Chapter 3

Discrete Sushila Distribution and Its Application

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3.1 Introduction

Shanker et al [47] introduced the following two parameter continuous Sushila distribution with parameter α and θ.

$$f(x;\ \theta,\alpha) = \frac{\theta^2}{\alpha(\theta+1)}\left(1+\frac{x}{\alpha}\right)e^{-\frac{\theta}{\alpha}x};\quad x>0,\theta>0,\alpha>0. \tag{3.1.1}$$

3.2. Discretization of Sushila Distribution

In this paper, our objective is to derive a new discrete distribution and to study some of their properties, which may be called discrete Sushila (DS) distribution based on the survival function of the continuous Sushila distribution. The survival function may be obtained as

$$S(x) = \int_x^{\infty} f(x;\ \theta,\alpha)dx$$

$$= \frac{e^{-\frac{\theta}{\alpha}x}\{\alpha(\theta+1)+\theta x\}}{\alpha(\theta+1)}, \tag{3.2.1}$$

hence,

$$S(x+1) = \frac{e^{-\frac{\theta}{\alpha}(x+)}\{\alpha(\theta+1)+\theta(x+1)\}}{\alpha(\theta+1)}. \tag{3.2.2}$$

3.2.1 Probability Mass Function (pmf)

The probability mass function (pmf) of DS distribution may be obtained as

$$P(X=x) = S(x) - S(x+1)$$

$$= \frac{e^{-\frac{\theta}{\alpha}x}\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)}, \quad x = 0, 1, 2, \dots \qquad (3.2.3)$$

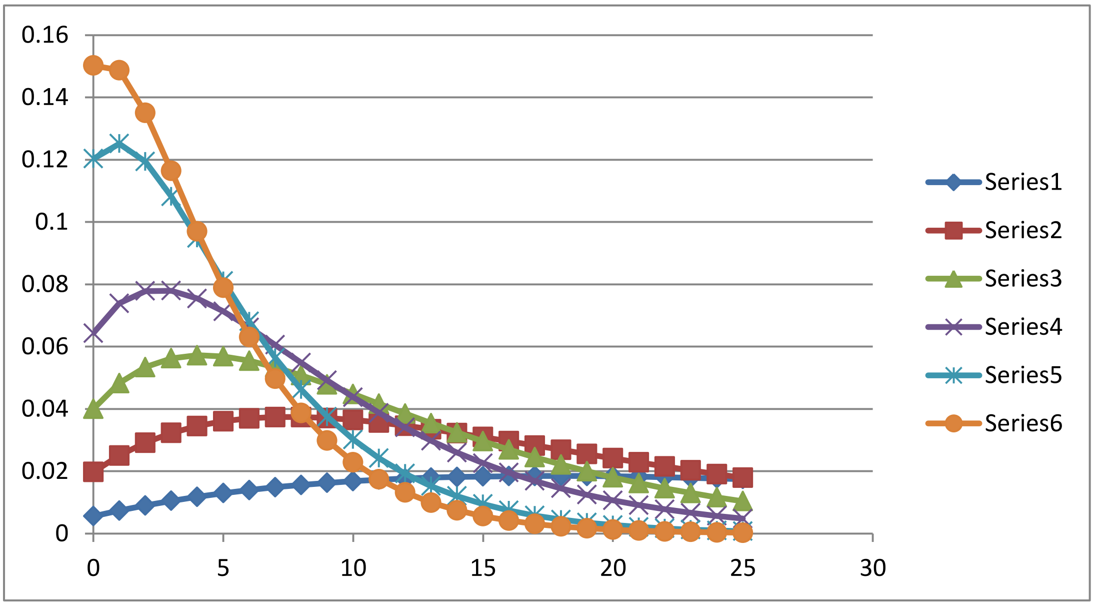

Figure 3: Probability graph for Discrete Sushila distribution $\alpha = 2, \theta = 0.1$ $(series1)\alpha = 2, \theta = 0.3$ $(series3)\alpha = 2, \theta = 0.4(series4)\alpha = 2, \theta = 0.4(series4)\alpha = 2, \theta = 0.5$ $(series5)\alpha = 2, \theta = 0.6(series6)$

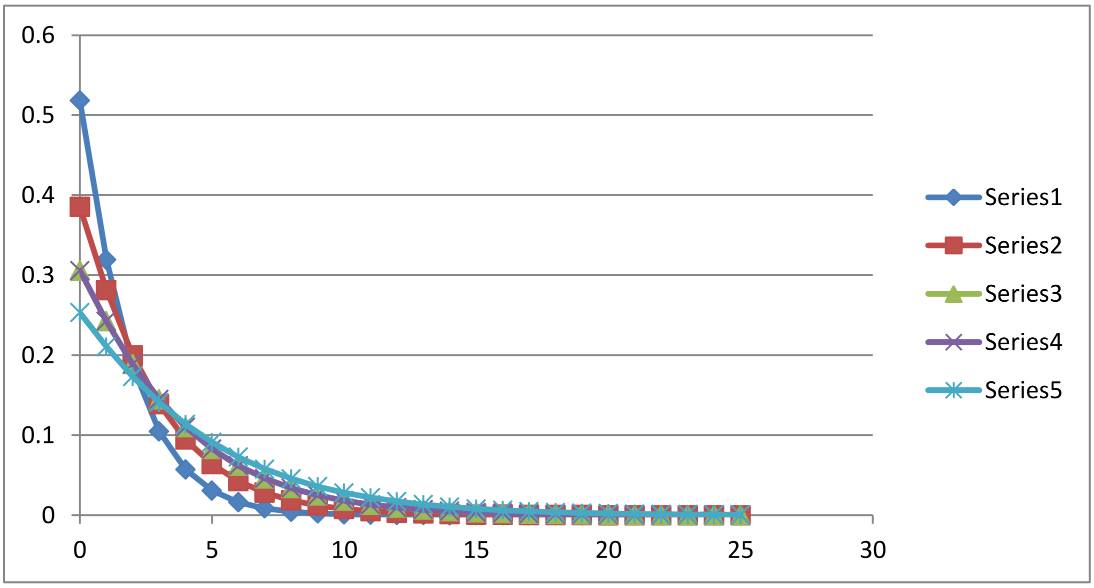

Figure 4: Probability graph for Discrete Sushila distribution $\theta = 1.5, \alpha = 2$ $(series1)\theta = 1.5, \alpha = 3$ $(series3)\theta = 1.5, \alpha = 4$ $(series4)\theta = 1.5, \alpha = 2$ $(series4)\theta = 1.5, \alpha = 5$ $(series5)\theta = 1.5, \alpha = 6(series6)$

3.2.2 Probability Generating Function

The pgf of DS distribution may be obtained as

$$G(t) = \sum_{x=0}^{\infty} t^x P_x$$

$$= \frac{\left[\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]\left(1-te^{-\frac{\theta}{\alpha}}\right)+\theta te^{-\frac{\theta}{\alpha}}\left(1-e^{-\frac{\theta}{\alpha}}\right)}{\alpha(\theta+1)\left(1-te^{-\frac{\theta}{\alpha}}\right)^2}, \quad \theta > 0, \alpha > 0. \quad (3.2.4)$$

3.2.3 Probability Recurrence Relation

Probability recurrence relation of DS distribution may be obtained as

$$P_r = e^{-\frac{\theta}{\alpha}}\left[2P_{r-1} - e^{-\frac{\theta}{\alpha}}P_{r-2}\right], \qquad \text{for} \quad r \geq 2 \qquad (3.2.5)$$

where,

$$P_0 = \frac{\left[\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)}, \text{ and} \qquad (3.2.6)$$

$$P_1 = \frac{e^{-\frac{\theta}{\alpha}}\left[\{\alpha(\theta+1)+\theta\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)}. \qquad (3.2.7)$$

3.2.4 Cumulative Distribution Function

The cumulative distribution function $F(x)$ of DS distribution may be obtain as

$$F(x) = \frac{\alpha(\theta+1)-e^{-\frac{\theta}{\alpha}(x+1)}[\alpha(\theta+1)+\theta(1+x)]}{\alpha(\theta+1)}. \qquad (3.2.8)$$

3.2.5 Survival Function

Hence the survival function of DS distribution may be written as

$$S_D(x) = \frac{e^{-\frac{\theta}{\alpha}(x+1)}[\alpha(\theta+1)+\theta(1+x)]}{\alpha(\theta+1)}. \qquad (3.2.9)$$

3.2.6 Failure Rate function

The corresponding failure or hazard rate $r(x)$ of DS distribution will be

$$r(x) = P_r(X < x|X < x-1) = \frac{P_r(X=x)}{P_r(X>x-1)}$$

$$= \frac{\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{[\alpha(\theta+1)+\theta x]}. \tag{3.2.10}$$

3.2.7 Reversed Hazard Rate Function

The corresponding reversed hazard rate of DS distribution will be

$$r^*(x) = \frac{P_r(X=x)}{P_r(X\leq x)}$$

$$= \frac{e^{-\frac{\theta}{\alpha}x}\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)-e^{-\frac{\theta}{\alpha}(x+1)}[\alpha(\theta+1)+\theta(1+x)]}. \tag{3.2.11}$$

3.2.8 Second Rate of Failure function

The corresponding second rate of DS distribution will be

$$r^{**} = \log\left[\frac{S(X)}{S(X+1)}\right] = \log\left[\frac{\alpha(\theta+1)+\theta(1+x)}{e^{-\frac{\theta}{\alpha}}[\alpha(\theta+1)+\theta(2+x)]}\right]. \tag{3.2.12}$$

3.2.9 Factorial Moment Generating Function

Factorial Moment Generating function may be obtained as

$$M(t) = \sum_{x=0}^{\infty} t^x P_x$$

$$= \frac{\left[\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]\left(1-e^{-\frac{\theta}{\alpha}}-te^{-\frac{\theta}{\alpha}}\right)+\theta(1+t)e^{-\frac{\theta}{\alpha}}\left(1-e^{-\frac{\theta}{\alpha}}\right)}{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}-te^{-\frac{\theta}{\alpha}}\right)^2}. \tag{3.2.13}$$

3.2.10 Factorial Moment's Recurrence Relation

Factorial moment's recurrence relation of DS distribution may be obtained as

$$\mu'_{[r]} = \frac{e^{-\frac{\theta}{\alpha}}}{\left(1-e^{-\frac{\theta}{\alpha}}\right)^2}\left[2\left(1-e^{-\frac{\theta}{\alpha}}\right)e^{-\frac{\theta}{\alpha}}r\mu'_{[r-1]} - e^{-\frac{2\theta}{\alpha}}r(r-1)\mu'_{[r-2]}\right], \quad r > 2$$

(3.2.14)

where

$$\mu'_{[1]} = \frac{e^{-\frac{\theta}{\alpha}}\left[\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)+\theta\right]}{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)^2}, \tag{3.2.15}$$

$$\mu'_{[2]} = \frac{2e^{-\frac{2\theta}{\alpha}}\left[\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)+2\theta\right]}{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)^3}, \tag{3.2.16}$$

The r^th factorial moment generating function may be obtained as

$$\mu'_{[r]} = \frac{r!e^{-\frac{r\theta}{\alpha}}\left[\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)+r\theta\right]}{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)^{r+1}}.. \qquad r = 1, 2, ... \tag{3.2.17}$$

The central moments μ_2, μ_3 *and* μ_4 of the distribution have been obtained as

$$\left.\begin{array}{c}\mu_2 = \mu'_{[2]} + \mu'_{[1]} - \mu'^2_{[1]} \\ \mu_3 = \mu'_{[3]} + 3\mu'_{[2]} + \mu'_{[1]} - 3\mu'_{[2]}\mu'_{[1]} - 3\mu'^2_{[1]} + 2\mu'^3_{[1]} \\ \mu_4 = \mu'_{[4]} + 6\mu'_{[3]} + 7\mu'_{[2]} + \mu'_{[1]} - 4\mu'_{[3]}\mu'_{[1]} - 12\mu'_{[2]}\mu'_{[1]} - 4\mu'^2_{[1]} + 6\mu'_{[2]}\mu'^2_{[1]} + 6\mu'^3_{[1]} - 3\mu'^4_{[1]}\end{array}\right],$$

(3.2.18)

where $\mu = \frac{e^{-\frac{\theta}{\alpha}}\left[\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)+\theta\right]}{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)^2}$ denotes the mean of the distribution.

3.3 Zero Truncated Discrete Sushila (ZTDS) Distribution

The pmf of Zero-truncated discrete Sushila (ZTDS) $P_z(x)$ distribution has been derived as

$$P_z(x) = \frac{P_x}{1-P_0}, \tag{3.3.1}$$

where P_x denotes the pmf of discrete Sushila distribution.

$$Hence, \quad P_z(x) = \frac{e^{-\frac{\theta}{\alpha}(x-1)}\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)+\theta}, \quad x = 1, 2, \ldots \qquad (3.3.2)$$

3.3.1 Probability Generating Function of ZTDS Distribution

Probability generating function $G_z(t)$ of ZTDS distribution may be obtained as

$$G_z(t) = \frac{t\left[\left\{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right\}\left(1-te^{-\frac{\theta}{\alpha}}\right)+\theta\left(1-e^{-\frac{\theta}{\alpha}}\right)\right]}{\{\alpha(\theta+1)+\theta\}\left(1-te^{-\frac{\theta}{\alpha}}\right)^2}. \qquad (3.3.3)$$

3.3.2 Probability Recurrence Relation of ZTDS Distribution

Probability recurrence relation for ZTDS Distribution

$$P_r = e^{-\frac{\theta}{\alpha}}\left[2P_{r-1} - e^{-\frac{\theta}{\alpha}}P_{r-2}\right], \quad for \;\; r \geq 3 \qquad (3.3.4)$$

where

$$P_1 = \frac{\left[\{\alpha(\theta+1)+\theta\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)+\theta}, \qquad (3.3.5)$$

$$P_2 = \frac{e^{-\frac{\theta}{\alpha}}\left[\{\alpha(\theta+1)+2\theta\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)+\theta}. \qquad (3.3.6)$$

3.3.3 Cumulative Distribution of ZTDS Distribution

The cumulative distribution of ZTDS Lindley distribution is given by

$$F_z(x) = \frac{(\alpha(\theta+1)+\theta)-[\alpha(\theta+1)+\theta(1+x)]e^{-\frac{\theta}{\alpha}x}}{\alpha(\theta+1)+\theta}. \qquad (3.3.7)$$

3.3.4 Survival function of ZTDS Distribution

The survival function of ZTDS distribution is given by

$$S_z(x) = \frac{[\alpha(\theta+1)+\theta(1+x)]e^{-\frac{\theta}{\alpha}x}}{\alpha(\theta+1)+\theta}. \qquad (3.3.8)$$

3.3.5 Failure Rate Function of ZTDS Distribution

The failure hazard ratefunction of ZTDS distribution is given by

$$r_z(x) = \frac{P(X=x)}{P(X \geq x-1)}$$

$$= \frac{\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}}{\alpha(\theta+1)+\theta x}. \qquad (3.3.9)$$

3.3.6 Reversed Failure Rate function of ZTDS Distribution

The reversed failure rate function of ZTDS distribution is given by

$$r_z^*(x) = \frac{P(X=x)}{P(X \leq x)}$$

$$= \frac{e^{-\frac{\theta}{\alpha}(x-1)}\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{(\alpha(\theta+1)+\theta)-[\alpha(\theta+1)+\theta(1+x)]e^{-\frac{\theta}{\alpha}x}}. \qquad (3.3.10)$$

3.3.7 Second Rate of Failure function of ZTDS Distribution

The second rate failure rate function of ZTDS distributionis given by

$$r_z^{**}(x) = log\left[\frac{s(x)}{s(x+1)}\right]$$

$$= log\left[\frac{\alpha(\theta+1)+\theta(1+x)}{e^{-\frac{\theta}{\alpha}}\{\alpha(\theta+1)+\theta(2+x)\}}\right]. \qquad (3.3.11)$$

3.3.8 Proportions of Probabilities of ZTDS Distribution

The proportions of probabilities of ZTDS Distribution is given by

$$\frac{P_z(x+1)}{P_z(x)} = e^{-\frac{\theta}{\alpha}}\left[1+\frac{\theta\left(1-e^{-\frac{\theta}{\alpha}}\right)}{\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}}\right]. \qquad (3.3.12)$$

3.3.9 Factorial Moment Recurrence Relation of ZTDS Distribution

Factorial Moment generating function of ZTDS distribution may be obtained as

$$M_z(t) = \frac{(1+t)\left[\left\{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right\}\left(1-e^{-\frac{\theta}{\alpha}}-te^{-\frac{\theta}{\alpha}}\right)+\theta\left(1-e^{-\frac{\theta}{\alpha}}\right)\right]}{\{\alpha(\theta+1)+\theta\}\left(1-e^{-\frac{\theta}{\alpha}}-te^{-\frac{\theta}{\alpha}}\right)^2} \qquad (3.3.13)$$

Factorial moment recurrence relation of ZTDS distribution may be obtained as

$$\mu'_{[r]} = \frac{e^{-\frac{\theta}{\alpha}}}{\left(1-e^{-\frac{\theta}{\alpha}}\right)^2}\left[2\left(1-e^{-\frac{\theta}{\alpha}}\right)e^{-\frac{\theta}{\alpha}} r\mu'_{[r-1]} - e^{-\frac{2\theta}{\alpha}} r(r-1)\mu'_{[r-2]}\right], \quad r \geq 2$$

(3.3.14)

where

$$\mu'_{[1]} = \frac{\left[(\theta+\alpha^2)\left(1-e^{-\frac{\theta}{\alpha}}\right)+\alpha\theta\right]}{(\theta+\alpha^2+\alpha\theta)\left(1-e^{-\frac{\theta}{\alpha}}\right)^2}, \qquad (3.3.15)$$

$$\mu'_{[2]} = \frac{2e^{-\frac{\theta}{\alpha}}\left[(\theta+\alpha^2)\left(1-e^{-\frac{\theta}{\alpha}}\right)+2\alpha\theta\right]}{(\theta+\alpha^2+\alpha\theta)\left(1-e^{-\frac{\theta}{\alpha}}\right)^3}. \qquad (3.3.16)$$

The general form of r^th ordered factorial moment may also be written as

$$\mu'_{[r]} = \frac{r!e^{-\frac{(r-1)\theta}{\alpha}}\left[\{\alpha(\theta+1)-\theta\}\left(1-e^{-\frac{\theta}{\alpha}}\right)+r\theta\right]}{\{\alpha(\theta+1)+\theta\}\left(1-e^{-\frac{\theta}{\alpha}}\right)^{r+1}}. \qquad r = 1, 2, 3, \ldots \qquad (3.3.17)$$

3.4 Size-Biased Discrete Sushila (SBDS) Distribution

In this section, the pmf of size-biased discrete Sushila (SBDS) distribution with parameter α and θ has been derived as

$$P_x^s = \frac{xe^{-\frac{\theta}{\alpha}(x-1)}\left(1-e^{-\frac{\theta}{\alpha}}\right)^2\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)+\theta}, \quad x = 1, 2, 3, \ldots \qquad (3.4.1)$$

where $\mu = \frac{e^{-\frac{\theta}{\alpha}}\left[\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)+\theta\right]}{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)^2}$ denotes the mean of DS distribution.

3.4.1 Probability Generating Function of SBDS Distribution

Probability generating function $G^s(t)$ for SBDS may be obtained as

$$G^s(t) = \frac{t\left(1-e^{-\frac{\theta}{\alpha}}\right)^2\left[\left\{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right\}\left(1-te^{-\frac{\theta}{\alpha}}\right)+\theta\left(1-e^{-\frac{\theta}{\alpha}}\right)\left(1+te^{-\frac{\theta}{\alpha}}\right)\right]}{\left\{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)+\theta\right\}\left(1-te^{-\frac{\theta}{\alpha}}\right)^3}, \qquad (3.4.2)$$

3.4.2 Probability Recurrence Relation of SBDS Distribution

Probability recurrence relation of SBDS distribution

$$P_r^s = e^{-\frac{\theta}{\alpha}}\left[3P_{r-1}^s - 3e^{-\frac{\theta}{\alpha}}P_{r-2}^s + e^{-\frac{2\theta}{\alpha}}P_{r-3}^s\right], \qquad \text{for} \quad r > 3 \qquad (3.4.3)$$

where,

$$P_1^s = \frac{\left(1-e^{-\frac{\theta}{\alpha}}\right)^2\left[(\alpha(\theta+1)+\theta)\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)+\theta}, \qquad (3.4.4)$$

$$P_2^s = \frac{2e^{-\frac{\theta}{\alpha}}\left(1-e^{-\frac{\theta}{\alpha}}\right)^2\left[(\alpha(\theta+1)+2\theta)\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)+\theta}. \qquad (3.4.5)$$

$$P_3^s = \frac{3e^{-2\frac{\theta}{\alpha}}\left(1-e^{-\frac{\theta}{\alpha}}\right)^2\left[\{\alpha(\theta+1)+3\theta\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)\left(1-e^{-\frac{\theta}{\alpha}}\right)+\theta} \qquad (3.4.6)$$

3.4.3 Factorial Moment Generating Function of SBDS distribution

Factorial Moment Generating Function of SBDS distribution may be obtained as

$$M_s(t) = \frac{(1+t)A^2\left[\left\{\alpha(\theta+1)A-\theta e^{-\frac{\theta}{\alpha}}\right\}\left(1-e^{-\frac{\theta}{\alpha}}-te^{-\frac{\theta}{\alpha}}\right)+\theta A\left(1+e^{-\frac{\theta}{\alpha}}+te^{-\frac{\theta}{\alpha}}\right)\right]}{\{\alpha(\theta+1)A+\theta\}\left(1-e^{-\frac{\theta}{\alpha}}-te^{-\frac{\theta}{\alpha}}\right)^3}. \qquad (3.4.7)$$

3.4.4 Factorial Moment Recurrence Relation of SBDS Distribution

Factorial moment recurrence relation of SBDS distribution

$$\mu'_{[r]} = \frac{e^{-\frac{\theta}{\alpha}}}{A^3}\left[3A^2 r\mu'_{[r-1]} - 3e^{-\frac{\theta}{\alpha}}Ar(r-1)\mu'_{[r-2]} + e^{-2\frac{\theta}{\alpha}}Ar(r-1)(r-2)\mu'_{[r-3]}\right] , \tag{3.4.8}$$

where, $A = 1 - e^{-\frac{\theta}{\alpha}}$.

3.5 Zero-Modified DS distribution

The Zero-modified of DS distribution is obtained as.

$$P^z[X = 0] = \omega + (1-\omega)P_0 ,$$

$$= \omega + (1-\omega)\left[\frac{\left[\{\alpha(\theta+1)\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\alpha(\theta+1)}\right], \tag{3.5.1}$$

where P_0 denotes probability of DS distribution at $x = 0$.

Hence the relationship will be

$$P^z[X = x] = (1-\omega)\lambda^x P(x), \quad x=1, 2, \dots , \tag{3.5.2}$$

$$\alpha \geq 0,\ 0 < \lambda < 1,\ \omega \geq \frac{-P_0}{1-P_0}$$

where $P(x)$ denotes the probability of NDQL distribution.

3.6 Estimation of Parameters

Discrete Sushila distribution has two parameters to be estimated. The mean and variance are used to get the initial guest values of the parameters α and θ. Newton - Raphson iterative method has been used to get the sufficiently close estimates $\hat{\theta}$ and $\hat{\alpha}$ for fitting of the distribution.

3.6.1 Maximum Likelihood Estimates

The likelihood function, L of the two parameter Sushila distribution (3.2.3) is given by

$$L = \prod_{x=1}^{k} P_x^{\;f_x}, \tag{3.6.1}$$

$$= \frac{e^{-\frac{\theta}{\alpha}n\bar{x}}}{\left(\alpha(\theta+1)\right)^n} \prod_{x=1}^{k} \left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]^{f_x}. \tag{3.6.2}$$

The log likelihood function is obtained as

$$logL = -\frac{\theta}{\alpha}n\bar{x} - nlog\alpha - nlog(\theta+1) + G. \tag{3.6.3}$$

where

$$G = \sum_{x=1}^{k} f_x log\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]$$

The derivative of log likelihood functions with respect to θ and α we have

$$\frac{\partial logL}{\partial\theta} = -\frac{1}{\alpha}n\bar{x} - \frac{n}{(\theta+1)} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\partial\theta}}{\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]} = 0. \tag{3.6.4}$$

$$\frac{\partial logL}{\partial\alpha} = \frac{\theta}{\alpha^2}n\bar{x} - \frac{2n}{\alpha} + \sum_{x=1}^{k} f_x \frac{\frac{\partial l\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\partial\alpha}}{\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]} = 0. \tag{3.6.5}$$

The above two equations (3.6.4) and (3.6.5) cannot be solved directly. However the Fisher's scoring method can be applied to solve these equations. The following three equations may be obtained by

$$\frac{\partial^2 logL}{\partial\theta^2} = -\frac{n}{(\theta+1)^2} + \frac{\partial}{\partial\theta}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\partial\theta}}{\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}. \tag{3.6.6}$$

$$\frac{\partial^2 logL}{\partial\theta\partial\alpha} = \frac{1}{\alpha^2}n\bar{x} + \frac{\partial}{\partial\alpha}\sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\partial\theta}}{\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}. \tag{3.6.7}$$

$$\frac{\partial^2 logL}{\partial\alpha^2} = -2\frac{\theta}{\alpha^3}n\bar{x} + \frac{n}{\alpha^2} + \sum_{x=1}^{k} f_x \frac{\frac{\partial\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}{\partial\alpha}}{\left[\{\alpha(\theta+1)+\theta x\}\left(1-e^{-\frac{\theta}{\alpha}}\right)-\theta e^{-\frac{\theta}{\alpha}}\right]}. \tag{3.6.8}$$

The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved

$$\begin{bmatrix} \frac{\partial^2 logL}{\partial\theta^2} & \frac{\partial^2 logL}{\partial\theta\partial\alpha} \\ \frac{\partial^2 logL}{\partial\alpha\partial\theta} & \frac{\partial^2 logL}{\partial\alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial logL}{\partial\theta} \\ \frac{\partial logL}{\partial\alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}, \quad (3.6.9)$$

where θ_0 and $\alpha_0$are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently close estimates of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

3.7 Goodness of Fit

Discrete Sushila distribution has been fitted to two sets of published data to which earlier the Poisson Lindley distribution with parameter θ presented by Sankaran [40] and two parameter Poisson-Janardan distribution with parameters $\theta \; and \; \alpha$ presented by Shanker et al. [45] have been fitted. The fitting of the distribution have been presented in the following tables.

Table 3.1 : Distribution of mistakes in copying groups of random digits.

No. of errors per group	Observed frequencies	Expected frequencies		
		Poisson-Lindley (θ)	Poisson-Lindley (α, θ)	DS distribution with parameter (α, θ)
0	35	33.1	32.4	32.91
1	11	15.3	15.8	14.98
2	8	6.8	7.0	6.91
3	4	2.9	2.9	3.53
4	2	1.2	1.9	1.67
Total	60 $d.f. = 2$	60.0 $\hat{\theta}$=1.7434 $\chi^2 = 2.20$ $d.f. = 2$ $p = 0.3499$	60.0 $\hat{\alpha} = 2.6120$ $\hat{\theta} = 5.223371$ $\chi^2 = 2.10$ $d.f. = 2$ $p = 0.3947$	60.0 $\hat{\alpha} = 98.0$ $\hat{\theta} = 74.0$ $\chi^2 = 1.49$ $d.f. = 2$ $p = 0.4747$

Table 3.2: Distribution of Pyrausta nublilalis in 1937.

No. of accidents	Observed frequencies	Expected frequencies		
		Poisson-Lindley (θ)	Poisson- Lindley (α,θ)	DS distribution with parameter (α,θ)
0	33	31.5	31.9	31.89
1	12	14.2	13.8	13.95
2	6	6.1	5.9	6.21
3	3	2.5	2.5	2.88
4	1	1.0	1.1	1.05
≥5	1	0.7	0.8	0.02
	56	56.0 $\hat{\theta}$=1.8081 $\chi^2 = 0.53$ $p = 0.8479$	56.0 $\hat{\alpha} = 0.2573$ $\hat{\theta} = 0.39249$ $d.f. = 2$ $\chi^2 = 0.50$ $p = 0.8579$	56 $\hat{\alpha} = 101.0$ $\hat{\theta} = 82.0$ $d.f. = 2$ $\chi^2 = 0.44$ $p = 0.86072$

3.8 Discussion

In this investigation, a two parameter continuous Sushila distribution proposed by Shanker et al. [47] has been discretized. The derived discrete distribution may be called 'discrete Sushila' distribution. Several properties of the distribution such as recurrence relations for probabilities moments have been investigated. Size biased and Zero-truncated forms of the distribution have been discussed. Finally, an application of the proposed distribution has been shown by fitting the distribution. Two sets of published data have been considered. The first set of data represents the mistakes in copying groups of random digits. The second set of data is regarding the distribution of Pyraust anublilalis in 1937. This proposed model seems to be simple and suitable for modelling different types of count data and thus provides a better alternative to discrete Poisson Lindley distribution proposed by Sankaran [40] and two parameter Poisson- Lindley distribution proposed by Shanker et al. [45].