Chapter 7

Discrete Gamma Lindley Distribution

7.1 Introduction

Two parameter continuous gamma Lindley (GL) distribution introduced by Zeghdoudi and Nedjar [53] with parameter θ and β is defined by its probability density function (pdf)

$$f(x:\theta,\beta) = \frac{\theta^2\big((\beta+\beta\theta-\theta)x+1\big)e^{-\theta x}}{\beta(1+\theta)} . \quad x > 0,\ \beta > \frac{\theta}{\theta+1},\ \theta > 0. \tag{7.1.1}$$

7.2 Derivation of Discrete Gamma-Lindley Distribution

The survival function of GL distribution may be obtained as

$$S(x) = \int_x^{\infty} f(x:\beta,\theta)dx,$$

$$= \frac{a(\ \theta x+1)+\theta}{a+\theta}\ e^{-\theta x}. \qquad \beta > \frac{\theta}{\theta+1},\ \theta > 0. \tag{7.2.1}$$

Here we say $a = (\beta + \beta\theta - \theta)$and $b = 1 - e^{-\theta}$.

hence

$$S(x+1) = \frac{a(\ \theta x+1+\theta)+\theta}{a+\theta}\ e^{-\theta(x+1)}. \tag{7.2.2}$$

7.2.1 Probability Mass Function (pmf)

The probability mass function (pmf) of DGL distribution may be obtained as

$$P(X = x) = S(x) - S(x + 1),$$

$$= \frac{(a(\theta x+1)+\theta)b - a\theta\, e^{-\theta}}{a+\theta}\, e^{-\theta x}, \qquad x = 0,1,2,3\ldots \qquad (7.2.3)$$

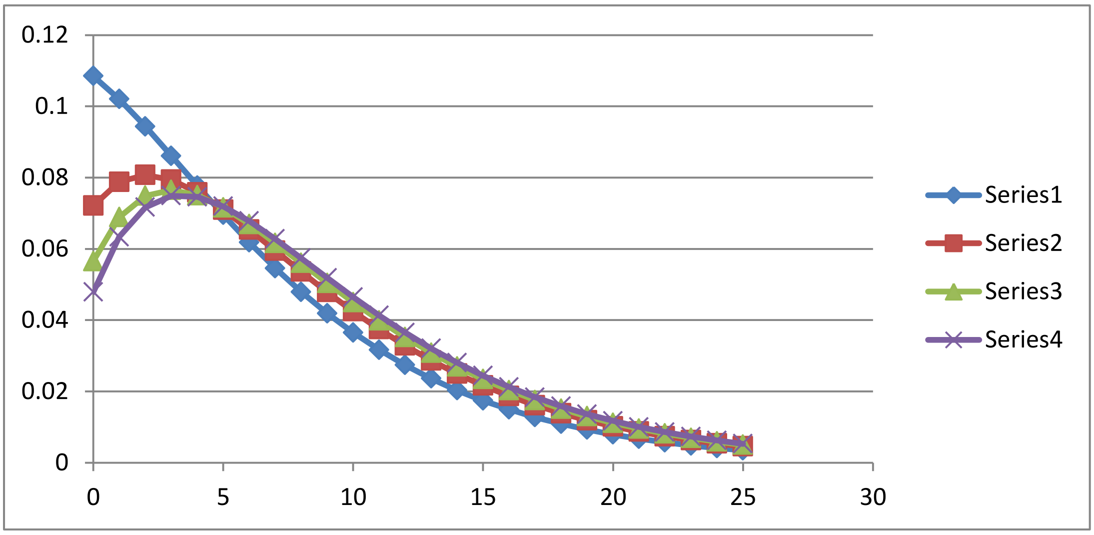

Figure11: Probability graph for two parameter discrete gamma Lindley distribution $\theta = 0.2, \beta = 0.3,\ (series1)\theta = 0.2, \beta = 0.5,\ (series2)\theta = 0.2\ \beta = 0.7,\ (series3). \theta = 0.2, \beta = 0.9, (series4)$

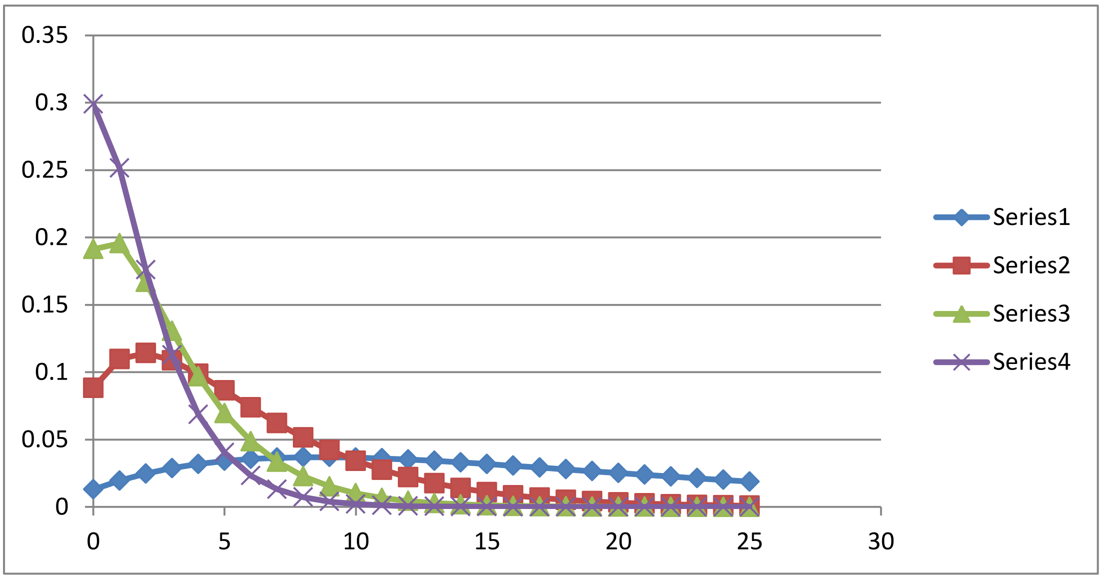

Figure12: Probability graph for two parameter discrete gamma Lindley distribution $\beta = 1, \theta = 0.1\ (series1)\ \beta = 1, \theta = 0.3\ (series2)\ \beta = 1, \theta = 0.5\ (series3). \beta = 1, \theta = 0.7 (series4)$

7.2.2 Probability Generating Function (pgf)

The probability generating function (pgf) of DGL distribution is given by

$$G(t) = \frac{[(a+\theta)b-a\theta\, e^{-\theta}](1-e^{-\theta}t)+a\theta\, e^{-\theta}t(1-e^{-\theta})}{(a+\theta)(1-e^{-\theta}t)^2}. \tag{7.2.4}$$

7.2.3 Cumulative Distribution Function

The cumulative distribution of DGL distribution is given by

$$F(x) = \frac{(a+\theta)b-[a\theta(1+xb)+(a+\theta)b-a\theta\, e^{-\theta}]e^{-\theta(x+1)}}{(a+\theta)b}, \tag{7.2.5}$$

7.2.4 Survival Function

The survival function of DGL distribution has obtained as

$$S(x) = \frac{[a\theta(1+xb)+(a+\theta)b-a\theta\, e^{-\theta}]e^{-\theta(x+1)}}{(a+\theta)b}. \tag{7.2.6}$$

7.2.5 Failure Hazard Rate Function

The failure hazard rate function of DGL distribution has obtained as

$$r(x) = \frac{P(X=x)}{P(X\geq x-1)},$$

$$= \frac{[(a(\theta x+1)+\theta)(b-a\theta\, e^{-\theta}]b}{a\theta(1+(x-1)b)+(a+\theta)b-a\theta\, e^{-\theta}}. \tag{7.2.7}$$

7.2.6 Reversed Failure Rate Function

The reversed failure rate function of DGL distribution has obtained as

$$r^*(x) = \frac{P(X=x)}{P(X\leq x)},$$

$$= \frac{[(a(\theta x+1)+\theta)b-a\theta\, e^{-\theta}]be^{-\theta x}}{(a+\theta)b-[a\theta(1+xb)+(a+\theta)b-a\theta\, e^{-\theta}]e^{-\theta(x+1)}}. \tag{7.2.8}$$

7.2.7 Second Rate of Failure Function

The second rate of failure function of DGL distribution has obtained as

$$r^*(x) = log\left[\frac{s(x)}{s(x+1)}\right],$$

$$= log\left[\frac{[a\theta(1+xb)+(a+\theta)b-a\theta\, e^{-\theta}]}{[a\theta(1+(x+1)b)+(a+\theta)b-a\theta\, e^{-\theta}]e^{-\theta}}\right]. \quad (7.2.9)$$

7.2.8 Proportions of Probabilities

The proportions of probabilities of DGL distribution has obtained as

$$\frac{P(x)}{P(x+1)} = e^{-\theta}\left[\frac{(a(\theta x+1+\theta)+\theta)b-a\theta\, e^{-\theta}}{(a(\theta x+1)+\theta)b-a\theta\, e^{-\theta}}\right]. \quad (7.2.10)$$

7.2.9 Probability Recurrence Relation

Probability recurrence relation of DGL distribution may be obtained as

$$P_{r+2} = e^{-\theta}(2P_{r+1} - e^{-\theta}P_r)\,,\ r \ \geq 2. \quad (7.2.11)$$

where,

$$P_0 = \frac{(a+\theta)b-a\theta\, e^{-\theta}}{a+\theta}\ , \quad (7.2.12)$$

$$P_1 = \frac{(a(\theta+1)+\theta)b-a\theta\, e^{-\theta}}{a+\theta}\ e^{-\theta}. \quad (7.2.13)$$

7.2.10 Factorial Moment Generating Function

Factorial moment generating function of DGL may be obtained as

$$M(t) = G(1+t),$$

$$= \frac{[(a+\theta)b-a\theta\, e^{-\theta}](b- e^{-\theta}t)+a\theta\, e^{-\theta}(1+t)}{(a+\theta)(b-e^{-\theta}t)^2}. \quad (7.2.14)$$

The r^{th} factorial moment may also be written as

$$\mu'_{[r]} = \frac{r!\, e^{-\theta r}[(a+\theta)b+a\theta r]}{(a+\theta)(1-e^{-\theta})^{r+1}}. \qquad r = 1, 2, 3, \ldots \quad (7.2.15)$$

where,

$$\mu'_{[1]} = \frac{e^{-\theta}[(a+\theta)b+a\theta]}{(a+\theta)(1-e^{-\theta})^2}, \quad (7.2.16)$$

$$\mu'_{[2]} = \frac{2\,e^{-2\theta}[(a+\theta)b+2a\theta]}{(a+\theta)\left(1-e^{-\theta}\right)^3}, \tag{7.2.17}$$

$$\mu'_{[3]} = \frac{6\,e^{-3\theta}[(a+\theta)b+3a\theta]}{(a+\theta)\left(1-e^{-\theta}\right)^4}. \tag{7.2.18}$$

The mean μ and the variance σ^2 of the DGL distribution may be obtained as

$$\mu \;= \frac{e^{-\theta}[(a+\theta)b+a\theta]}{(a+\theta)\left(1-e^{-\theta}\right)^2}, \text{ and} \tag{7.2.19}$$

$$\sigma^2 \;= \frac{(a+\theta)^2(1-e^{-\theta})^2e^{-2\theta}+(a+\theta)(1-e^{-\theta})\beta\theta e^{-\theta}\left(1+e^{-\theta}\right)-e^{-2\theta}\beta^2\theta^2}{(a+\theta)^2(1-e^{-\theta})^4}. \tag{7.2.20}$$

respectively.

7.3 Zero Truncated DGL Distribution

The pmf$P_z(x)$ of Zero-truncated discrete Gamma Lindley ZTDGL distribution has been derived as

$$P_z(x) = \frac{P_x}{1-P_0}, \tag{7.3.1}$$

where P_x denotes the pmf of DGL distribution

hence

$$P_z(x) = \frac{(a(\,\theta x+1)+\theta)b-a\theta\, e^{-\theta}}{a+\theta+a\theta}\; e^{-\theta(x-1)}, \quad x = 1,\ 2, 3, \ldots \tag{7.3.2}$$

7.3.1 Probability Generating Function of ZTDGL Distribution

Probability generating function $G_z(t)$ of ZTDGL distribution may be obtained as

$$G_z(t) = \sum_{x=1}^{\infty} t^x P_x,$$

$$= \frac{t\left[\{(a+\theta)b-a\theta e^{-\theta}\}\left(1-te^{-\theta}\right)+a\theta b\,\right]}{(a+\theta+a\theta)\left(1-te^{-\theta}\right)^2}. \tag{7.3.3}$$

7.3.2 Probability Recurrence Relation of ZTDGL Distribution

Probability recurrence relation ZTDGL distribution may obtained as

$$P_r = e^{-\theta}\left[2P_{r-1} - e^{-\theta}P_{r-2}\right], r > 2 \tag{7.3.4}$$

where,

$$P_1 = \frac{\left(a(\theta+1)+\theta\right)b - a\theta\, e^{-\theta}}{a+\theta+a\theta}\ , \tag{7.3.5}$$

$$P_2 = \frac{\left(a(2\theta+1)+\theta\right)b - a\theta\, e^{-\theta}}{a+\theta+a\theta}\ e^{-\theta}, \tag{7.3.6}$$

7.3.3 Cumulative Distribution of ZTDGL Distribution

The cumulative distribution of discrete ZTDGL distribution is given by

$$F_z(x) = \frac{(a+\theta+a\theta)-[a+\theta+a\theta(1+x)]e^{-\theta x}}{(a+\theta+a\theta)}. \tag{7.3.7}$$

7.3.4 Survival Function of ZTDGL Distribution

The survival function of ZTDGL distribution is given by

$$S_z(x) = \frac{[a+\theta+a\theta(1+x)]e^{-\theta x}}{(a+\theta+a\theta)}. \tag{7.3.8}$$

7.3.5 Failure Hazard Rate of ZTDGL Distribution

The failure hazard rate of ZTDGL distribution is given by

$$r_z(x) = \frac{P(X=x)}{P(X \geq x-1)}.$$

$$= \frac{\left(a(\theta x+1)+\theta\right)b - a\theta\, e^{-\theta}}{a+\theta+a\theta x}. \tag{7.3.9}$$

7.3.6 Reversed Failure Rate of ZTDGL Distribution

The reversed failure rate of ZTDGL distribution is given by

$$r_z^*(x) = \frac{P(X=x)}{P(X \leq x)}\ .$$

$$= \frac{\left[\left(a(\theta x+1)+\theta\right)b - a\theta\, e^{-\theta}\right]e^{-\theta(x-1)}}{(a+\theta+a\theta)-[a+\theta+a\theta(1+x)]e^{-\theta x}}. \tag{7.3.10}$$

7.3.7 The Second Rate of Failure of ZTDGL Distribution

The second rate of failure function of ZTDGL distribution is given by

$$r_z^{**}(x) = log\left[\frac{s(x)}{s(x+1)}\right].$$

$$= log\left[\frac{a+\theta+a\theta(1+x)}{e^{-\theta}(a+\theta+a\theta(2+x))}\right]. \qquad (7.3.11)$$

7.3.8 Proportions of Probabilities of ZTDGL Distribution

The proportions of probabilities of ZTDGL distribution is given by

$$\frac{P_z(x+1)}{P_z(x)} = e^{-\theta}\left[1+\frac{a\theta\, e^{-\theta}}{(a(\,\theta x+1)+\theta)b-a\theta\, e^{-\theta}}\right]. \qquad (7.3.12)$$

7.3.9 Factorial Moment Generating function for ZTDGL Distribution

Factorial moment generating function $M_z(t)$ of ZTDGL distribution may be obtained as

$$M_z(t) = \frac{(1+t)[\{(a+\theta)b-a\theta e^{-\theta}\}(b-te^{-\theta})+a\theta b\,]}{(a+\theta+a\theta)(b-te^{-\theta})^2}. \qquad (7.3.13)$$

The rth factorial moment may be obtained as

$$\mu'_{[r]} = \frac{r!\, e^{-\theta(r-1)}[(a+\theta)b+a\theta r]}{\beta(1-e^{-\theta})^{r+1}}. \qquad r = 1,2,3,\ldots \qquad (7.3.14)$$

Factorial moment recurrence relation of ZTDGL distribution may be obtained as

$$\mu'_{[r]} = \frac{e^{-\theta}}{b^2}\left[2br - e^{-\theta}\mu'_{[r-1]} - r(r-1)e^{-\theta}\mu'_{[r-2]}\right], \quad r \geq 2 \qquad (7.3.15)$$

where,

$$\mu'_{[1]} = \frac{[(a+\theta)b+a\theta]}{\beta(1-e^{-\theta})^2}, \qquad (7.3.16)$$

$$\mu'_{[2]} = \frac{2\, e^{-\theta}[(a+\theta)b+2a\theta]}{\beta(1-e^{-\theta})^3}, \qquad (7.3.17)$$

$$\mu'_{[3]} = \frac{6\, e^{-2\theta}[(a+\theta)b+3a\theta]}{\beta\left(1-e^{-\theta}\right)^4}. \tag{7.3.18}$$

7.4 Size-Biased DGL Distribution

In this section, the pmf of size-biased discrete Gamma Lindley SBDGL distribution

$$P_x^s(x) = \frac{x(1-e^{-\theta})^2\left[\left(a(\ \theta x+1)+\theta\right)b-a\theta\, e^{-\theta}\right]}{(a+\theta)b+a\theta}\ e^{-\theta(x-1)}, \qquad x = 1, 2, 3, \ldots \tag{7.4.1}$$

where,

$$\mu = \frac{e^{-\theta}[(a+\theta)b+a\theta]}{(a+\theta)b^2}$$, denotes the mean of DGL distribution.

7.4.1 Probability Generating Function of SBDGL Distribution

The probability generating function for SBDGL distribution may be obtained as

$$G_x^s(\mathrm{t}) = \frac{\mathrm{tb}^2\left[\left\{\left(\theta^2+\alpha\right)\mathrm{b}-\theta\alpha\mathrm{e}^{-\theta}\right\}\left(1-\mathrm{te}^{-\theta}\right)+\theta\alpha\mathrm{b}\left(1+\mathrm{te}^{-\theta}\right)\right]}{\{(\theta^2+\alpha)b+\alpha\theta\}\left(1-\mathrm{te}^{-\theta}\right)^3}. \tag{7.4.2}$$

7.4.2 Recurrence Relation of SBDGL Distribution

Probability recurrence relation of SBDGL distribution may be obtained as

$$P_r^s = e^{-\theta}\left[3P_{r-1}^s - 3e^{-\theta}P_{r-2}^s + e^{-2\theta}P_{r-3}^s\right],\ \text{for} r > 2. \tag{7.4.3}$$

where,

$$P_1^s = \frac{(1-e^{-\theta})^2\left[\left(\theta^2+\alpha+\ \alpha\theta\right)(1-e^{-\theta})-\theta\ \alpha e^{-\theta}\right]}{(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta}. \tag{7.4.4}$$

$$P_2^s = \frac{2(1-e^{-\theta})^2\left[\left(\theta^2+\alpha+\ 2\alpha\theta\right)(1-e^{-\theta})-\theta\ \alpha e^{-\theta}\right]}{(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta}\ e^{-\theta}. \tag{7.4.5}$$

7.4.3 Factorial Moment Generating Function of SBDGL Distribution

Factorial Moment Generating function of SBDGL distribution may be obtained as

$$\mathrm{M_s(t)} = \frac{(1+\mathrm{t})\mathrm{b}^2\left[\left\{(\theta^2+\alpha)(\mathrm{b})-\theta\alpha\mathrm{e}^{-\theta}\right\}\left(1-\mathrm{e}^{-\theta}-\mathrm{te}^{-\theta}\right)+\theta\alpha\mathrm{b}\left(1+\mathrm{e}^{-\theta}+\mathrm{te}^{-\theta}\right)\right]}{\left\{(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta\right\}\left(1-\mathrm{e}^{-\theta}-\mathrm{te}^{-\theta}\right)^3}. \tag{7.4.6}$$

7.4.4.Factorial Moment Recurrence Relation of SBDGL Distribution

Factorial moment recurrence relation of SBDGL distribution may be obtained as

$$\mu'_{[r]} = \frac{e^{-\theta}}{b^3}\left[3b^2 r\mu'_{[r-1]} - 3e^{-\theta}br(r-1)\mu'_{[r-2]} + e^{-2\theta}br(r-1)(r-2)\mu'_{[r-3]}\right]. \tag{7.4.7}$$

The r^th factorial moment generating function of SBDGL distribution may be obtained as

$$\mu'_{[r]} = \frac{r!\, e^{-\theta(r-1)}\left[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta r\right]}{((\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta)(1-e^{-\theta})^{r+3}}, \quad r = 1, 2, 3, \ldots \tag{7.4.8}$$

where,

$$\mu'_{[1]} = \frac{\left[(\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta\right]}{((\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta)(1-e^{-\theta})^4}, \tag{7.4.9}$$

$$\mu'_{[2]} = \frac{2\, e^{-\theta}\left[(\theta^2+\alpha)(1-e^{-\theta})+2\alpha\theta\right]}{((\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta)(1-e^{-\theta})^5}, \tag{7.4.10}$$

$$\mu'_{[3]} = \frac{6\, e^{-2\theta}\left[(\theta^2+\alpha)(1-e^{-\theta})+3\alpha\theta\right]}{((\theta^2+\alpha)(1-e^{-\theta})+\alpha\theta)(1-e^{-\theta})^6}. \tag{7.4.11}$$

7.5. Zero-Modified of DGL Distribution

Zero modified of DGL distribution is obtained as

$$P_z[X = 0] = \omega + (1-\omega)P_0 .$$

$$= \omega + (1-\omega)\left[\frac{(\theta^2+\alpha)(1-e^{-\theta})-\theta\,\alpha e^{-\theta}}{\theta^2+\alpha}\right]. \tag{7.5.1}$$

$$P_z[X = x] = (1-\omega)\lambda^x P(x), \quad x = 1, 2, \ldots \tag{7.5.2}$$

$$\alpha \geq 0,\ 0 < \lambda < 1,\ \omega \geq \frac{-P_0}{1-P_0}.$$

where $P(x)$ denotes the probability Discrete gamma Lindley Distribution.

7.6 Parameter Estimation

7.6.1Estimation of λ in terms of mean and variance

The mean of DGL distribution may be written as

$$\mu = \frac{e^{-\theta}[(a+\theta)b + a\theta]}{(a+\theta)b^2}. \tag{7.6.1}$$

The value of $\alpha\theta$

$$a\theta = e^{\theta}\mu(a+\theta)b^2 - [(a+\theta)b]. \tag{7.6.2}$$

Putting value of $a\theta$ in $\sigma^2 = \frac{(a+\theta)^2 b^2 e^{-2\theta} + (a+\theta)b\beta\theta e^{-\theta}(1+e^{-\theta}) - e^{-2\theta}\beta^2\theta^2}{(a+\theta)^2 b^4}$ the variance of DGL distribution, the quadratic equation in $\lambda = e^{-\theta}$ may be obtained as

$$\lambda^2 A - 2\lambda B + C = 0. \tag{7.6.3}$$

There are two values of λ had solving equation (7.6.3). We choose that the value λ had which minimizes the value of χ^2 static in table 7.1- 7.3, column 5.

$$\hat{\lambda} = \frac{B \pm \sqrt{B^2 - AC}}{A}, \tag{7.6.4}$$

where $A = \sigma^2 + \mu^2 + 3\mu + 2$, $B = \sigma^2 + \mu^2 + \mu$ and $C = \sigma^2 + \mu^2 - \mu$.
by selecting appropriate value of $\hat{\lambda}$ from (7.6.4)
Putting the value θ in mean μ we can α as following

$$\alpha = \frac{1}{1+\theta}\left[\frac{\theta(1-e^{-\theta})(e^{-\theta} - \mu(1-e^{-\theta}))}{\mu(1-e^{-\theta})^2 - e^{-\theta}(1-e^{-\theta}) - \theta e^{-\theta}} + \theta\right]. \tag{7.6.5}$$

7.6.2. Maximum Likelihood Estimates

The likelihood function, L of the two parameter discrete gamma-Lindley distribution (7.2.3) is given by

$$L = \prod_{x=1}^{k} P_x{}^{f_x}. \tag{7.6.6}$$

$$L = \frac{e^{-\theta n\bar{x}}}{(\beta+\beta\theta)^n} \prod_{x=1}^{k} \left[\left((\beta + \beta\theta - \theta)(\theta x + 1) + \theta\right)(1 - e^{-\theta}) - (\beta + \beta\theta - \theta)\theta\, e^{-\theta}\right]^{f_x}. \tag{7.6.7}$$

And so the likelihood function is obtained as

$$logL = -\theta n\bar{x} - nlog(\beta + \beta\theta) + G \;. \tag{7.6.8}$$

where,

$$G = \sum_{x=1}^{k} f_x \left[\left((\beta + \beta\theta - \theta)(\theta x + 1) + \theta\right)(1 - e^{-\theta}) - (\beta + \beta\theta - \theta)\theta\, e^{-\theta}\right]$$

The two log likelihood equations are thus obtained as

$$\frac{\partial logL}{\partial\theta} = -n\bar{x} - \frac{n}{(1+\theta)} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}{\partial\theta}}{\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}=0. \tag{7.6.9}$$

$$\frac{\partial logL}{\partial\alpha} = -\frac{n}{(1+\theta)} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}{\partial\beta}}{\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]} = 0. \tag{7.6.10}$$

The two equations (7.6.9) and (7.6.10) do not seem to be solved directly. However the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 logL}{\partial\theta^2} = -\frac{n}{(1+\theta)^2} + \frac{\partial}{\partial\theta}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}{\partial\theta}}{\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}. \tag{7.6.11}$$

$$\frac{\partial^2 logL}{\partial\theta\partial\beta} = \frac{\partial}{\partial\theta}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}{\partial\beta}}{\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}. \tag{7.6.12}$$

$$\frac{\partial^2 logL}{\partial\beta^2} = \frac{n}{\beta^2} + \frac{\partial}{\partial\beta}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}{\partial\beta}}{\left[\left((\beta+\beta\theta-\theta)(\theta x+1)+\theta\right)(1-e^{-\theta})-(\beta+\beta\theta-\theta)\theta\, e^{-\theta}\right]}. \tag{7.6.13}$$

The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved

$$\begin{bmatrix} \frac{\partial^2 logL}{\partial\theta^2} & \frac{\partial^2 logL}{\partial\theta\partial\alpha} \\ \frac{\partial^2 logL}{\partial\alpha\partial\theta} & \frac{\partial^2 logL}{\partial\alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial logL}{\partial\theta} \\ \frac{\partial logL}{\partial\alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}, \tag{7.6.14}$$

Where θ_0 and $\alpha_0$are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently close estimates of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

7.7 Goodness of Fit

The fittings of the two-parameter DGL distribution based on three data-sets have been presented in the following tables. The expected frequencies according to the one parameter Poisson- Lindley with parameter θ in Table 7.1 presented by Sankaran [40], two parameter Poisson- Lindley distributions with parameter θ and α in Table 7.2 presented by Shanker et al. [45] and two parameter discrete gamma with parameter k and θ in Table 7.3 presented by Chakraborty an Chakravarty [10] have also been given for ready comparison with DGL distribution. The estimates of the parameters have been obtained by the method of moments.

Table 7.1 Observed and expected frequencies for mistakes in copying groups of random digits.

No. of errors per group	Observed frequencies	Expected frequencies		
		Poisson-Lindley (θ)	Poisson-Lindley (θ, α)	DGL (θ, β)
0	35	33.1	32.4	31.33
1	11	15.3	15.8	15.67
2	8	6.8	7.0	7.61
3	4	2.9	2.9	3.54
4	2	1.2	1.9	1.53
	60	60	60	60
		$\hat{\theta}$ =1.743	$\hat{\alpha} = 2.61204$ $\hat{\theta} = 5.22337$	$\hat{\theta} = 0.534399$ $\beta = 0.258753$
		$d.f. = 3$	$d.f. = 3$	$d.f. = 3$
		$\chi^2 = 2.20$	$\chi^2 = 2.11$	$\chi^2 = 1.935$
		$p = 0.1380$	$p = 0.3482$	$p = 0.5861$

Table 7.2 Observed and expected frequencies for distribution of Pyraustan ublilalis in 1937.

No. of accidents	Observed frequencies	Expected frequencies		
		Poisson-Lindley (θ)	Poisson-Lindley (θ, α)	DGL (θ, β)
0	33	31.49	31.9	29.43
1	12	14.16	13.8	14.72
2	6	6.09	5.9	7.12
3	3	2.54	2.5	3.28
4	1	1.04	1.1	1.39
5	1	0.42	0.8	0.50
	56	56 $\hat{\theta} = 1.8082$ $d.f. = 3$ $\chi^2 = 4.82$ $p = 0.1855$	56 $\hat{\alpha} = 0.2573$ $\hat{\theta} = 0.39249$ $d.f. = 3$ $\chi^2 = 0.36$ $p = 0.8353$	56 $\hat{\theta} = 0.526543$ $\hat{\beta} = 0.251160$ $d.f. = 3$ $\chi^2 = 1.062$ $p = 0.7862$

Table 7.3 Distribution of number of European red mites on apple leaves

European red mites	Observed frequencies	Expected frequencies		
		$d\gamma(k,\theta)$	NBD(r,p)	DGL (θ,β)
0	70	69.67	69.49	66.63
1	38	37.49	37.6	37.86
2	17	20.02	20.1	21.25
3	10	10.67	10.7	11.71
4	9	5.69	5.69	6.39
5	3	3.03	3.02	3.36
6	2	1.61	1.6	1.69
7	1	.86	0.85	0.79
8	0	.96	0.95	0.32
	150	150 $\hat{k} = 1.0078$ $\hat{\theta} = 1.5830$ $\chi^2 = 2.89$ $p = 0.7169$	150 $\hat{r} = 1.0245$ $\hat{p} = 0.5281$ $\chi^2 = 2.91$ $p = 0.7139$	150 $\hat{\theta} = 0.454736$ $\hat{\beta} = 0.245548$ $\chi^2 = 2.341$ $p = 0.8002$

7.8 Conclusion

Gamma Lindley (DJ)distribution has been discretized and the parameters of discrete gamma-Lindley (DG) distribution has been estimated using method of moment. Three well known data-sets have been considered and presented in the above tables. The expected frequencies according to DGL distribution and one parameter Poisson Lindley distribution, two parameter Poisson–Lindley have been shown for ready comparison. Based on χ^2 goodness of fit test and p–value, it may be concluded that the DGL distribution gives much closer fit.