Chapter 6

Two Parameter Discrete Lindley Distribution

6.1 Introduction

Two-parameter continuous Lindley distribution (TPL) introduced by Shankar et. al [46] with parameter θ and β is defined by its probability density function (pdf)

$$f(x:\theta,\beta) = \frac{\theta^2(1+\beta x)e^{-\theta x}}{\theta+\beta}. \qquad x \geq 0, \theta > 0,\ \beta > 0. \tag{6.1.1}$$

6.2.Discretization of TPL Distribution

In this paper, our objective is to derive a new discrete distribution and to study some of their properties, which may be called two parameter discrete–Lindley (TPDL) distribution based on the survival function of the distribution. Putting $\beta = 1$, discrete Lindley distribution may be obtained as a limiting form of the proposed distribution.

The survival function of the TPL distribution, which may be obtained as

$$S(x) = \int_x^\infty f(x:\theta,\beta)dx,$$

$$= \frac{[\theta+\beta(1+x\theta)]e^{-\theta x}}{\theta+\beta} \ . \tag{6.2.1}$$

Hence,

$$S(x+1) = \frac{[\theta+\beta(1+x\theta+\theta)]e^{-\theta(x+1)}}{\theta+\beta} \ . \tag{6.2.2}$$

6.2.1 Probability Mass Function (pmf)

The probability mass function (pmf) of TPDL distribution may be obtained as

$$P(X = x) = S(x) - S(x+1),$$

$$= \frac{[\theta+\beta(1+x\theta)](1-e^{-\theta})-\theta\beta e^{-\theta}}{\theta+\beta} e^{-\theta x}, \qquad x = 0,1,2,\ 3\ldots, \quad (6.2.3)$$

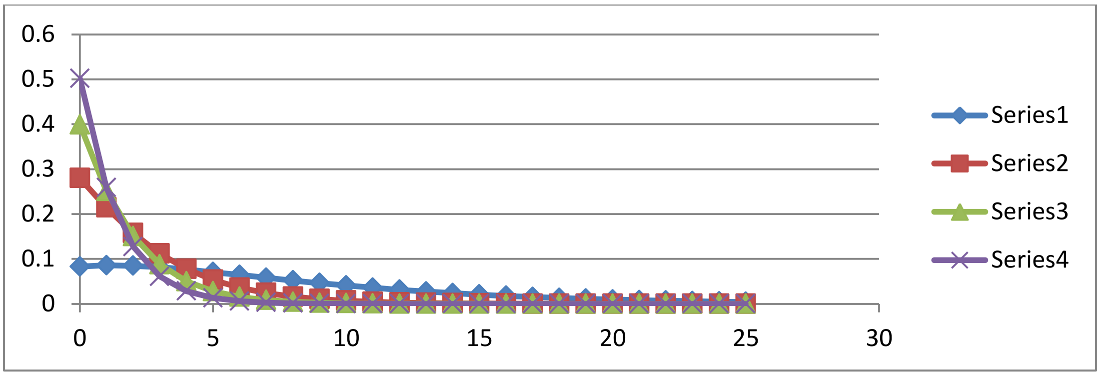

Figure 9: Probability graph for Two parameter discrete Lindley distribution $\beta = 0.5, \theta = 0.2\ (series1)\ \ \beta = 0.5, \theta = 0.5\ (series2)\ \ \beta = 0.5, \theta = 0.7\ \ (series3).\ \beta = 0.5, \theta = 0.9 (series4)$

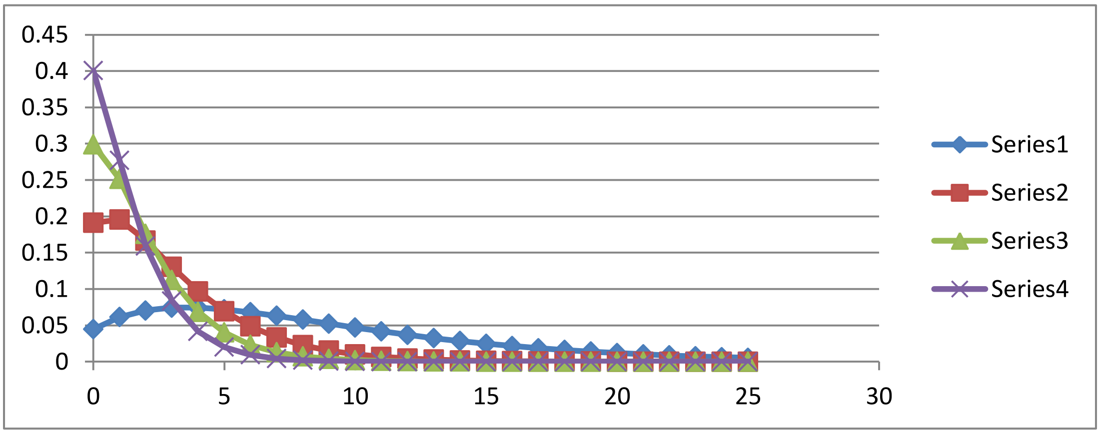

Figure10: Probability graph for Two parameter discrete Lindley distribution $\beta = 1, \theta = 0.2\ (series1)\ \ \beta = 1, \theta = 0.5\ (series2)\ \ \beta = 1, \theta = 0.7\ \ (series3).\ \beta = 1, \theta = 0.9 (series4)$

6.2.2 Probability Generating Function (pgf)

The probability generating function (pgf) of TPDL distribution is given by

$$G(t) = \frac{[(\theta+\beta)(1- e^{-\theta})-\theta\beta\, e^{-\theta}](1- e^{-\theta}t)+\beta\theta\, e^{-\theta}t(1- e^{-\theta})}{(\theta+\beta)(1-e^{-\theta}t)^2}\,. \tag{6.2.4}$$

6.2.3 Cumulative Distribution Function

The cumulative distribution of TPDLD distribution is given by

$$F(x) = 1 - \frac{[\theta+\beta+(x+1)\beta\theta]\, e^{-\theta(x+1)}}{(\theta+\beta)}\,. \tag{6.2.5}$$

6.2.4 Survival Function

The survival function of TPDL distribution has been obtained as

$$S(x) = \frac{[\theta+\beta+(x+1)\beta\theta]\, e^{-\theta(x+1)}}{(\theta+\beta)}\,. \tag{6.2.6}$$

6.2.5 Failure Rate Function

The Failure rate function of TPDL distribution may be obtained as

$$r(x) = \frac{P(X=x)}{P(X\geq x-1)}$$

$$= \frac{[\theta+\beta(1+x\theta)](1- e^{-\theta})-\theta\beta e^{-\theta}}{\theta+\beta+x\beta\theta}\,. \tag{6.2.7}$$

6.2.6 Reversed Failure Rate

The reversed hazard rate function of TPDL distribution may be obtained as

$$r^*(x) = \frac{P(X=x)}{P(X\leq x)}$$

$$= \frac{\left[[\theta+\beta(1+x\theta)](1- e^{-\theta})-\theta\beta e^{-\theta}\right]e^{-\theta x}}{(\theta+\beta)-[\theta+\beta+(x+1)\beta\theta]\, e^{-\theta(x+1)}}\,. \tag{6.2.8}$$

6.2.7 Second Rate of Failure

The second rate function of TPDL distribution may be obtained as

$$r^*(x) = log\left[\frac{s(x)}{s(x+1)}\right]$$

$$= log\left[\frac{[\theta+\beta+(x+1)\beta\theta]}{[\theta+\beta+(x+2)\beta\theta]e^{-\theta}}\right]. \quad (6.2.9)$$

6.2.8 Proportions of Probabilities

The proportions of TPDL distribution may be obtained as

$$\frac{P(x+1)}{P(x)} = e^{-\theta}\left[1 + \frac{1-2\theta\beta\, e^{-\theta}}{[\theta+\beta(1+x\theta)](1-e^{-\theta})-\theta\beta e^{-\theta}}\right]. \quad (6.2.10)$$

6.2.9 Probability Recurrence Relation

Probability recurrence relation of TPDL distribution may be obtained as

$$P_{r+2} = e^{-\theta}(2P_{r+1} - e^{-\theta}P_r),\ r \ \geq 1 \quad (6.2.11)$$

where

$$P_0 = \frac{[\theta\alpha+\beta](1-e^{-\theta})-\theta\beta e^{-\theta}}{\theta+\beta}\ , \quad (6.2.12)$$

$$P_1 = \frac{[\theta+\beta(1+\theta)](1-e^{-\theta})-\theta\beta e^{-\theta}}{\theta+\beta}\ e^{-\theta}. \quad (6.2.13)$$

6.2.10 Factorial Moment Generating Function

Factorial moment generating function of TPDL distribution may be obtained as

$$M(t) = G(1+t)$$

$$=\frac{[(\alpha\theta+\beta)(1-e^{-\theta})-\theta\beta\, e^{-\theta}](1-e^{-\theta}-e^{-\theta}t)+\beta\theta\, e^{-\theta}(1+t)(1-e^{-\theta})}{(\alpha\theta+\beta)(1-e^{-\theta}-e^{-\theta}t)^2}. \quad (6.2.14)$$

The first four factorial moments may be obtained as

$$\mu'_{[1]} = \frac{e^{-\theta}[(\theta+\beta)(1-e^{-\theta})+\beta\theta]}{(\theta+\beta)(1-e^{-\theta})^2}, \quad (6.2.15)$$

$$\mu'_{[2]} = \frac{2\,e^{-2\theta}\left[(\theta+\beta)\left(1-e^{-\theta}\right)+2\beta\theta\right]}{(\theta+\beta)(1-e^{-\theta})^3}\,, \tag{6.2.16}$$

$$\mu'_{[3]} = \frac{6\,e^{-3\theta}\left[(\theta+\beta)\left(1-e^{-\theta}\right)+3\beta\theta\right]}{(\theta+\beta)(1-e^{-\theta})^4}\,, \tag{6.2.17}$$

$$\mu'_{[4]} = \frac{12\,e^{-4\theta}\left[(\theta+\beta)\left(1-e^{-\theta}\right)+4\beta\theta\right]}{(\theta+\beta)(1-e^{-\theta})^5}. \tag{6.2.18}$$

The mean μ and the variance σ^2 of the TPDL distribution may be obtained as

$$\mu \;= \frac{e^{-\theta}\left[(\theta+\beta)\left(1-e^{-\theta}\right)+\beta\theta\right]}{(\theta+\beta)(1-e^{-\theta})^2}\,, \text{ and} \tag{6.2.19}$$

$$\sigma^2 = \frac{(\theta+\beta)^2(1-e^{-\theta}t)^2e^{-\theta}+(\theta+\beta)(1-e^{-\theta})\beta\theta e^{-\theta}\left(1+e^{-\theta}\right)-e^{-2\theta}\beta^2\theta^2}{(\theta+\beta)^2(1-e^{-\theta})^4}. \tag{6.2.20}$$

respectively.

The general form of factorial moment may also be written as

$$\mu'_{[r]} = \left[\frac{d^r M(t)}{dt^r}\right]_{t=0}$$

$$= \frac{r!\,e^{-\theta r}\left[(\theta+\beta)\left(1-e^{-\theta}\right)+r\beta\theta\right]}{(\theta+\beta)(1-e^{-\theta})^{r+1}}, \tag{6.2.21}$$

which may be verified by putting $\mathrm{r} = 1, 2, 3, \ldots$ etc.

6.3 Zero Truncated Discrete TPDL Distribution

The pmf of Zero-truncated two parameter discrete Lindley (ZTTPDL) $P_z(x)$distribution has been derived as

$$P_z(x) = \frac{P_x}{1-P_0}, \tag{6.3.1}$$

whereP_x denotes the pmf of discrete TPDL distribution

hence

$$P_z(x) = \frac{[\theta+\beta(1+x\theta)]\left(1-e^{-\theta}\right)-\theta\beta e^{-\theta}}{\theta+\beta+\theta\beta}\; e^{-\theta(x-1)}, \quad x = 1,\, 2, \ldots \tag{6.3.2}$$

6.3.1 Probability Generating Function $G_z(t)$ of ZTTDL Distribution

Probability generating function $G_z(t)$ of ZTTDL distribution may be obtained as

$$G_z(t) = \sum_{x=1}^{\infty} t^x P_x,$$

$$= \frac{[(\theta+\beta)(1- e^{-\theta})-\theta\beta\, e^{-\theta}]t(1- e^{-\theta}t)+\beta\theta\, t(1- e^{-\theta})}{(\theta+\beta+\theta\beta)(1-e^{-\theta}t)^2}. \quad (6.3.3)$$

Probability recurrence relation ZTTDL distribution may obtained as

$$P_r = e^{-\theta}\left[2P_{r-1} - e^{-\theta}P_{r-2}\right], \quad r > 2 \quad (6.3.4)$$

where,

$$P_1 = \frac{[\theta+\beta(1+\theta)](1- e^{-\theta})-\theta\beta e^{-\theta}}{\theta+\beta+\theta\beta}, \quad (6.3.5)$$

$$P_2 = \frac{[\theta+\beta(1+2\theta)](1- e^{-\theta})-\theta\beta e^{-\theta}}{\theta+\beta+\theta\beta}\, e^{-\theta}, \quad (6.3.6)$$

6.3.2 Cumulative Distribution of ZTTDL Distribution

The cumulative distribution of discrete ZTTDL distribution is given by

$$F_z(x) = 1 - \frac{[\theta+\beta+\beta\theta(1+x)]e^{-\theta x}}{\theta+\beta+\theta\beta}. \quad (6.3.7)$$

6.3.3 Survival Function of ZTTDL Distribution

The survival function of ZTTDL distribution is given by

$$S_z(x) = \frac{[\theta+\beta+\beta\theta(1+x)]e^{-\theta x}}{\theta+\beta+\theta\beta}. \quad (6.3.8)$$

6.3.4 Failure Hazard Rate of ZTTDL Distribution

The failure hazard rate of ZTTDL distribution is given by

$$r_z(x) = \frac{P(X=x)}{P(X\geq x-1)},$$

$$= \frac{[\theta+\beta(1+x\theta)](1- e^{-\theta})-\theta\beta e^{-\theta}}{\theta+\beta+\beta\theta x}. \quad (6.3.9)$$

6.3.5 Reversed Failure Rate of ZTTDL Distribution

The reversed hazard rate of ZTTDL distribution is given by

$$r_z^*(x) = \frac{P(X=x)}{P(X\leq x)},$$

$$= \frac{\left[[\theta+\beta(1+x\theta)](1- e^{-\theta})-\theta\beta e^{-\theta}\right]e^{-\theta(x-1)}}{\theta+\beta+\theta\beta-[\theta+\beta+\beta\theta(1+x)]e^{-\theta x}}. \quad (6.3.10)$$

6.3.6 Second Rate of Failure of ZTTDL Distribution

The second rate of failure rate function of ZTTDL distribution is given by

$$r_z^{**}(x) = log\left[\frac{s(x)}{s(x+1)}\right],$$

$$= log\left[\frac{\theta+\beta+\beta\theta(1+x)}{e^{-\theta}(\theta+\beta+\beta\theta(2+x))}\right]. \quad (6.3.11)$$

6.3.7 Proportions of Probabilities of ZTTDL Distribution

The proportions of probabilities of ZTTDL distribution is given by

$$\frac{P_z(x+1)}{P_z(x)} = e^{-\theta}\left[1+\frac{\theta\beta(1- e^{-\theta})}{[\theta+\beta(1+x\theta)](1- e^{-\theta})-\theta\beta e^{-\theta}}\right]. \quad (6.3.12)$$

6.3.8 Factorial Moment Generating Function of ZTTDL Distribution

Factorial moment generating function $M_z(t)$ of ZTTDL distribution may be obtained as

$$M_z(t) = \frac{[(\theta+\beta)(1- e^{-\theta})-\theta\beta\, e^{-\theta}](1+t)(1-e^{-\theta}- e^{-\theta}t)+\beta\theta\,(1+t)(1- e^{-\theta})}{(\theta+\beta+\theta\beta)(1-e^{-\theta}-e^{-\theta}t)^2}. \quad (6.3.13)$$

Factorial moment recurrence relation of ZTTDL may be obtained as

$$\mu'_{[r]} = \frac{e^{-\theta}}{\left(1-e^{-\theta}\right)^2}\left[2(1-e^{-\theta})r - e^{-\theta}\mu'_{[r-1]} - r(r-1) - e^{-\theta}\mu'_{[r-2]}\right], \quad r \geq 2. \tag{6.3.14}$$

where

$$\mu'_{[1]} = \frac{\left[(\theta+\beta)\left(1-e^{-\theta}\right)+\beta\theta\right]}{(\theta+\beta+\theta\beta)(1-e^{-\theta})^2}, \tag{6.3.15}$$

$$\mu'_{[2]} = \frac{2\,e^{-\theta}\left[(\theta+\beta)\left(1-e^{-\theta}\right)+2\beta\theta\right]}{(\theta+\beta+\theta\beta)(1-e^{-\theta})^3}, \tag{6.3.16}$$

$$\mu'_{[3]} = \frac{6e^{-2\theta}\left[(\theta+\beta)\left(1-e^{-\theta}\right)+3\beta\theta\right]}{(\theta+\beta+\theta\beta)(1-e^{-\theta})^4}. \tag{6.3.17}$$

The mean μ and the variance σ^2 of the ZTTDL distribution may be obtained as

$$\mu = \frac{\left[(\theta+\beta)\left(1-e^{-\theta}\right)+\beta\theta\right]}{(\theta+\beta+\theta\beta)(1-e^{-\theta})^2}, \tag{6.3.18}$$

$$\sigma^2 = \frac{(\theta+\beta)(\theta+\beta+\theta\beta)(1-e^{-\theta})^2\left(1+e^{-\theta}\right)+\beta\theta(\theta+\beta+\theta\beta)(1-e^{-\theta})(1+3e^{-\theta})-\left[(\theta+\beta)(1+e^{-\theta})+\beta\theta\right]^2}{(\theta+\beta+\theta\beta)^2(1-e^{-\theta})^4}. \tag{6.3.19}$$

The r[th] factorial moment of ZTTDL distribution may be written as

$$\mu'_{[r]} = \left[\frac{d^r M(t)}{dt^r}\right]_{t=0}$$

$$= \frac{r!\,e^{-\theta(r-1)}\left[(\theta+\beta)\left(1-e^{-\theta}\right)+r\beta\theta\right]}{(\theta+\beta+\theta\beta)(1-e^{-\theta})^{r+1}}. \qquad r = 1, 2, 3, \ldots \tag{6.3.20}$$

6.4 Size-Biased Two Parameter Discrete Lindley Distribution

In this section, the pmf of size-biased two parameter Lindley (SBTDL) distribution with parameter α and θ has been derived as

$$P_x^s = \frac{x(1-e^{-\theta})^2\left[\left(\theta^2+1+\theta x\right)(1-e^{-\theta})-\theta\,e^{-\theta}\right]}{(\theta^2+1)(1-e^{-\theta})+\theta}\; e^{-\theta(x-1)}, \quad x = 1, 2, 3, \ldots \tag{6.4.1}$$

where $\mu = \frac{e^{-\theta}\left[(\theta^2+1)(1-e^{-\theta})+\theta\right]}{(\theta^2+1)(1-e^{-\theta})^2}$ denotes the mean of TDL distribution.

6.4.1 Probability Generating Function of SBTDL Distribution

The probability generating function of SBTDL distribution may be obtained as

$$\mathrm{G}^{s}(\mathrm{t})=\frac{\mathrm{t}\left(1-\mathrm{e}^{-\theta}\right)^{2}\left[\left\{\left(\theta^{2}+1\right)\left(1-\mathrm{e}^{-\theta}\right)-\theta 1\mathrm{e}^{-\theta}\right\}\left(1-\mathrm{te}^{-\theta}\right)+\theta\left(1-\mathrm{e}^{-\theta}\right)\left(1+\mathrm{te}^{-\theta}\right)\right]}{\left\{(\theta^{2}+1)(1-e^{-\theta})+\theta\right\}\left(1-\mathrm{te}^{-\theta}\right)^{3}}. \qquad (6.4.2)$$

6.4.2 Recurrence Relation of SBTDL Distribution

Probability recurrence relation of SBTDL distribution may be obtained as

$$P_{r}^{s}=e^{-\theta}\left[3P_{r-1}^{s}-3e^{-\theta}P_{r-2}^{s}+e^{-2\theta}P_{r-3}^{s}\right], \quad \text{for} \quad r>2. \qquad (6.4.3)$$

$$P_{1}^{s}=\frac{(1-e^{-\theta})^{2}\left[\left(\theta^{2}+1+\theta\right)(1-e^{-\theta})-\theta\, e^{-\theta}\right]}{(\theta^{2}+1)(1-e^{-\theta})+\theta}. \qquad (6.4.4)$$

$$P_{2}^{s}=\frac{2(1-e^{-\theta})^{2}\left[\left(\theta^{2}+1+2\theta\right)(1-e^{-\theta})-\theta\, e^{-\theta}\right]}{(\theta^{2}+1)(1-e^{-\theta})+\theta}\, e^{-\theta}. \qquad (6.4.5)$$

6.4.3 Factorial Moment Generating Function of SBTDL Distribution

Factorial moment generating function for SBTDL distribution may be obtained as

$$\mathrm{M_{s}}(\mathrm{t})=\frac{\mathrm{t}(1+\mathrm{t})\left(1-\mathrm{e}^{-\theta}\right)^{2}\left[\left\{\left(\theta^{2}+1\right)\left(1-\mathrm{e}^{-\theta}\right)-\theta\mathrm{e}^{-\theta}\right\}\left(1-\mathrm{e}^{-\theta}-\mathrm{te}^{-\theta}\right)+\theta\left(1-\mathrm{e}^{-\theta}\right)\left(1+\mathrm{e}^{-\theta}+\mathrm{te}^{-\theta}\right)\right]}{\left\{(\theta^{2}+1)(1-e^{-\theta})+\theta\right\}\left(1-\mathrm{e}^{-\theta}-\mathrm{te}^{-\theta}\right)^{3}}. \qquad (6.4.6)$$

6.4.4 Factorial Moment Recurrence Relation SBTDL Distribution

Factorial moment recurrence relation of SBTDL distribution may be obtained as

$$\mu'_{[r]}=\frac{e^{-\theta}}{b^{3}}\left[3b^{2}r\mu'_{[r-1]}-3e^{-\theta}br(r-1)\mu'_{[r-2]}+e^{-2\theta}br(r-1)(r-2)\mu'_{[r-3]}\right]. \qquad (6.4.7)$$

where,

$b=1-e^{-\theta}$.

The r[th] Factorial Moment Generating Function of SBTDL distribution may be obtained as

$$\mu'_{[r]} = \frac{r!\, e^{-\theta(r-1)}\left[(\theta^2+1)(1-e^{-\theta})+\theta r\right]}{((\theta^2+1)(1-e^{-\theta})+\theta)(1-e^{-\theta})^{r+3}} r = 1, 2, 3 \ldots \quad (6.4.8)$$

where,

$$\mu'_{[1]} = \frac{\left[(\theta^2+1)(1-e^{-\theta})+\theta\right]}{((\theta^2+1)(1-e^{-\theta})+\theta)(1-e^{-\theta})^4}, \quad (6.4.9)$$

$$\mu'_{[2]} = \frac{2\, e^{-\theta}\left[(\theta^2+1)(1-e^{-\theta})+2\theta\right]}{((\theta^2+1)(1-e^{-\theta})+\theta)(1-e^{-\theta})^5}, \quad (6.4.10)$$

$$\mu'_{[3]} = \frac{6\, e^{-2\theta}\left[(\theta^2+1)(1-e^{-\theta})+3\theta\right]}{((\theta^2+1)(1-e^{-\theta})+\theta)(1-e^{-\theta})^6}. \quad (6.4.11)$$

6.5 Zero-Modified Two Parameter Discrete Lindley Distribution

Zero-Modified TPDL distribution may be obtained as

$$P_z[X=0] = \omega + (1-\omega)P_0,$$

$$= \omega + (1-\omega)\left[\frac{(\theta^2+1)(1-e^{-\theta})-\theta\, e^{-\theta}}{\theta^2+1}\right], \quad (6.5.1)$$

$$P_z[X=x] = (1-\omega)\lambda^x P(x). \qquad x=1, 2, \ldots \quad (6.5.2)$$

$$\alpha \geq 0,\ 0 < \lambda < 1,\ \omega \geq \frac{-P_0}{1-P_0}.$$

Where $P(x)$ denotes the probability of A new Discrete Quasi Lindley Distribution.

6.6 Estimation of Parameters

6.6.1 Estimation of Parameters in terms of mean and variance of TPDL distribution

The mean of TPDL distribution may be written as

$$\mu = \frac{e^{-\theta}\left[(\theta+\beta)(1-e^{-\theta})+\beta\theta\right]}{(\theta+\beta)(1-e^{-\theta})^2}, \quad (6.6.1)$$

Now, putting value of

$$\beta\theta = e^{\theta}\mu(\theta+\beta)\left(1-e^{-\theta}\right)^2 - (\theta+\beta)\left(1-e^{-\theta}\right), \quad (6.6.2)$$

In

$$\sigma^2 \ = \frac{(\theta+\beta)^2(1-e^{-\theta}t)^2e^{-\theta}+(\theta+\beta)(1-e^{-\theta})\beta\theta e^{-\theta}(1+e^{-\theta})-e^{-2\theta}\beta^2\theta^2}{(\theta+\beta)^2(1-e^{-\theta})^4}, \qquad (6.6.3)$$

the variance of TPDL distribution may be obtained in quadratic equation in $\lambda = e^{-\theta}$ as

$$\lambda^2 A - 2\lambda B + C = 0\ . \qquad (6.6.4)$$

There are two values of λ had solving equation (6.6.4). We choose that the value λ had which minimizes the value of χ^2 static in table 2.1- 2.3, column 5.

$$\hat{\lambda} = \frac{B \pm \sqrt{B^2 - AC}}{A}, \qquad (6.6.5)$$

where,

$$A = \sigma^2 + \mu^2 + 3\mu + 2, \quad B = \sigma^2 + \mu^2 + \mu, \quad C = \sigma^2 + \mu^2 - \mu\ . \qquad (6.6.6)$$

Putting the value θ in mean μ we can α as following

$$\beta = \frac{\mu\theta(1-e^{-\theta})^2 - \mu(1-e^{-\theta})e^{-\theta}}{\theta e^{-\theta} - \mu(1-e^{-\theta})^2 - e^{-\theta}(1-e^{-\theta})}\ . \qquad (6.6.7)$$

6.6.2 Maximum Likelihood Estimates

The likelihood function, L of the two-parameter discrete gamma (6.2.3) is given by

$$L = \prod_{x=1}^{k} {P_x}^{f_x}, \qquad (6.6.8)$$

$$L = \frac{e^{-\theta n\bar{x}}}{(\theta+\beta)^n} \prod_{x=1}^{k} \left[[\theta + \beta(1 + x\theta)](1 - e^{-\theta}) - \theta\beta e^{-\theta} \right]^{f_x}. \qquad (6.6.9)$$

The log likelihood function is obtained as

$$logL = -\theta n\bar{x} - nlog(\theta + \beta) + G. \qquad (6.6.10)$$

where,

$$G = \sum_{x=1}^{k} f_x \left[[\theta + \beta(1 + x\theta)](1 - e^{-\theta}) - \theta\beta e^{-\theta} \right]$$

The three log likelihood equations are thus obtained as

$$\frac{\partial logL}{\partial\theta} = -n\bar{x} - \frac{n}{(\theta+\beta)} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[[\theta+\beta(1+x\theta)](1-e^{-\theta})-\theta\beta e^{-\theta}\right]}{\partial\theta}}{\left[[\theta\alpha+\beta(1+x\theta)](1-e^{-\theta})-\theta\beta e^{-\theta}\right]} = 0. \qquad (6.6.11)$$

$$\frac{\partial logL}{\partial \beta} = -\frac{n}{(\theta+\beta)} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[[\theta+\beta(1+x\theta)]\left(1-e^{-\theta}\right)-\theta\beta e^{-\theta}\right]}{\partial\beta}}{\left[[\theta+\beta(1+x\theta)](1-e^{-\theta})-\theta\beta e^{-\theta}\right]} = 0. \qquad (6.6.12)$$

Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 logL}{\partial \theta^2} = \frac{n}{(\theta+\beta)^2} + \frac{\partial}{\partial\theta}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}{\partial\theta}}{\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}. \quad (6.6.13)$$

$$\frac{\partial^2 logL}{\partial \theta\partial\beta} = \frac{n}{(\theta+\beta)^2} + \frac{\partial}{\partial\theta}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}{\partial\beta}}{\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}. \quad (6.6.14)$$

$$\frac{\partial^2 logL}{\partial \beta^2} = \frac{n}{(\theta+\beta)^2} + \frac{\partial}{\partial\beta}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}{\partial\beta}}{\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}. \quad (6.6.15)$$

The following equations for $\hat{\theta}$ and $\hat{\beta}$ can be solved

$$\begin{bmatrix} \frac{\partial^2 logL}{\partial\theta^2} & \frac{\partial^2 logL}{\partial\theta\partial\beta} \\ \frac{\partial^2 logL}{\partial\theta\partial\beta} & \frac{\partial^2 logL}{\partial\beta^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\beta}=\beta_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\beta}-\beta_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial logL}{\partial\theta} \\ \frac{\partial logL}{\partial\beta} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\beta}=\beta_0}}, \qquad (6.6.16)$$

where θ_0, and β_0 are the initial values of θ and β respectively. These equations are solved iteratively till sufficiently close estimates of $\hat{\theta}$, and $\hat{\beta}$ are obtained.

6.7 Goodness of Fit

The fittings of the two-parameter DL distribution based on three data-sets have been presented in the following tables. The expected frequencies according to the one parameter Poisson- Lindley with parameter θ in Table 6.1 presented by Sankaran [40] two parameter Poisson- Lindley distributions with parameter θ and α in Table 6.2 presented by Shanker et al. [45] have also been given for ready comparison with TPDL distribution. The estimates of the parameters have been obtained by the method of moments.

Table 6.1 Observed and expected frequencies for distribution of Pyrausta nublilalis in 1937.

No. of ccidents	Observed frequencies	Expected frequencies		
		Poisson-Lindley (θ)	Poisson-Lindley (θ, α)	TPDL (θ, β)
0	33	31.49	31.9	29.42
1	12	14.16	13.8	16.52
2	6	6.09	5.9	6.66
3	3	2.54	2.5	2.36
4	1	1.04	1.1	0.78
≥5	1	0.42	0.8	0.26
	56	56	56	56
		$\hat{\theta} = 1.8082$	$\hat{\alpha} = 0.2573$	$\beta = 2.063118$
			$\hat{\theta} = 0.39249$	$\theta = 1.335001$
		$\chi^2 = 4.82$	$\chi^2 = 0.36$	$\chi^2 = 1.70$
		$p = 0.1855$	$p = 0.8353$	$p = 0.366$

Table 6.2 Distribution of number of epileptic seizure counts

European red mites	Observed frequencies	Expected frequencies		
		GPL (α, θ)	NBD (r,p)	TPDL (θ, β)
0	126	121.51	91.0	125.33
1	80	95.81	86.60	89.00
2	59	59.81	63.37	57.02
3	42	34.49	42.57	34.45
4	24	19.24	27.60	20.05
5	8	10.59	17.60	11.36
6	5	5.81	10.50	6.32
7	4	3.18	6.52	3.46
8	3	3.88	5.00	1.87
	351	351 $\hat{\alpha} = 1.139$ $\hat{\theta} = 1.292$ $\chi^2 = 5.94$	351 $\hat{r} = 1.757$ $\hat{p} = .463$ $\chi^2 = 22.53$	351 $\theta = 0.717397$ $\beta = 0.569288$ $\chi^2 = 2.40$

6.8 Conclusion

Two-parameter DL distribution has been introduced. Several properties of the TPDL distribution, such as moments, recurrence relation, estimation of parameters by the method of maximum likelihood and the method of moments have been discussed. The properties of size-biased and Zero- truncated version of TPDL distribution have also been investigated. Finally, the proposed distribution has been fitted to a number of data sets. It is observed that two-parameter DL provides better fits than those by the DL distribution and hence it should be preferred to the TPDL while modeling count data-sets.