)$$

The first four factorial moments may be obtained as

$$\mu'_{[1]} = \frac{e^{-\theta}[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta]}{(\theta^2+\alpha)(1-e^{-\theta})^2} \quad , \tag{5.2.15}$$

$$\mu'_{[2]} = \frac{2\,e^{-2\theta}[(\theta^2+\alpha)(1-e^{-\theta})+2\alpha\theta]}{(\theta^2+\alpha)(1-e^{-\theta})^3} , \tag{5.2.16}$$

$$\mu'_{[3]} = \frac{6e^{-3\theta}[(\theta^2+\alpha)(1-e^{-\theta})+3\alpha\theta]}{(\theta^2+\alpha)(1-e^{-\theta})^4} , \tag{5.2.17}$$

$$\mu'_{[4]} = \frac{12\,e^{-4\theta}[(\theta^2+\alpha)(1-e^{-\theta})+4\alpha\theta]}{(\theta^2+\alpha)(1-e^{-\theta})^5} . \tag{5.2.18}$$

The **mean** μ and the **variance** σ^2 of the distribution may be obtained as

$$\mu \; = \frac{e^{-\theta}[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta]}{(\theta^2+\alpha)(1-e^{-\theta})^2}. \tag{5.2.19}$$

$$\sigma^2 \; = \frac{e^{-\theta}\left[(\theta^2+\alpha)^2(1-e^{-\theta})^2+(\theta^2+\alpha)(1-e^{-\theta})(1+e^{-\theta})\alpha\theta-e^{-\theta}\theta^2\alpha^2\right]}{(\theta^2+\alpha)^2(1-e^{-\theta})^4}. \tag{5.2.20}$$

The r[th] factorial moment may be obtained as

$$\mu'_{[r]} = \frac{r!\,e^{-\theta r}[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta r]}{(\theta^2+\alpha)(1-e^{-\theta})^{r+1}}, \tag{5.2.21}$$

which may be verified by putting r $= 1,2,3,$ ... etc.

5.3 Zero Truncated of NDQL Distribution

The pmf of Zero-truncated new discrete Quasi Lindley (ZTNDQL) $P_z(x)$distribution has been derived as

$$P_z(x) = \frac{P_x}{1-P_0}, \tag{5.3.1}$$

$where\ P_x$ denotes the pmf of discrete Quasi-Lindley distribution.

$$Hence, \quad P_z(x) = \frac{(\theta^2+\alpha+\,\alpha\theta x)(1-e^{-\theta})-\theta\alpha e^{-\theta}}{\theta^2+\alpha+\alpha\theta}\; e^{-\theta(x-1)}, \qquad x = 1,\, 2, \dots \tag{5.3.2}$$

5.3.1 Probability Generating Function of ZTNDQL Distribution

Probability generating function $G_z(t)$ of ZTNDQL distribution may be obtained as

$$G_z(t) = \sum_{x=1}^{\infty} t^x P_z(x)$$

$$= \frac{t[\{(\theta^2+\alpha)(1-e^{-\theta})-\alpha\theta e^{-\theta}\}(1-e^{-\theta}t)+\alpha\theta\,(1-e^{-\theta})]}{(\theta^2+\alpha+\alpha\theta)(1-e^{-\theta}t)^2}. \qquad (5.3.3)$$

5.3.2 Probability Recurrence Relation of ZTNDQL Distribution

Probability recurrence relation ZTNDQL distribution distribution may obtained as

$$P_r = e^{-\theta}\left[2P_{r-1} - e^{-\theta}P_{r-2}\right], \quad r > 2. \qquad (5.3.4)$$

Where

$$P_1 = \frac{(\theta^2+\alpha+\alpha\theta)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}{\theta^2+\alpha+\alpha\theta}, \qquad (5.3.5)$$

$$P_2 = \frac{(\theta^2+\alpha+2\alpha\theta)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}{\theta^2+\alpha+\alpha\theta}\,e^{-\theta}, \qquad (5.3.6)$$

5.3.3 Cumulative Distribution of ZTNDQL Distribution

The cumulative distribution of ZTNDQL Lindley distribution is given by

$$F_z(x) = \frac{(\theta^2+\alpha+\alpha\theta)-[\theta^2+\alpha+\alpha\theta(1+x)]e^{-\theta x}}{(\theta^2+\alpha+\alpha\theta)}. \qquad (5.3.7)$$

5.3.4 Survival Function of ZTNDQL Distribution

The survival function of Zero truncated of ZTNDQL distribution is given by

$$S_z(x) = \frac{[\theta^2+\alpha+\alpha\theta(1+x)]e^{-\theta x}}{(\theta^2+\alpha+\alpha\theta)}. \qquad (5.3.8)$$

5.3.5 Failure Rate Function of ZTNDQL Distribution

The failure hazard rate function of Zero truncated of a new discrete Quasi Lindley Distribution is given by

$$r_z(x) = \frac{P(X=x)}{P(X\geq x-1)}$$

$$= \frac{(\theta^2+\alpha+\alpha\theta x)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}{\theta^2+\alpha+\alpha\theta x}. \qquad (5.3.9)$$

5.3.6 Reversed Failure Rate of ZTNDQL Distribution

The reversed failure rate function of Zero truncated of a new discrete Quasi Lindley Distribution is given by

$$r_z^*(x) = \frac{P(X=x)}{P(X\le x)}$$

$$= \frac{[(\theta^2+\alpha+\alpha\theta x)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}]e^{-\theta(x-1)}}{(\theta^2+\alpha+\alpha\theta)-[\theta^2+\alpha+\alpha\theta(1+x)]e^{-\theta x}}. \qquad (5.3.10)$$

5.3.7 Second Rate of Failure of ZTNDQL Distribution

The second rate failure rate function of Zero truncated of a new discrete Quasi Lindleyis given by

$$r_z^{**}(x) = log\left[\frac{s(x)}{s(x+1)}\right]$$

$$= log\left[\frac{\theta^2+\alpha+\alpha\theta(1+x)}{e^{-\theta}(\theta^2+\alpha+\alpha\theta(2+x))}\right]. \qquad (5.3.11)$$

5.3.8 Proportions of Probabilities of ZTNDQL Distribution

The proportions of probabilities of Zero truncated of a new discrete Quasi Lindley Distribution is given by

$$\frac{P_z(x+1)}{P_z(x)} = e^{-\theta}\left[1+\frac{\alpha\theta(1-e^{-\theta})}{(\theta^2+\alpha+\alpha\theta x)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}\right]. \qquad (5.3.12)$$

5.3.9 Factorial Moment Generating Function ZTNDQL Distribution

Factorial moment generating function $M_z(t)$ of ZTNDQL distribution may be obtained as

$$M_z(t) = \frac{(1+t)[\{(\theta^2+\alpha)(1-e^{-\theta})-\alpha\theta e^{-\theta}\}(1-e^{-\theta}-e^{-\theta}t)+\alpha\theta\,(1-e^{-\theta})]}{(\theta^2+\alpha+\alpha\theta)(1-e^{-\theta}-e^{-\theta}t)^2}. \qquad (5.3.13)$$

Factorial moment recurrence relation of zero-truncated of ZTNDQL distribution may be obtained as

$$\mu'_{[r]} = \frac{e^{-\theta}}{(1-e^{-\theta})^2}\left[2(1-e^{-\theta})r - e^{-\theta}\mu'_{[r-1]} - r(r-1)e^{-\theta}\mu'_{[r-2]}\right], \qquad r \ge 2.$$

(5.3.14)

where

$$\mu'_{[1]} = \frac{[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta]}{(\theta^2+\alpha+\alpha\theta)(1-e^{-\theta})^2}, \qquad (5.3.15)$$

$$\mu'_{[2]} = \frac{2\,e^{-\theta}[(\theta^2+\alpha)(1-e^{-\theta})+2\alpha\theta]}{(\theta^2+\alpha+\alpha\theta)(1-e^{-\theta})^3}, \qquad (5.3.16)$$

$$\mu'_{[3]} = \frac{6\,e^{-2\theta}[(\theta^2+\alpha)(1-e^{-\theta})+3\alpha\theta]}{(\theta^2+\alpha+\alpha\theta)(1-e^{-\theta})^4}. \qquad (5.3.17)$$

The mean μ and the variance σ^2 of the distribution may be obtained as

$$\mu = \frac{[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta]}{(\theta^2+\alpha+\alpha\theta)(1-e^{-\theta})^2}. \qquad (5.3.18)$$

$$\sigma^2 =.\, \mu'_{[2]}+\mu'_{[1]}-\mu'^2_{[1]}. \qquad (5.3.19)$$

The rth factorial moment is obtained from moment as

$$\mu'_{[r]} = \left[\frac{d^r M(t)}{dt^r}\right]_{t=0}$$

$$= \frac{r!\,e^{-\theta(r-1)}[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta r]}{(\theta^2+\alpha+\alpha\theta)(1-e^{-\theta})^{r+1}}. \qquad r = 1, 2, 3, \ldots \qquad (5.3.20)$$

5.4. Size-Biased New Discrete Quasi–Lindley (SBNDQL) Distribution

In this section, the pmf of SBNDQL distribution with parameter α and θ has been derived as

$$P_x^s = \frac{x(1-e^{-\theta})^2[(\theta^2+\alpha+\alpha\theta x)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}]}{(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta}\, e^{-\theta(x-1)}, \quad x = 1, 2, 3, \ldots \qquad (5.4.1)$$

where $\mu = \frac{e^{-\theta}[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta]}{(\theta^2+\alpha)(1-e^{-\theta})^2}$ denotes the mean of NDQL distribution.

5.4.1 Probability Generating Function of SBNDQL Distribution

The probability generating function for SBNDQL distribution may be obtained as

$$G^s(\mathrm{t}) = \frac{\mathrm{t}(1-\mathrm{e}^{-\theta})^2[\{(\theta^2+\alpha)(1-\mathrm{e}^{-\theta})-\theta\alpha\mathrm{e}^{-\theta}\}(1-\mathrm{te}^{-\theta})+\theta\alpha(1-\mathrm{e}^{-\theta})(1+\mathrm{te}^{-\theta})]}{\{(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta\}(1-\mathrm{te}^{-\theta})^3}. \qquad (5.4.2)$$

5.4.2 Recurrence Relation of SBNDQL Distribution

Probability recurrence relation for SBNDQL distribution may be obtained as

$$P_r^s = e^{-\theta}\left[3P_{r-1}^s - 3e^{-\theta}P_{r-2}^s + e^{-2\theta}P_{r-3}^s\right], \quad \text{for} \quad r > 2. \qquad (5.4.3)$$

where

$$P_1^s = \frac{(1-e^{-\theta})^2\left[(\theta^2+\alpha+\alpha\theta)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}\right]}{(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta}, \qquad (5.4.4)$$

$$P_2^s = \frac{2(1-e^{-\theta})^2\left[(\theta^2+\alpha+2\alpha\theta)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}\right]}{(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta}\, e^{-\theta}. \qquad (5.4.5)$$

5.4.3 Factorial Moment Generating Function of SBNDQL Distribution

Factorial moment generating function for SBNDQL distribution is obtained as

$$M^s(\mathrm{t}) = \frac{\mathrm{t}(1+)(1-\mathrm{e}^{-\theta})^2[\{(\theta^2+\alpha)(1-\mathrm{e}^{-\theta})-\theta\alpha\mathrm{e}^{-\theta}\}(1-\mathrm{e}^{-\theta}-\mathrm{te}^{-\theta})+\theta\alpha(1-\mathrm{e}^{-\theta})(1+\mathrm{e}^{-\theta}+\mathrm{te}^{-\theta})]}{\{(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta\}(1-\mathrm{e}^{-\theta}-\mathrm{te}^{-\theta})^3}.$$

(5.4.6)

5.4.4. Factorial Moment Recurrence Relation of SBNDQL Distribution

Factorial moment recurrence relation for SBNDQL distribution is obtained as

$$\mu'_{[r]} = \frac{e^{-\theta}}{A^3}\left[3Ar\mu'_{[r-1]} - 3e^{-\theta}Ar(r-1)\mu'_{[r-2]} + e^{-2\theta}Ar(r-1)(r-2)\mu'_{[r-3]}\right] \qquad (5.4.7)$$

where, $r > 3$, $A = (1 - e^{-\theta})$.

The r[th] moment can be derived from Moment Generating Function as

$$\mu'_{[r]} = \frac{r!\, e^{-\theta(r-1)}\left[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta r\right]}{((\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta)(1-e^{-\theta})^{r+3}}. \qquad r = 1, 2, 3, \dots \qquad (5.4.8)$$

Where,

$$\mu'_{[1]} = \frac{\left[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta\right]}{((\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta)(1-e^{-\theta})^4}, \qquad (5.4.9)$$

$$\mu'_{[2]} = \frac{2\, e^{-\theta}\left[(\theta^2+\alpha)(1-e^{-\theta})+2\alpha\theta\right]}{((\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta)(1-e^{-\theta})^5}, \tag{5.4.10}$$

$$\mu'_{[3]} = \frac{6\, e^{-2\theta}\left[(\theta^2+\alpha)(1-e^{-\theta})+3\alpha\theta\right]}{((\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta)(1-e^{-\theta})^6}. \tag{5.4.11}$$

5.5 Zero-Modified of NDQL Distribution

The Zero-modified of new discrete quasi–Lindley (ZMNDQL) distribution is obtained as.

$$P^z[X=0] = \omega + (1-\omega)P_0$$

$$= \omega + (1-\omega)\left[\frac{(\theta^2+\alpha)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}{\theta^2+\alpha}\right], \tag{5.5.1}$$

where P_0 denotes probability of NDQL distribution at $x = 0$.
Hence the relationship will be

$$P^z[X=x] = (1-\omega)\lambda^x P(x), \mathrm{x} = 1, 2, \ldots, \tag{5.5.2}$$

$$\alpha \geq 0,\ \ 0 < \lambda < 1,\ \ \omega \geq \frac{-P_0}{1-P_0}.$$

where $P(x)$ denotes the pdf of NDQL distribution.

5.6 Estimation of Parameters of NDQL Distribution

5.6.1. Estimation of Parameters in terms of mean and variance

The mean $\mu$of NDQL distribution may be written as

$$\mu = \frac{e^{-\theta}\left[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta\right]}{(\theta^2+\alpha)(1-e^{-\theta})^2}. \tag{5.6.1}$$

The value of $\alpha\theta$

$$\alpha\theta = e^{\theta}\mu(\theta^2+\alpha)\left(1-e^{-\theta}\right)^2 - (\theta^2+\alpha)(1-e^{-\theta}) \tag{5.6.2}$$

putting in $\sigma^2 = \frac{e^{-\theta}\left[(\theta^2+\alpha)^2\left(1-e^{-\theta}\right)^2+(\theta^2+\alpha)\left(1-e^{-\theta}\right)\left(1+e^{-\theta}\right)\alpha\theta-e^{-\theta}\theta^2\alpha^2\right]}{(\theta^2+\alpha)^2(1-e^{-\theta})^4}$, the variance of NDQL distribution may be obtained from the quadratic equation in $\lambda = e^{-\theta}$.

$$\lambda^2 A - 2\lambda B + C = 0. \tag{5.6.3}$$

There are two values of λ had solving equation (5.6.3). We choose that the value λ had which minimizes the value of χ^2 static in table 5.1- 5.3, column 5.

$$\hat{\lambda} = \frac{B \pm \sqrt{B^2 - AC}}{A}, \tag{5.6.4}$$

where $A = \sigma^2 + \mu^2 + 3\mu + 2$, $B = \sigma^2 + \mu^2 + \mu$ and $C = \sigma^2 + \mu^2 - \mu$.
Putting the value θ in mean μ we can have α as following

$$\alpha = \frac{\theta^2(1-e^{-\theta})(e^{-\theta} - \mu(1-e^{-\theta}))}{\mu(1-e^{-\theta})^2 - e^{-\theta}(1-e^{-\theta}) - \theta e^{-\theta}}. \tag{5.6.5}$$

5.6.2 Maximum Likelihood Estimates

The likelihood function, L of the two parameter new discrete Quasi-Lindley distribution (5.2.3) is given by

$$L = \prod_{x=1}^{k} {P_x}^{f_x}. \tag{5.6.6}$$

$$L = \frac{e^{-\theta n\bar{x}}}{(\theta^2+\alpha)^n} \prod_{x=1}^{k} \left[\left(\theta^2 + \alpha + \alpha\theta x\right)(1 - e^{-\theta}) - \theta\, \alpha e^{-\theta}\right]^{f_x}. \tag{5.6.7}$$

And so the likelihood function is obtained as

$$logL = -\theta n\bar{x} - nlog(\theta^2 + \alpha) + G. \tag{5.6.8}$$

where,

$$G = \sum_{x=1}^{k} f_x log\left[\left(\theta^2 + \alpha + \alpha\theta x\right)(1 - e^{-\theta}) - \theta\, \alpha e^{-\theta}\right]$$

The two log likelihood equations are thus obtained as

$$\frac{\partial logL}{\partial \theta} = -n\bar{x} - \frac{2n\theta}{(\theta^2+\alpha)} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left(\theta^2+\alpha+\alpha\theta x\right)\left(1-e^{-\theta}\right) - \theta\, \alpha e^{-\theta}\right]}{\partial\theta}}{\left[\left(\theta^2+\alpha+\alpha\theta x\right)\left(1-e^{-\theta}\right) - \theta\, \alpha e^{-\theta}\right]} = 0. \tag{5.6.9}$$

$$\frac{\partial logL}{\partial \alpha} = -\frac{n}{(\theta^2+\alpha)} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left(\theta^2+\alpha+\alpha\theta x\right)(1-e^{-\theta}) - \theta\, \alpha e^{-\theta}\right]}{\partial\alpha}}{\left[\left(\theta^2+\alpha+\alpha\theta x\right)(1-e^{-\theta}) - \theta\, \alpha e^{-\theta}\right]} = 0. \tag{5.6.10}$$

The two equations (5.6.9) and (5.6.10) do not seem to be solved directly. However the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 logL}{\partial \theta^2} = \frac{n(\theta^2-\alpha)}{(\theta^2+\alpha)^2} + \frac{\partial}{\partial\theta} \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left(\theta^2+\alpha+\alpha\theta x\right)(1-e^{-\theta}) - \theta\, \alpha e^{-\theta}\right]}{\partial\theta}}{\left[\left(\theta^2+\alpha+\alpha\theta x\right)(1-e^{-\theta}) - \theta\, \alpha e^{-\theta}\right]}. \tag{5.6.11}$$

$$\frac{\partial^2 logL}{\partial\theta\partial\alpha} = \frac{2n\theta}{(\theta^2+\alpha)^2} + \frac{\partial}{\partial\alpha}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[(\theta^2+\alpha+\ \alpha\theta x)(1-e^{-\theta})-\theta\ \alpha e^{-\theta}\right]}{\partial\theta}}{\left[(\theta^2+\alpha+\ \alpha\theta x)(1-e^{-\theta})-\theta\ \alpha e^{-\theta}\right]}. \qquad (5.6.12)$$

$$\frac{\partial^2 logL}{\partial\alpha^2} = \frac{n}{(\theta^2+\alpha)^2} + \frac{\partial}{\partial\alpha}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[(\theta^2+\alpha+\ \alpha\theta x)(1-e^{-\theta})-\theta\ \alpha e^{-\theta}\right]}{\partial\alpha}}{\left[(\theta^2+\alpha+\ \alpha\theta x)(1-e^{-\theta})-\theta\ \alpha e^{-\theta}\right]}. \qquad (5.6.13)$$

The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved

$$\begin{bmatrix} \frac{\partial^2 logL}{\partial\theta^2} & \frac{\partial^2 logL}{\partial\theta\partial\alpha} \\ \frac{\partial^2 logL}{\partial\alpha\partial\theta} & \frac{\partial^2 logL}{\partial\alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial logL}{\partial\theta} \\ \frac{\partial logL}{\partial\alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}, \qquad (5.6.14)$$

Where θ_0 and $\alpha_0$are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently close estimates of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

5.7 Goodness of Fit

The fittings of the two-parameter NDQL distribution based on three data-sets have been presented in the following tables. The expected frequencies according to the one parameter Poisson- Lindley with parameter θ in Table 5.1 presented by Sankaran [40], two parameter Poisson- Lindley distributions with parameter θ and α in Table 5.2 presented by Shanker et ai. [45 have also been given for ready comparison with NDQL distribution. The estimates of the parameters have been obtained by the method of moments.

Table 5.1 Observed and expected frequencies for mistakes in copying groups of random digits.

No.of errors per group	Observed frequencies	Expected frequencies		
		Poisson-Lindley (θ)	Poisson-Lindley (θ, α)	NDQL (α, λ)
0	35	33.1	32.4	31.45
1	11	15.3	15.8	17.66
2	8	6.8	7.0	7.31
3	4	2.9	2.9	2.69
4	2	1.2	1.9	.89
	60	60 $\hat{\theta}$ =1.743 $d.f. = 3$ $\chi^2 = 2.20$ $p = 0.138$	60 $\hat{\alpha} = 2.61204$ $\hat{\theta} = 5.22337$ $d.f. = 3$ $\chi^2 = 2.11$ $p = 0.3482$	60 $\hat{\alpha} = 2.610$ $\hat{\theta} = 1.3189$ $d.f. = 3$ $\chi^2 = 4.613$ $p = 0.2024$

Table 5.2 Observed and expected frequencies for distribution of Pyrausta nublilalis in 1937.

No. of accidents	Observed frequencies	Expected frequencies		
		Poisson-Lindley (θ)	Poisson-Lindley (θ, α)	NDQL (α, λ)
0	33	31.49	31.9	30.97
1	12	14.16	13.8	15.73
2	6	6.09	5.9	6.14
3	3	2.54	2.5	2.14
4	1	1.04	1.1	0.70
≥5	1	0.42	0.8	0.32
Total	56	56 $\hat{\theta} = 1.808$ $d.f. = 3$ $\chi^2 = 4.82$ $p = 0.1855$	56 $\hat{\alpha} = 0.257$ $\hat{\theta} = 0.392$ $d.f. = 3$ $\chi^2 = 0.36$ $p = 0.8353$	56 $\hat{\alpha} = 1.9542$ $\hat{\lambda} = 1.3350$ $d.f. = 3$ $\chi^2 = 2.092$ $p = 0.5535$

5.8 Conclusion

Two-parameter NDQL distribution has been introduced. Several properties of the two-parameter NDQL distribution have been discussed. Estimation of parameters by the method of maximum likelihood and the method of moments have been discussed. The properties of size-biased and Zero- truncated version of NDQL distribution have also been investigated. Finally, the proposed distribution has been fitted to a number of data sets. It is observed that two-parameter NDQL provides better fits