Chapter 2

Discrete Janardan Distribution and Its Applications

2.1 Introduction

A two parameter continuous Janardan distribution introduced by Shanker et. al. [48] with parameter α and θ is defined by its probability density function (pdf)

$$f(x;\ \theta,\alpha) = \frac{\theta^2}{\alpha(\theta+\alpha^2)}(1+\alpha x)e^{-\frac{\theta}{\alpha}x},\ x>0,\theta>0,\alpha>0. \tag{2.1.1}$$

2.2 Discretization of Janardan Distribution

In this paper, our objective is to derive a new discrete distribution and to study some of their properties, which may be called Discrete Janardan (DJ) distribution based on the survival function of the continuous Janardan distribution. The survival function of the Janardan distribution is

$$s(x) = \int_x^\infty f(x;\theta,\alpha)dx = \frac{e^{-\frac{\theta}{\alpha}x}[(\theta+\alpha\theta x+\alpha^2)]}{(\theta+\alpha^2)} \tag{2.2.1}$$

$$s(x+1) = \frac{e^{-\frac{\theta}{\alpha}(x+1)}[(\theta+\alpha\theta(x+1)+\alpha^2)]}{(\theta+\alpha^2)}. \tag{2.2.2}$$

2.2.1 The probability Mass Function (pmf)

The probability mass function (pmf) of discrete Janardan (DJ) distribution may be obtained as

$$P(X=x) = S(x) - S(x+1)$$

$$= \frac{e^{-\frac{\theta}{\alpha}x}\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha^2)}, \qquad x = 0,1,2, \ldots, \qquad (2.2.3)$$

where, $A = 1 - e^{-\frac{\theta}{\alpha}}$.

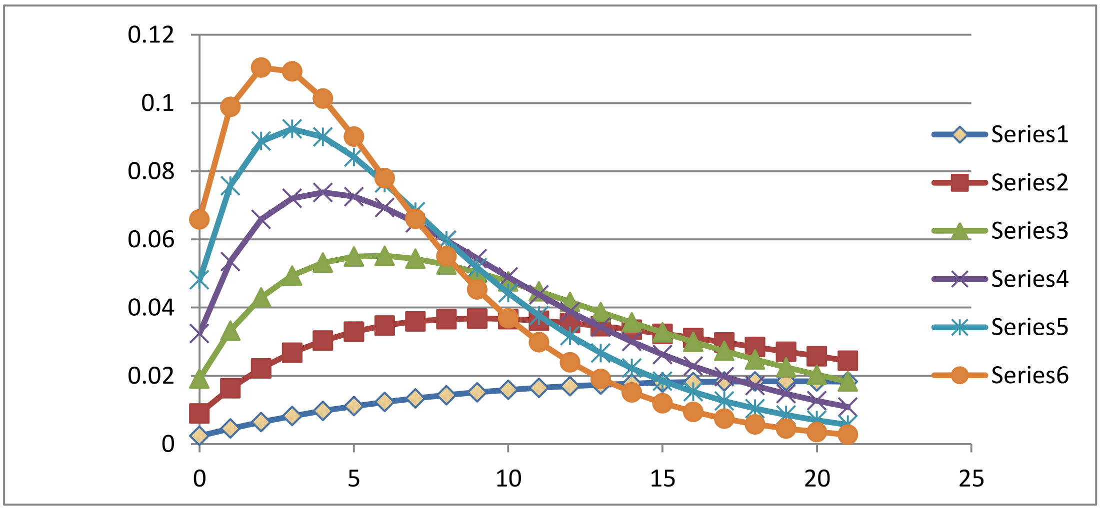

Figure1: Probability graph for Discrete Janardan distribution $\alpha = 2, \theta = 0.1\ (series1) \alpha = 2, \theta = 0.3\ (series3) \alpha = 2, \theta = 0.4 (series4) \alpha = 2, \theta = 0.4 (series4) \alpha = 2, \theta = 0.5\ (series5) \alpha = 2, \theta = 0.6 (series6)$

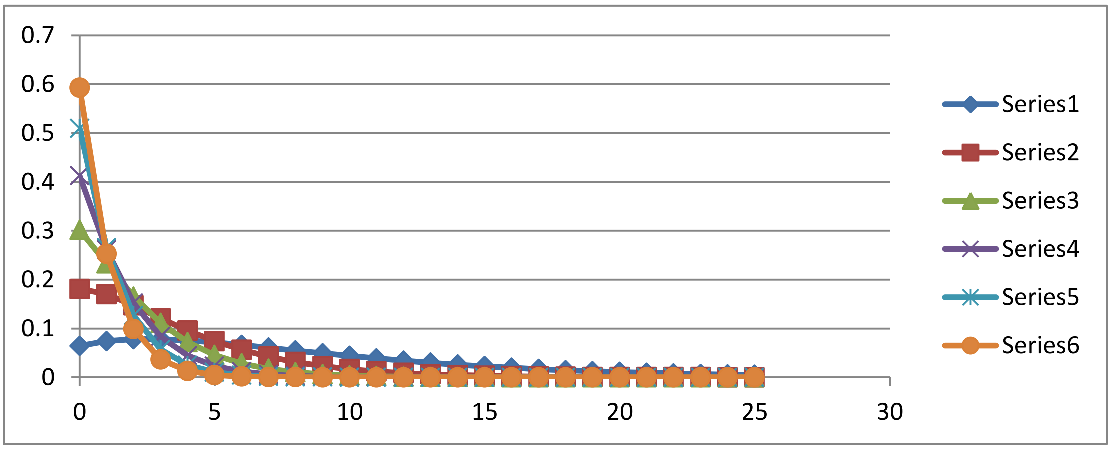

Figure1: Probability graph for Discrete Janardan distribution $\alpha = 0.5, \theta = 0.1\ (series1) \alpha = 0.5, \theta = 0.3\ (series3) \alpha = 0.5, \theta = 0.4 (series4) \alpha = 0.5, \theta = 0.4 (series4) \alpha = 0.5, \theta = 0.5\ (series5) \alpha = 0.5, \theta = 0.6 (series6)$

2.2.2 Probability Generating Function

Probability generating function of DJ distribution may be obtained as

$$G(t) = \sum_{x=0}^{\infty} t^x P_x \ , \quad x = 0,1,2, \ldots \ , \ \theta > 0, \alpha > 0$$

$$= \frac{\left[(\theta+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]\left(1-te^{-\frac{\theta}{\alpha}}\right)+\alpha\theta A\, t\, e^{-\frac{\theta}{\alpha}}}{(\theta+\alpha^2)\left(1-te^{-\frac{\theta}{\alpha}}\right)^2}, \qquad (2.2.4)$$

where $A = 1 - e^{-\frac{\theta}{\alpha}}$.

2.2.3 Probability Recurrence Relation

Probability recurrence relation of DJ distribution may be obtained as

$$P_r = e^{-\frac{\theta}{\alpha}}\left[2P_{r-1} - e^{-\frac{\theta}{\alpha}}P_{r-2}\right], \qquad r \geq 2 \qquad (2.2.5)$$

where

$$P_0 = \frac{\left[(\theta+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha^2)}, \text{ and} \qquad (2.2.6)$$

$$P_1 = \frac{e^{-\frac{\theta}{\alpha}}\left[(\theta+\alpha\theta+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha^2)}. \qquad (2.2.7)$$

2.2.4 Cumulative Distribution Function

Cumulative distribution function $F(x)$ of DJ distribution may be obtained as

$$F(x) = \frac{(\theta+\alpha^2)-e^{-\frac{\theta}{\alpha}(x+1)}[(\theta+\alpha^2)+\alpha\theta(1+x)]}{(\theta+\alpha^2)}. \qquad (2.2.8)$$

2.2.5 Survival Function

Survival function $S(x)$ of DJ distribution may be obtained as

$$S(x) = \frac{e^{-\frac{\theta}{\alpha}(x+1)}[(\theta+\alpha^2)+\alpha\theta(1+x)]}{(\theta+\alpha^2)}. \qquad (2.2.9)$$

2.2.6 Failure or Hazard Rate

Failure rate of DJ distribution may be obtained as

$$r(x)=\frac{(\theta+\alpha\theta x+\alpha^2)\left(1-e^{-\frac{\theta}{\alpha}}\right)-\alpha\theta e^{-\frac{\theta}{\alpha}}}{(\theta+\alpha^2+\alpha\theta x)}\,. \tag{2.2.10}$$

2.2.7 Reverse Hazard Rate

The reverse hazard rate function of DJ distribution is obtained as

$$r^*(x)=\frac{e^{-\frac{\theta}{\alpha}x}\left[(\theta+\alpha\theta x+\alpha^2)\left(1-e^{-\frac{\theta}{\alpha}}\right)-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha^2)-e^{-\frac{\theta}{\alpha}(x+1)}[(\theta+\alpha^2)+\alpha\theta(1+x)]}\,. \tag{2.2.11}$$

2.2.8 Second Rate of Failure

The second rate of failure rate function of DJ distribution is obtained as

$$r^{**}(x)=\log\left[\frac{[(\theta+\alpha^2)+\alpha\theta(1+x)]}{e^{-\frac{\theta}{\alpha}}[(\theta+\alpha^2)+\alpha\theta(2+x)]}\right]. \tag{2.2.12}$$

2.2.9 Factorial Moment Generating Function

Factorial moment generating function (fmgf) of DJ distribution may be obtained as

$$M(t)=\frac{\left[(\theta+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]\left(A-te^{-\frac{\theta}{\alpha}}\right)+\alpha\theta(1+t)A\,e^{-\frac{\theta}{\alpha}}}{(\theta+\alpha^2)\left(A-te^{-\frac{\theta}{\alpha}}\right)^2}\,, \tag{2.2.13}$$

where $A\;=\;1-e^{-\frac{\theta}{\alpha}}$.

2.2.10 Factorial Moment Recurrence Relation

The recurrence relation for fmgf may be obtained as

$$\mu'_{[r]}=\frac{e^{-\frac{\theta}{\alpha}}}{A^{\,2}}\left[2\,A\,r\mu'_{[r-1]}-r(r-1)\,e^{-\frac{\theta}{\alpha}}\mu'_{[r-2]}\right],\;r>1 \tag{2.2.14}$$

where

$$\mu'_{[1]} = \frac{e^{-\frac{\theta}{\alpha}}[(\theta+\alpha^2)A+\alpha\theta]}{(\theta+\alpha^2)A^2} \text{and} \tag{2.2.15}$$

$$\mu'_{[2]} = \frac{2e^{-\frac{2\theta}{\alpha}}[(\theta+\alpha^2)A+2\alpha\theta]}{(\theta+\alpha^2)A^3}. \tag{2.2.16}$$

The r^th factorial moment is obtained from moment as

$$\mu'_{[r]} = \left[\frac{d^r M(t)}{dt^r}\right]_{t=0}$$

$$\mu'_{[r]} = \frac{r! e^{-\frac{r\theta}{\alpha}}[(\theta+\alpha^2)A+r\alpha\theta]}{(\theta+\alpha^2)A^{r+1}}. \quad r = 1,2,... \tag{2.2.17}$$

The central moments μ_2, μ_3 and μ_4 of DJ distribution may be obtained in terms of its factorial moments as

$$\mu_2 = \mu'_{[2]} + \mu'_{[1]} - \mu'^2_{[1]} \quad , \tag{2.2.18}$$

$$\mu_3 = \mu'_{[3]} + 3\mu'_2 + \mu'_{[1]} - 3\mu'_{[2]}\mu'_{[1]} - 3\mu'^2_{[1]} + 2\mu'^3_{[1]} , \tag{2.2.19}$$

$$\mu_4 = \mu'_{[4]} + 6\mu'_{[3]} + 7\mu'_{[2]} + \mu'_{[1]} - 4\mu'_{[3]}\mu'_{[1]} - 12\mu'_{[2]}\mu'_{[1]} - 4\mu'^2_{[1]}$$
$$+6\mu'_{[2]}\mu'^2_{[1]} + 6\mu'^3_{[1]} - 3\mu'^4_{[1]} , \tag{2.2.20}$$

where $\mu = \mu'_{[1]} = \frac{e^{-\frac{\theta}{\alpha}}[(\theta+\alpha^2)A+\alpha\theta]}{(\theta+\alpha^2)A^2}$ denotes the mean of the distribution.

2.3 Zero Truncated Discrete Janardan (ZTDJ) Distribution

The pmf of Zero-truncated discrete Janardan (ZTNDJ) $P_z(x)$ distribution has been derived as

$$P_z(x) = \frac{P_x}{1-P_0}, \tag{2.3.1}$$

where P_x denotes the pmf of discrete Janardan distribution.

Hence,

$$P_z(x) = \frac{e^{-\frac{\theta(x-1)}{\alpha}}\left[(\theta+\alpha^2+\alpha\theta x)A - \alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha\theta+\alpha^2)}, \qquad x = 1,2,... \tag{2.3.2}$$

2.3.1 Probability Generating function for ZTDJ Distribution

Probability generating function $G_z(t)$ of ZTDJ may be obtained as

$$G_z(t) = \frac{t\left[\left\{(\theta+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right\}\left(1-te^{-\frac{\theta}{\alpha}}\right)+\alpha\theta A\right]}{(\theta+\alpha\theta+\alpha^2)\left(1-te^{-\frac{\theta}{\alpha}}\right)^2} \quad . \tag{2.3.3}$$

2.3.2 Probability Recurrence Relation for ZTDJ Distribution

Probability recurrence relation for ZTDJ distribution may obtained as

$$P_r = e^{-\frac{\theta}{\alpha}}\left[2P_{r-1} - e^{-\frac{\theta}{\alpha}}P_{r-2}\right], \quad r > 2 \tag{2.3.4}$$

Where

$$P_1 = \frac{\left[(\theta+\alpha^2+\alpha\theta)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha\theta+\alpha^2)}, \tag{2.3.5}$$

$$P_2 = \frac{e^{-\frac{\theta}{\alpha}}\left[(\theta+\alpha^2+2\alpha\theta)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha\theta+\alpha^2)}. \tag{2.3.6}$$

2.3.3 Cumulative Distribution of ZTDJ Distribution

The cumulative distribution of ZTDJ Lindley distribution is given by

$$F_z(x) = \frac{(\theta+\alpha\theta+\alpha^2)-[\theta+\alpha^2+\alpha\theta(1+x)]e^{-\frac{\theta}{\alpha}x}}{(\theta+\alpha\theta+\alpha^2)} \, . \tag{2.3.7}$$

2.3.4 Survival Function of ZTDJ Distribution

The survival function of ZTDJ distribution is given by

$$S_z(x) = \frac{[\theta+\alpha^2+\alpha\theta(1+x)]e^{-\frac{\theta}{\alpha}x}}{(\theta+\alpha\theta+\alpha^2)}. \tag{2.3.8}$$

2.3.5 Failure Hazard Rate Function of ZTDJ Distribution

The failure hazard rate function of ZTDJ distribution is given by

$$r_z(x) = \frac{P(X=x)}{P(X\geq x-1)}$$

$$= \frac{(\theta+\alpha^2+\alpha\theta x)A-\alpha\theta e^{-\frac{\theta}{\alpha}}}{\theta+\alpha^2+\alpha\theta x}. \tag{2.3.9}$$

2.3.6 Reversed Failure Rate of ZTDJ Distribution

The reversed failure rate function of ZTDJ distribution is given by

$$r_z^*(x) = \frac{P(X=x)}{P(X\leq x)}$$

$$= \frac{e^{-\frac{\theta(x-1)}{\alpha}}\left[(\theta+\alpha^2+\alpha\theta x)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha\theta+\alpha^2)-[\theta+\alpha^2+\alpha\theta(1+x)]e^{-\frac{\theta}{\alpha}x}}. \tag{2.3.10}$$

2.3.7 Second Rate of Failure of ZTDJ Distribution

The second rate failure rate function of ZTDJ distributionis given by

$$r_z^{**}(x) = log\left[\frac{s(x)}{s(x+1)}\right]$$

$$= log\left[\frac{\theta+\alpha^2+\alpha\theta(1+x)}{e^{-\theta}\theta+\alpha^2+\alpha\theta(2+x)}\right]. \tag{2.3.11}$$

2.3.8 Proportions of Probabilities of ZTDJ Distribution

The proportions of probabilities of ZTDJ Distribution is given by

$$\frac{P_z(x+1)}{P_z(x)} = e^{-\frac{\theta}{\alpha}}\left[1+\frac{\alpha\theta A}{(\theta+\alpha^2+\alpha\theta x)A-\alpha\theta e^{-\frac{\theta}{\alpha}}}\right] \tag{2.3.12}$$

2.3.9 Factorial Moment Recurrence Relation for ZTDJ Distribution

Factorial moment generating function for ZTDJ distribution may be obtained as

$$M(t) = \frac{(1+t)\left[\left\{(\theta+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right\}\left(A-te^{-\frac{\theta}{\alpha}}\right)+\alpha\theta A\right]}{(\theta+\alpha\theta+\alpha^2)\left(A-te^{-\frac{\theta}{\alpha}}\right)^2}. \tag{2.3.13}$$

Factorial moment recurrence relation for ZTDJ distribution may be obtained as

$$\mu'_{[r]} = \frac{e^{-\frac{\theta}{\alpha}}}{\left(1-e^{-\frac{\theta}{\alpha}}\right)^2}\left[2A\, r\, e^{-\frac{\theta}{\alpha}}\mu'_{[r-1]} - r(r-1)e^{-\frac{2\theta}{\alpha}}\mu'_{[r-2]}\right], \quad r > 2 \qquad (2.3.14)$$

$$\mu'_{[1]} = \frac{[(\theta+\alpha^2)A+\alpha\theta]}{(\theta+\alpha^2+\alpha\theta)A^2}, \qquad (2.3.15)$$

$$\mu'_{[2]} = \frac{2e^{-\frac{\theta}{\alpha}}[(\theta+\alpha^2)A+2\alpha\theta]}{(\theta+\alpha^2+\alpha\theta)A^3}. \qquad (2.3.16)$$

The r^th factorial moment is obtained from moment as

$$\mu'_{[r]} = \left[\frac{d^r M(t)}{dt^r}\right]_{t=0}$$

$$\mu'_{[r]} = \frac{r!e^{-\frac{(r-1)\theta}{\alpha}}[(\theta+\alpha^2)A+r\alpha\theta]}{(\theta+\alpha^2+\alpha\theta)A^{r+1}}. \qquad r = 1, 2, 3, \ldots \qquad (2.3.17)$$

2.4 Size-Biased Discrete Janardan (SBDJ) Distribution

If a random variable X have DJ distribution with parameter θ and α then the pmf of the size-biased distribution may be derived as

$$P_x^s = \frac{xP_x}{\mu}, \qquad (2.4.1)$$

where P_x and μ denote respectively pmf and the mean of DJ distribution.

The pmf P_x^s of size-biased discrete Janardan (SBDJ) distribution with parameters θ and α may be derived from (2.2.3) as

$$P_x^s = \frac{x\, A^2 e^{-\frac{\theta}{\alpha}(x-1)}\left[(\theta+\alpha^2+\alpha\theta x)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha^2)A+\alpha\theta}, \quad for \quad x = 1, 2, 3, \ldots \qquad (2.4.2)$$

where A = $1 - e^{-\frac{\theta}{\alpha}}$.

2.4.1 Probability Generating Function of SBDJ distribution

Probability generating function $G^s(t)$ for SBDJ may be obtained as

$$G^s(t) = \frac{t\,A^2\left[\left\{(\theta+\alpha^2)\,A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right\}\left(1-te^{-\frac{\theta}{\alpha}}\right)+\alpha\theta A\left(1+te^{-\frac{\theta}{\alpha}}\right)\right]}{\{(\theta+\alpha^2)\,A-\alpha\theta\}\left(1-te^{-\frac{\theta}{\alpha}}\right)^3}, \qquad (2.4.3)$$

where A = $1 - e^{-\frac{\theta}{\alpha}}$

2.4.2 Probability Recurrence Relation of SBDJ distribution

Probability recurrence relation of SBDJ distribution may be obtained as

$$P_r = e^{-\frac{\theta}{\alpha}}\left[3P_{r-1} - 3e^{-\frac{\theta}{\alpha}}P_{r-2} + e^{-\frac{2\theta}{\alpha}}P_{r-3}\right], \qquad \text{for } r > 2 \qquad (2.4.4)$$

where

$$P_0 = P(X=0) = \frac{\left[(\theta+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha^2)}, \qquad (2.4.5)$$

$$P_1 = \frac{A^2\left[(\theta+\alpha^2+\alpha\theta)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha^2)A+\alpha\theta} \quad \text{and} \qquad (2.4.6)$$

$$P_2 = \frac{2e^{-\frac{\theta}{\alpha}}\,A^2\left[(\theta+\alpha^2+2\alpha\theta)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{(\theta+\alpha^2)A+\alpha\theta}. \qquad (2.4.7)$$

2.4.3 Factorial Moment Recurrence of SBDJ distribution

Factorial moment generating function $M_s(t)$ of SBDJ may be obtained as

$$M^s(t) = \frac{(1+t)A^2\left[\left\{(\theta+\alpha^2)\,A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right\}\left(A-te^{-\frac{\theta}{\alpha}}\right)+\alpha\theta\,A\left(A+te^{-\frac{\theta}{\alpha}}\right)\right]}{\{(\theta+\alpha^2)\,A-\alpha\theta\}\left(A-te^{-\frac{\theta}{\alpha}}\right)^3}, \qquad (2.4.8)$$

where A = $1 - e^{-\frac{\theta}{\alpha}}$.

Factorial moment recurrence relation of SBDJ distribution may be obtained as

$$\mu'_{[r]} = \frac{e^{-\frac{\theta}{\alpha}}}{A^3}\left[3A^2 r\mu'_{[r-1]} - 3Ar(r-1)e^{-\frac{\theta}{\alpha}}\mu'_{[r-2]} + Ar(r-1)(r-2)e^{-2\frac{\theta}{\alpha}}\mu'_{[r-3]}\right], \qquad (2.4.9)$$

where A = $1 - e^{-\frac{\theta}{\alpha}}$.

2.5 Zero-Modified DJ Distribution

The Zero-modified of DJ distribution may be obtained as.

$$P^z[X=0] = \omega + (1-\omega)P_0$$

$$= \omega + (1-\omega)\left[\frac{(\theta+\alpha^2)A - \alpha\theta e^{-\frac{\theta}{\alpha}}}{(\theta+\alpha^2)}\right], \qquad (2.5.1)$$

Where P_0 denotes probability of DJ distribution at $x = 0$.
Hence the relationship will be

$$P^z[X=x] = (1-\omega)\lambda^x P(x), \qquad x=1, 2, \ldots, \alpha \geq 0,\ 0 < \lambda < 1,\ \omega \geq \frac{-P_0}{1-P_0}, \qquad (2.5.2)$$

where $P(x)$ denotes the probability of DJ distribution.

2.6 Estimation of Parameters of DJ Distribution

2.6.1 Estimation based on first three relative frequencies and mean

Probability recurrence relation of distribution DJD is given as

$$P_r = e^{-\frac{\theta}{\alpha}}\left[2P_{r-1} - e^{-\frac{\theta}{\alpha}}P_{r-2}\right], \qquad r \geq 2 \qquad (2.6.1)$$

where $P_r = P(X=r)$denotes the r^{th} order probability of the distribution

Now putting $\lambda = e^{-\frac{\theta}{\alpha}}$ and r = 2 in (2.5.1), we have

$$\lambda^2 P_0 - 2\lambda P_1 + P_2 = 0\ . \qquad (2.6.2)$$

There are two values of λ had solving equation (2.6.2). We choose that the value λ had which minimizes the value of χ^2 static in table 2.1- 2.3, column 5.

The appropriate root C_1 (say) of the quadratic equation

$$\lambda = \frac{P_1 \pm \sqrt{(P_1{}^2 - P_0P_2)}}{P_0} \quad \text{may be selected. Hence} \quad C_1 = e^{-\frac{\theta}{\alpha}} \qquad (2.6.3)$$

Now let us $\frac{\theta}{\alpha} = log\ \frac{1}{C_1} = C_2$ (say) (2.6.4)

The mean μ_1' of distribution DJ can be written as

$$\mu_1' = \frac{C_1 \ [(C_2 \ +\alpha)(1-C_1)+\alpha C_2 \ ]}{(C_2 \ +\alpha)(1-C_1)^2} \tag{2.6.5}$$

$Hence\ \alpha$ can be estimated from (2.6.5) as

$$\alpha = \frac{C_1 C_2(1-C_1)-(1-C_1)^2 C_2\, \mu_1'}{(1-C_1)^2\mu_1'-C_1(1-C_1)-C_2C_1} \tag{2.6.6}$$

and θ can be estimated from (2.6.4) as $\theta = \alpha C_2$

2.6.2 Estimation based on first four relative frequencies and mean

From Probability recurrence relation (2.2.10) putting r = 2 and r = 3 respectively, we have

$$\lambda^2 P_0 - 2\lambda P_1 + P_2 = 0 \tag{2.6.7}$$

$$\lambda^2 P_1 - 2\lambda P_2 + P_3 = 0 \tag{2.6.8}$$

Solving equations (2.6.7) and (2.6.8) for λ , we have

$$\lambda = \frac{P_1P_2-P_0P_3}{2(P_1^2-P_0P_2)} = D_1 \quad \text{(say) , provided} \qquad P_1^2 - P_0P_2 > 0. \tag{2.6.9}$$

$$Now\ let\ us\ take\, \frac{\theta}{\alpha} = log\ \frac{1}{D_1} = D_2 \ \ (say)$$

The mean μ_1' of distribution DJ can be written as

$$\mu_1' = \frac{D_1\,[(D_2 \ +\alpha)(1-D_1)+\alpha D_2 \ ]}{(D_2 \ +\alpha)(1-D_1)^2} \tag{2.6.10}$$

$Now\ \alpha$ can be estimated from (5.9) as

$$\hat{\alpha} = \frac{D_1 D_2(1-D_1)-(1-D_1)^2 D_2\, \mu_1'}{(1-D_1)^2\mu_1'-D_1(1-D_1)-D_2D_1} \tag{2.6.11}$$

Ultimately θ can be estimated as $\hat{\theta} = \alpha D_2$.

2.6.3. Maximum Likelihood Estimates

The likelihood function, L of the two parameter Lindley distribution (2.2.3) is given by

$$L = \prod_{x=1}^{k} P_x^{\,f_x},$$

$$L = \frac{e^{-\frac{\theta}{\alpha}n\bar{x}}}{(\theta+\alpha^2)^n} \prod_{x=1}^{k} \left[(\theta + \alpha\theta x + \alpha^2)A - \alpha\theta e^{-\frac{\theta}{\alpha}}\right]^{f_x}. \quad (2.6.12)$$

The log likelihood function may be obtained as

$$logL = -\frac{\theta}{\alpha}n\bar{x} - nlog(\theta + \alpha^2) + \sum_{x=1}^{k} f_x log\left[(\theta + \alpha\theta x + \alpha^2)A - \alpha\theta e^{-\frac{\theta}{\alpha}}\right]$$

. (2.6.13)

The derivatives of *log* likelihood equations are thus obtained as

$$\frac{\partial logL}{\partial\theta} = -\frac{1}{\alpha}n\bar{x} - \frac{n}{(\theta+\alpha^2)} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{\partial\theta}}{\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]} = 0. \quad (2.6.14)$$

$$\frac{\partial logL}{\partial\alpha} = \frac{\theta}{\alpha^2}n\bar{x} - \frac{2n\alpha}{(\theta+\alpha^2)} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{\partial\alpha}}{\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]} = 0. \quad (2.6.15)$$

These two equations (2.6.14) and (2.6.15) cannot be solved directly. However the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 logL}{\partial\theta^2} = -\frac{n}{(\theta+\alpha^2)^2} + \frac{\partial}{\partial\theta}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{\partial\theta}}{\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}. \quad (2.6.16)$$

$$\frac{\partial^2 logL}{\partial\theta\partial\alpha} = \frac{1}{\alpha^2}n\bar{x} + \frac{2n\alpha}{(\theta+\alpha^2)^2} + \frac{\partial}{\partial\alpha}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{\partial\theta}}{\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}. \quad (2.6.17)$$

$$\frac{\partial^2 logL}{\partial\alpha^2} = -2\frac{\theta}{\alpha^3}n\bar{x} + \frac{2n(\theta-\alpha^2)}{(\theta+\alpha^2)^2} + \sum_{x=1}^{k} f_x \frac{\frac{\partial l\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}{\partial\alpha}}{\left[(\theta+\alpha\theta x+\alpha^2)A-\alpha\theta e^{-\frac{\theta}{\alpha}}\right]}. \quad (2.6.18)$$

The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved

$$\begin{bmatrix} \frac{\partial^2 logL}{\partial\theta^2} & \frac{\partial^2 logL}{\partial\theta\partial\alpha} \\ \frac{\partial^2 logL}{\partial\alpha\partial\theta} & \frac{\partial^2 logL}{\partial\alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial logL}{\partial\theta} \\ \frac{\partial logL}{\partial\alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}, \quad (2.6.19)$$

where θ_0 and $\alpha_0$are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently close estimates of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

2.7 Goodness of Fit of DJ Distribution

The fitting of the DJD to three data-sets have been presented in the following tables. the table is due to Beall (1940) regarding the distribution of Pyrausta nublialis in 1937. The expected frequencies according to the discrete two parameter PLD have also been given in these tables for ready comparison with those obtained by DJD.

Table 2.1 Distribution of Pyrausta nublilalis in 1937.

No. of Insects	Observed frequency	Two parameter PL	Fitted DJ
0	213	228.6	213.12
1	128	101.5	110.93
2	37	43.5	47.45
3	18	17.9	18.43
4	3	6.8	6.79
5	1	2.2	2.34
6	0	0.6	0.94
Total	400	400	400
		$\hat{\theta} = 0.9151$ $\hat{\alpha} = -0.2498$ $d.f. = 3$ $\chi^2 = 12.3$ $p = 0.015$	$\hat{\theta} = 1.24518$ $\hat{\alpha}= 1.03475$ $d.f. = 3$ $\chi^2 = 8.59$ $p = 0.072$

2.8 Discussion

In this investigation, the discrete Janardan (DJ) distribution has been introduced by discretizing the continuous Janardan distribution. A few useful properties of DJ distribution have been discussed. The estimation of parameters by the method of moments have been discussed. In this paper, an attempt has been made for the fitting of DJ distribution to a real data set by estimating its parameters based on first four relative frequencies and mean. Based on χ^2 goodness of fit test, it may be concluded that DJ distribution provides better fit than the two parameter Poisson–Lindley distribution. Besides, the proposed model seems to be suitable for modeling not only numbers of claims but also other count models.