Chapter 4

Topology generated by fuzzy normed linear spaces

4.1 Introduction

Fuzzy set theory has been systematically applied to generalize concepts in topology and functional analysis. In this Chapter, applying the method of neighborhood we study some properties of the topology induced by a fuzzy norm according to the change of right norm. We develop Schauder basis with the help of summation of infinite series in FNLSs. We also explore the space of convergent sequences in an FNLS to obtain results as regards to its completeness.

4.2 Topology induced by a fuzzy norm

Let $(X, \parallel . \parallel, L, R)$ be an FNLS. Let $\alpha \in (0, 1]$ and $\varepsilon > 0$. For each $x \in X$, let us define the (ε, α)-neighborhood of x as the set

$$N_x(\varepsilon, \alpha) = \{y \in X : \parallel x - y \parallel_\alpha^2 < \varepsilon\}.$$

The (ε, α)-neighborhood base of x is the collection

$$\mathcal{N}_x = \{N_x(\varepsilon, \alpha) : \varepsilon > 0, \alpha \in (0, 1]\}$$

Şencimen and Pehlivan [63] showed that the topology $\tau = \bigcup_{x \in X} \mathcal{N}_x$ induced by the fuzzy norm $\parallel . \parallel$ is a Hausdorff and first countable topology for X.

We prove the following result:

Theorem 4.2.1. *The induced topology* τ *is a vector topology for the FNLS* X *with* $\lim_{a \to 0^+} R(a, a) = 0$*, i.e., the vector space operations are continuous in the topology* τ.

Proof. As the family $\{N_x(\varepsilon, \alpha) : \varepsilon, \alpha \text{ are rational numbers}\}$ is a countable (ε, α)-neighborhood base for each $x \in X$, therefore τ is a first countable Hausdorff topology on X. Thus it is sufficient to show that the vector space operations are sequentially continuous, i.e., fuzzy norm continuous in τ.

Consider two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$ in (X, τ). This gives $\lim_{n \to \infty} \| x_n - x \|_\alpha^2 = 0$ and $\lim_{n \to \infty} \| y_n - y \|_\alpha^2 = 0$ for each $\alpha \in (0, 1]$. As $\lim_{a \to 0^+} R(a, a) = 0$, there is $\beta \in (0, \alpha]$ such that

$$\| (x_n + y_n) - (x + y) \|_\alpha^2 \leq \| x_n - x \|_\beta^2 + \| y_n - y \|_\beta^2$$

As $n \to \infty$, $\| (x_n + y_n) - (x + y) \|_\alpha^2 \to 0$. Thus, $\lim_{n \to \infty} (x_n + y_n) = x + y$.
Further, let $\lim_{n \to \infty} \lambda_n = \lambda$ in $\mathbb{R}$ or $\mathbb{C}$, then

$$\begin{aligned} \| \lambda_n x_n - \lambda x \|_\alpha^2 &= \| \lambda_n x_n - \lambda x_n + \lambda x_n - \lambda x \|_\alpha^2 \\ &= \| x_n(\lambda_n - \lambda) + \lambda(x_n - x) \|_\alpha^2 \\ &\leq \| x_n(\lambda_n - \lambda)_\beta^2 + \| \lambda(x_n - x) \|_\beta^2 \\ &= \| x_n \|_\beta^2 | \lambda_n - \lambda | + | \lambda | \| x_n - x \|_\beta^2 \end{aligned}$$

As $n \to \infty$, $\| \lambda_n x_n - \lambda x \|_\alpha^2 \to 0$. Therefore, $\lim_{n \to \infty} \lambda_n x_n = \lambda x$. This completes the proof. □

Remark 4.2.2. Clearly $x + N(\varepsilon, \alpha) = N_x(\varepsilon, \alpha)$ for any $\alpha \in (0, 1]$ and $\varepsilon > 0$. For:
$y \in x + N(\varepsilon, \alpha)$

$\Leftrightarrow y = x + z_\circ,$ for some $z_\circ \in N(\varepsilon, \alpha)$

$\Leftrightarrow y - x = z_\circ \in N(\varepsilon, \alpha)$

$\Leftrightarrow \| y - x \|_\alpha^2 \leq \varepsilon$

$\Leftrightarrow y \in N_x(\varepsilon, \alpha)$

Therefore, Definition 1.3.24 of closure point and interior point can also be given with the help of the set $N_x(\varepsilon, \alpha)$ in the following way:

Definition 4.2.3. *Let $(X, \| \,.\, \|, L, R)$ be an FNLS. Consider a set $A \subseteq X$. A point $x_\circ \in X$ is called a point of closure of A if $N_{x_\circ}(\alpha, \alpha) \cap A \neq \phi$ for every $\alpha \in (0, 1]$. The point $x_\circ$ is called an interior point of A if there exists $\varepsilon_\circ$ and $\alpha_\circ$ such that $N_{x_\circ}(\varepsilon_\circ, \alpha_\circ) \subseteq A$.*

Remark 4.2.4. If $A \subseteq X$ is fuzzy open then *Int* A=A. Therefore each point $x \in A$ is an interior point. Thus, by Definition 4.2.3 for each x in A there exists an (ε, α)-neighborhood $N_x(\varepsilon, \alpha)$ such that $N_x(\varepsilon, \alpha) \subseteq A$. Therefore, A is fuzzy open if and only if for each $x \in A$ there exists $N_x(\varepsilon, \alpha)$ such that $N_x(\varepsilon, \alpha) \subseteq A$.

Lemma 4.2.5. *Let $(X, \| \,.\, \|, L, R)$ be an FNLS with $\lim\limits_{a \to 0^+} R(a, a) = 0$. Then for $\varepsilon > 0$ and each $\alpha \in (0, 1]$, there exists $\beta \in (0, \alpha]$ such that $\overline{N_x(\frac{\varepsilon}{2}, \beta)} \subseteq N_x(\varepsilon, \alpha)$.*

Proof. As $\lim\limits_{a \to 0^+} R(a, a) = 0$, for $\alpha \in (0, 1]$, there exists $\beta \in (0, \alpha]$ such that

$$\| x + y \|_\alpha^2 \leq \| x \|_\beta^2 + \| y \|_\beta^2 \text{ for x, y} \in X$$

For $\varepsilon > 0$, let $y \in \overline{N_x(\frac{\varepsilon}{2}, \beta)}$. Therefore $N_x(\frac{\varepsilon}{2}, \beta) \cap N_y(\frac{\varepsilon}{2}, \beta) \neq \phi$. Thus there exists $z \in N_x(\frac{\varepsilon}{2}, \beta) \cap N_y(\frac{\varepsilon}{2}, \beta)$. This gives:

$$\| x - y \|_\alpha^2 \leq \| x - z \|_\beta^2 + \| z - y \|_\beta^2 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, $y \in N_x(\varepsilon, \alpha)$ and thus $\overline{N_x(\frac{\varepsilon}{2}, \beta)} \subseteq N_x(\varepsilon, \alpha)$. □

Theorem 4.2.6. *A subset A of an FNLS $(X, \| \,.\, \|, L, R)$ with $\lim\limits_{a \to 0^+} R(a, a) = 0$ is rare (or, nowhere dense) if and only if every nonempty fuzzy open set in X contains an open ball whose closure is disjoint from A*

Proof. Let A be a rare in X. Let B be a nonempty fuzzy open subset of X. Then $B \cap \overline{A} = \phi$. Let $x \in B$. Then, there exists $\varepsilon > 0$ and each $\alpha \in (0, 1]$ such that $N_x(\varepsilon, \alpha) \subset U$. By Lemma 4.2.5 there exists $\beta \in (0, \alpha]$ such that $\overline{N_x(\frac{\varepsilon}{2}, \beta)} \subseteq N_x(\varepsilon, \alpha)$. Thus $\overline{N_x(\frac{\varepsilon}{2}, \beta)} \subseteq B$ and $\overline{N_x(\frac{\varepsilon}{2}, \beta)} \cap A = \phi$.

Conversely, suppose A is not rare. Therefore $Int(\overline{A}) \neq \phi$, so there exists a nonempty fuzzy open set U such that $U \subset \overline{A}$. For any $x \in U$ and $\varepsilon > 0$ and each $\alpha \in (0, 1]$, let $N_x(\varepsilon, \alpha) \subset U$. Then $\overline{N_x(\varepsilon, \alpha)} \cap A \neq \phi$. This is a contradiction. Hence the proof follows. □

Theorem 4.2.7. *Let $(X, \| \cdot \|, L, R)$ be a separable FNLS with $\lim_{a \to 0^+} R(a, a) = 0$. Then every subspace of X is separable.*

Proof. Let Y be a subspace of the FNLS X. Let $A = \{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X, therefore, $\overline{A} = X$. For $x \in X$ and $\alpha \in (0, 1]$ we can find $k \in \mathbb{N}$ such that $\| x_n - x \|^2_{\frac{1}{k}} \leq \frac{1}{2k}$. Consider the set B of all such x_ns and denote its elements by x_{n_k}. Then B is countable. Now we show $Y \subset \overline{B}$. Let $y \in Y$. Since A is dense in X, there exists x_m such that $\| x_m - y \|^2_{\frac{1}{k}} \leq \frac{1}{2k}$. By the definition of B, there exists $x_{m_k} \in A$ such that $\| x_{m_k} - x_m \|^2_{\frac{1}{k}} \leq \frac{1}{2k}$. Now

$$
\begin{aligned}
\| x_{m_k} - y \|^2_\alpha &\leq \| x_{m_k} - y \|^2_{\frac{1}{k}} \\
&\leq \| x_{m_k} - x_m \|^2_{\frac{1}{k}} + \| x_m - y \|^2_{\frac{1}{k}} \\
&\leq \frac{1}{2k} + \frac{1}{2k} \leq \frac{1}{k}.
\end{aligned}
$$

Thus, $x_{m_k} \in N_x(\frac{1}{k}, \alpha)$. Therefore $y \in \overline{B}$ and hence Y is separable. □

Definition 4.2.8. *Let $(X, \| \cdot \|, L, R)$ be an FNLS with $\lim_{a \to 0^+} R(a, a) = 0$ and $\{x_n\} \in X$. A point $x \in X$ is said to be cluster point of $\{x_n\}$ if every (ε, α)-neighborhood of x contains infinitely many points of $\{x_n\}$.*

In other words, $\{x_n\}$ has a subsequence that converges to x.

Lemma 4.2.9. *Let $(X, \| \, . \, \|, L, R)$ be an FNLS with $\lim\limits_{a \to 0^+} R(a, a) = 0$. If a Cauchy sequence $\{x_n\}$ has a cluster point $x \in X$ then $\{x_n\}$ converges to x.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X and $x \in X$ be a cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $\lim\limits_{n\to\infty} x_m = x$. Thus

$$\lim_{m\to\infty} \| \, x_m - x \, \|_\alpha^2 = 0, \text{ for any } \alpha \in (0,1] \tag{4.2.1}$$

Since $\{x_n\}$ is Cauchy, therefore,

$$\lim_{n,m\to\infty} \| \, x_n - x_m \, \|_\alpha^2 = 0 \tag{4.2.2}$$

For each $\alpha \in (0,1]$, we can find $\beta \in (0, \alpha]$ such that

$$\| \, x_n - x \, \|_\alpha^2 \leq \| \, x_n - x_m \, \|_\beta^2 + \| \, x_m - x \, \|_\beta^2 \tag{4.2.3}$$

As $n \to \infty$, $\| \, x_n - x \, \|_\alpha^2 \to \infty$ for any $\alpha \in (0,1]$. Hence, the Cauchy sequence $\{x_n\}$ converges to x. This completes the proof. □

4.3 Schauder basis in an FNLS

In this section, we shall introduce the notion of Schauder basis in FNLS.

4.3.1 Summable family in FNLSs

As mentioned in Chapter 1, Felbin introduced the convergence of a sequence and Cauchy sequence in FNLSs. In the classical case, one of the important notions of convergence is given in terms of a summable family. We introduce the same notion in FNLSs.

Definition 4.3.1. *Let $(X, \| \, . \, \|, L, R)$ be an FNLS with $\lim\limits_{a \to 0^+} R(a, a) = 0$. Let $\{x_n\} \subseteq X$. Associate the sequence $\{s_n\}$ with $\{x_n\}$ where*

$$s_n = x_1 + x_2 + x_3 + ... + x_n$$

$\{s_n\}$ is called the sequence of partial sums of $\{x_n\}$. If the sequence $\{s_n\}$ is convergent, say,

$$\lim_{n\to\infty} s_n = s, \text{ that is, } \lim_{n\to\infty} \| s_n - s \|_\alpha^2 = 0$$

for any $\alpha \in (0,1]$, then the infinite series $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \ldots$ is said to be fuzzy convergent and s is called the sum of the series.

We write:

$$\lim_{n\to\infty} \| \sum_{k=1}^{n} x_k - s \|_\alpha^2 = 0, \text{ or, } \sum_{n=1}^{\infty} x_n = s$$

Thus, for any $\varepsilon > 0$ and $\alpha \in (0,1]$ there exists $n_\circ = n_\circ(\varepsilon, \alpha)$ such that

$$\| \sum_{k=1}^{n} x_k - s \|_\alpha^2 < \varepsilon, \forall n \geq n_\circ$$

Definition 4.3.2. *An infinite series $\sum_{n=1}^{\infty} x_n$ is said to be absolutely fuzzy convergent if the series $\sum_{n=1}^{\infty} \| x_n \|_\alpha^2 = \| x_1 \|_\alpha^2 + \| x_2 \|_\alpha^2 + \| x_3 \|_\alpha^2 + \ldots$ is convergent for any $\alpha \in (0,1]$.* In fact, if $\sum_{n=1}^{\infty} x_n$ is absolutely fuzzy convergent, then $\sum_{n=1}^{\infty} \| x_n \|_\alpha^2 < \infty$ for any $\alpha \in (0,1]$

Theorem 4.3.3. *Let $(X, \| \,.\, \|, L, R)$ be an FNLS with $R \leq \max$. Then X is complete if and only if every absolutely fuzzy convergent series is fuzzy convergent.*

Proof. Let X be complete. Let $\sum_{n=1}^{\infty} x_n$ be absolutely fuzzy convergent. Consider the sequence $\{s_n\}$, where $s_n = \sum_{k=1}^{n} x_k$, of partial sums of $\sum_{n=1}^{\infty} x_n$. Since $\sum_{n=1}^{\infty} x_n$ is absolutely fuzzy convergent, therefore $\sum_{n=1}^{\infty} \| x_n \|_\alpha^2 < \infty$. Thus there exists $n_\circ \in \mathbb{N}$ such that

$$\sum_{k=1}^{n} \| x_n \|_\alpha^2 < \varepsilon, \quad \forall n \geq n_\circ$$

This gives $\| s_n - s_m \|_\alpha^2 = \| \sum_{k=m+1}^{n} \| x_k \|_\alpha^2$ for all $n > m \geq n_\circ$. As $R \leq$ max, using Lemma 1.3.19, we have:

$$\| s_n - s_m \|_\alpha^2 = \| \sum_{k=m+1}^{n} \| x_k \|_\alpha^2 \leq \sum_{k=m+1}^{n} \| x_k \|_\alpha^2 < \varepsilon, \qquad n > m \geq n_\circ$$

Hence $\{s_n\}$ is a Cauchy sequence. Since X is complete, therefore, $\{s_n\}$ converges and hence $\sum_{n=1}^{\infty} x_n$ is fuzzy convergent.

Conversely, let $\{x_n\}$ be a Cauchy sequence. For each $k \in \mathbb{N}$, we can choose n_k such that for $\alpha \in (0, 1]$

$$\| x_m - x_n \|_\alpha^2 \leq \frac{1}{2^k}, \qquad \forall m, n \geq n_k$$

In particular,

$$\| x_{n_{k+1}} - x_{n_k} \|_\alpha^2 \leq \frac{1}{2^k}.$$

Write $y_1 = x_{n_1}$, $y_2 = x_{n_2} - x_{n_1}$, . . ., $y_{k+1} = x_{n_{k+1}} - x_{n_k}$. It gives

$$\begin{aligned}\sum_{n=1}^{\infty} \| y_n \|_\alpha^2 &= \| x_{n_1} \|_\alpha^2 + \sum_{k=1}^{\infty} \| x_{n_{k+1}} - x_{n_k} \|_\alpha^2 \\ &\leq \| x_{n_1} \|_\alpha^2 + \sum_{k=1}^{\infty} \frac{1}{2^k} \\ &= \| x_{n_1} \|_\alpha^2 + 1\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} y_n$ is absolutely fuzzy convergent and thus $\sum_{n=1}^{\infty} y_n$ is fuzzy convergent. Therefore the subsequence $\{x_{n_k}\}$ is convergent and by Lemma 4.2.9 the Cauchy sequence $\{x_n\}$ is convergent. Hence X is complete. □

4.3.2 Schauder basis

The concept of fuzzy convergence of a series can be used to define Schauder basis of an FNLS. It is well known that Schauder bases are important in the structural investigation of Banach spaces of infinite dimensions. Şencimen and Pehlivan [63] defined convergence of a sequence in a different way which was referred to as "strong convergence".

Definition 4.3.4. [63]*Let $(X, \| \,.\, \|, L, R)$ be an FNLS. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ with respect to the fuzzy norm on X, denoted by $x_n \overset{FN}{\longrightarrow} x$, provided $\lim_{n\to\infty} \| x_n - x \| = \bar{0}$. That is, for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $\sup_{\alpha\in[0,1]} \| x_n - x \|_\alpha^2 = \| x_n - x \|_0^2 < \varepsilon$.*

In terms of (ε, α)-neighborhoods, $x_n \overset{\text{FN}}{\longrightarrow} x$, provided that for any $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $x_n \in N_x(\varepsilon, 0)$ for all $n \geq N$. Thus the convergence is uniform in α.

Remark 4.3.5. The concept of strong convergence was first introduced by Fang [18] in fuzzy metric space of Kaleva and Seikkala type. This definition is different from Felbin's notion of convergence (refer to Definition 1.3.21), where the convergence is in the sense of α-level sets called levelwise convergence.

In correspondence to strong convergence strong Cauchy sequence, strong closure of a set and strongly complete FNLS can be defined.
Clearly if a sequence $\{x_n\}$ is strongly convergent then $\{x_n\}$ is convergent to the same point, but not conversely.

Example 4.3.6. Let $X = \mathbb{R}$ and consider the fuzzy norm

$$\| x \| (t) = \begin{cases} \frac{|x|}{t+|x|}, & t > | x |, x \neq 0, \\ 1, & t = | x | = 0; \\ 0, & \text{otherwise} \end{cases}$$

It can be verified that $\| \,.\, \|$ is a fuzzy norm on $\mathbb{R}$ with $R = \max$ and $L = \min$. The α-level sets of $\| \,.\, \|$ are given by $[\| x \|]_\alpha = [| x |, \frac{1-\alpha}{\alpha} \mid x \mid]$. We show that the sequence $\{x_n\} = \{\frac{1}{n}\}$ is convergent, but not strongly convergent. In fact, we have $\| x_n \|_\alpha^2 = \frac{1-\alpha}{\alpha}\frac{1}{n}$. As $n \to \infty$, $x_n \to 0$. However this convergence is not strong convergence as for a given $\varepsilon > 0$,

$$\| x_n \|_\alpha^2 = \frac{1-\alpha}{\alpha} \mid x_n \mid < \varepsilon \Leftrightarrow \frac{1-\alpha}{\alpha\varepsilon} < n$$

Since $\frac{1-\alpha}{\alpha\varepsilon} \to \infty$ as $\alpha \to 0$, therefore we cannot find the required $N(\varepsilon) \in \mathbb{N}$.

Definition 4.3.7. *Let $\{x_n\}$ be a sequence in an FNLS $(X, \| \,.\, \|, L, R)$ with $R \leq$ max.*

(i) *$\{x_n\}$ is said to be a fuzzy Schauder basis, namely fuzzy basis of X if and only if for every $x \in X$, there is a unique sequence $\{a_n\}$ of scalars such that*

$$\sum_{k=1}^{n} a_k x_k \to x$$

This means, for each $\alpha \in (0, 1]$ and $\varepsilon > 0$ there exists $n_\circ = n_\circ(\alpha, \varepsilon)$ such that for $n \geq n_\circ$, we have

$$\| x - \sum_{k=1}^{n} a_k x_k \|_\alpha^2 < \varepsilon, \;\; i.e. \;\; \lim_{n\to\infty} \sum_{k=1}^{n} a_k x_k = x$$

In this case, we say that x has fuzzy expansion with respect to $\{x_n\}$ of the form $x = \sum_{k=1}^{\infty} a_k x_k$.

(ii) *$\{x_n\}$ is said to be a strong fuzzy Schauder basis, namely strong fuzzy basis of X if and only if for every $x \in X$, there is a unique sequence $\{a_n\}$ of scalars such that*

$$\sum_{k=1}^{n} a_k x_k \xrightarrow{FN} x$$

This means, for any $\varepsilon > 0$ there exists $n_\circ = n_\circ(\varepsilon)$ such that

$$\| x - \sum_{k=1}^{n} a_k x_k \|_0^2 < \varepsilon, \forall n \geq n_\circ \;\; i.e. \;\; \lim_{n\to\infty} \sum_{k=1}^{n} a_k x_k = x$$

In this case, x is said to have a strong fuzzy expansion with respect to $\{x_n\}$ of the form $x = \sum_{k=1}^{\infty} a_k x_k$.

It is clear that if $\{x_n\}$ is a strong fuzzy basis of X, then it is a fuzzy basis, but not conversely.

Example 4.3.8. Let $X = c_\circ$, the classical Banach space with the norm $\| x \|_\infty = \sup \| x_n \|$ where $x = \{x_n\}$.

Define a fuzzy norm $\| . \|$ on $c_\circ$ *as*

$$\| x \| (t) = \begin{cases} \frac{\|x\|_\infty}{t+\|x\|_\infty}, & t \geq \| x \|_\infty, x \neq 0, \\ 1, & t = \| x \|_\infty = 0; \\ 0, & \text{otherwise} \end{cases} \tag{4.3.1}$$

The α-level sets are $[\| x \|]_\alpha = [\| x \|_\infty, \frac{1-\alpha}{\alpha} \| x \|_\infty]$.

Then the sequence $e_1 = (1, 0, 0, 0, . \ , . \ , . \ )$, $e_2 = (0, 1, 0, 0, . \ , . \ , . \ ), . \ . \ .$ is a fuzzy basis for the FNLS $(c_\circ, \| . \|, L, R)$ as for each $\alpha \in (0, 1]$,

$$\lim_{n\to\infty} \| x - \sum_{k=1}^{n} a_k x_k \|_\alpha^2 = \frac{1-\alpha}{\alpha} \lim_{n\to\infty} \| x - \sum_{k=1}^{n} a_k x_k \|_\infty = 0,$$

However since $\frac{1-\alpha}{\alpha\varepsilon} \to \infty$ as $\alpha \to 0$, the convergence is not uniform in α. Thus no sequence in $(c_\circ, \| . \|, L, R)$ can be a strong fuzzy basis.

But, if we consider the fuzzy norm $\| . \|^\sim$ on $c_\circ$ as

$$\| x \|^\sim (t) = \begin{cases} 1, & t = \| x \|_\infty; \\ 0, & \text{otherwise} \end{cases}$$

Then $(c_\circ, \| . \|^\sim)$ is an FNLS with $L = \min$ and $R = \max$. Also, $\{e_n\}$ as defined above is a strong fuzzy basis as $\| x \|_\alpha^{\sim 1} = \| x \|_\alpha^{\sim 2} = \| x \|_\infty$ for each $\alpha \in (0, 1]$.

Remark 4.3.9. In case of finite dimensional FNLS, the definition of fuzzy basis is independent of the fuzzy norm and hence coincides with the classical definition of a basis (i.e. Hamel basis) in a vector space.

A classical normed linear space having a Schauder basis is separable. Let us now investigate the fuzzy analogue of this result.

Theorem 4.3.10. *Let $(X, \| . \|, L, R)$ be an FNLS with $R \leq \max$. If X has a fuzzy basis then X is separable.*

Proof. Let $\{x_n\}$ be a fuzzy basis of the FNLS X. Let us define a set $M \subset X$ as follows:

$$M = \{\sum_{k=1}^{n} b_k x_k : b_k \text{ is a (real or complex) rational number}\}$$

Clearly M is countable. We show M is dense in τ.

Let $x \in X$ be an arbitrary element. Hence, there exists a unique sequence $\{a_n\}$ of scalars such that $\sum_{n=1}^{\infty} a_n x_n = x$. By definition, for any $\alpha \in (0,1]$ and $\varepsilon > 0$, there exists $n_\circ \in \mathbb{N}$ such that

$$\| x - \sum_{k=1}^{n} a_k x_k \|_\alpha^2 < \varepsilon, \forall n \geq n_\circ$$

That is, for all $n \geq n_\circ$,

$$\sum_{k=1}^{n} a_k x_k \in N_x(\varepsilon, \alpha)$$

On the other hand, for each scalar a_k we can construct a sequence $\{b_{k_i}\}_{i=1}^{\infty}$ of rational scalars converging to a_k. Hence, by the continuity of vector space operations (Theorem 4.2.1), the sequence $\{\sum_{k=1}^{n} b_{k_i} x_k\}_{i=1}^{\infty}$ converges to $\sum_{k=1}^{n} a_k x_k$. Therefore, every (ε, α)-neighborhood of x in τ contains an element $\sum_{k=1}^{n} b_{k_i} x_k$ of the set M. Hence M is dense in X. □

4.4 Sequence spaces in FNLSs

In Example 4.3.8, it was mentioned that the classical Banach space $c_\circ$ can be made an FNLS with the fuzzy norm defined as (4.3.1). We now proceed to construct the space of convergent sequences in an FNLS.

In the sequel we consider right norm $R \leq \max$ and left norm $L \geq \min$, unless and otherwise specified. Following definitions are due to Felbin.

Definition 4.4.1. [20] *Two FNLSs $(X, \| \,.\, \|, L, R)$ and $(Y, \| \,.\, \|, L, R)$ are called isometric if there exists a one to one mapping ϕ from X onto Y such that for every $x, y \in X$,*

$$\| x - y \| = \| \phi(x) - \phi(y) \| .$$

The mapping ϕ is called an isometry from X onto Y. If ϕ is an isometry of X onto Y, ϕ^{-1} is an isometry of Y onto X.

Definition 4.4.2. [20] *An isometry ϕ is called a linear isometry of X onto Y if ϕ is a linear mapping of the linear space X onto the linear space Y.*

Definition 4.4.3. [20] *Two FNLSs $(X, \| \,.\, \|, L, R)$ and $(Y, \| \,.\, \|, L, R)$ are called congruent if there exists a linear isometry of $(X, \| \,.\, \|, L, R)$ onto $(Y, \| \,.\, \|, L, R)$*

Lemma 4.4.4. *If two FNLSs X and Y are isometric then completeness of X implies completeness of Y.*

Proof. Let ϕ be a linear isometry of a complete FNLS X onto an FNLS Y. We show that Y is also complete.

Let $\{y_n\}$ be a Cauchy sequence in Y. Since ϕ is onto, for each y_n there exists $x_n \in X$ such that $\phi(x_n) = y_n$, $\forall n \in \mathbb{N}$. Clearly $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, therefore, $\lim_{n\to\infty} x_n = x$ for $x \in X$. As ϕ is an isometry,

$$\| \phi(x_n) - \phi(x) \| = \| x_n - x \|, \forall n \in \mathbb{N}$$

Also, $\phi(x) = y(\text{say}) \in Y$. Therefore, for $\alpha \in (0, 1]$ and $n \in \mathbb{N}$, we get: $\| y_n - y \|_\alpha^2 = \| x_n - x \|_\alpha^2$. As $n \to \infty$, it gives $\| y_n - y \|_\alpha^2 \to 0$ and therefore, $\lim_{n\to\infty} y_n = y$. Hence Y is complete. □

Let us now consider the set of convergent sequences in X by partitioning the set. Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences in X. Let us call $\{x_n\}$ and $\{y_n\}$ to be equivalent, denoted by $\{x_n\} \sim \{y_n\}$, if and only if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, i.e., if and only if $\lim_{n\to\infty} \| x_n \|_\alpha^2 = \lim_{n\to\infty} \| y_n \|_\alpha^2$ for each $\alpha \in (0, 1]$.

Let C denote the set of all convergent sequences in X. Then, clearly $\sim$ is an equivalence relation on C. Let $\mathcal{C}$ be the collection of all equivalent classes of C determined by $\sim$. We define addition and scalar multiplication on $\mathcal{C}$ as follows:

$$x^* + y^* = \{\{x_n + y_n\} : \{x_n\} \in x^* \text{ and } \{y_n\} \in y^*\}$$

and

$$rx^* = \{\{rx_n\} : r \in \mathbb{R} \text{ and } \{x_n\} \in x^*\}$$

for every x^* and $y^* \in \mathcal{C}$.

Lemma 4.4.5. *The set $\mathcal{C}$ is a linear space together with the operation of addition and scalar multiplication.*

Proof. Let x^* and $y^* \in \mathcal{C}$. Suppose $\{x_n\}$, $\{x'_n\} \in x^*$ and $\{y_n\}$, $\{y'_n\} \in y^*$. Then, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x'_n$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} y'_n$. For any $\alpha \in (0, 1]$, we have

$$\begin{aligned} | \| x_n + y_n \|_\alpha^2 - \| x'_n + y'_n \|_\alpha^2 | &\leq \| x_n + y_n \|_\alpha^2 + \| x'_n + y'_n \|_\alpha^2 \\ &\leq \| x_n \|_\alpha^2 + \| y_n \|_\alpha^2 + \| x'_n \|_\alpha^2 + \| y'_n \|_\alpha^2 \end{aligned}$$

As $n \to \infty$, $| \| x_n + y_n \|_\alpha^2 - \| x'_n + y'_n \|_\alpha^2 | \to 0$.

Then, $\lim_{n\to\infty} \| x_n + y_n \|_\alpha^2 = \lim_{n\to\infty} \| x'_n + y'_n \|_\alpha^2$. Similarly, we get

$$\lim_{n\to\infty} \| x_n + y_n \|_\alpha^1 = \lim_{n\to\infty} \| x'_n + y'_n \|_\alpha^1, \text{ for any } \alpha \in (0, 1]$$

Therefore, $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty}(x'_n + y'_n)$, so $\{x_n + y_n\} \sim \{x'_n + y'_n\}$. As each of the sequences $\{x_n + y_n\}$ and $\{x'_n + y'_n\}$ is convergent, therefore $x^* + y^*$ is the equivalence class containing sequences of the form $\{x_n + y_n\}$, where $\{x_n\} \in x^*$ and $\{y_n\} \in y^*$. Thus, $x^* + y^* \in \mathcal{C}$ for every x^* and $y^* \in \mathcal{C}$.

Again, for $r \in \mathbb{R}$ and $\{x_n\}$, $\{x'_n\} \in x^*$,

$$\lim_{n\to\infty} \| rx_n \|_\alpha^2 = \lim_{n\to\infty} | r | \| x_n \|_\alpha^2 = \lim_{n\to\infty} | r | \| x'_n \|_\alpha^2 = \lim_{n\to\infty} \| rx'_n \|_\alpha^2$$

Similarly, $\lim_{n\to\infty} \| rx_n \|_\alpha^1 = \lim_{n\to\infty} \| rx'_n \|_\alpha^1$.

Thus, $\lim_{n\to\infty} rx_n = \lim_{n\to\infty} rx'_n$.

Therefore, the class rx^* is well defined and it contains sequences of the form $\{rx_n\}$, where $\{x_n\} \in x^*$. Hence, $rx^* \in \mathcal{C}$ for $r \in \mathbb{R}$ and $x^* \in \mathcal{C}$.

Therefore, the set $\mathcal{C}$ together with the operations of addition and scalar multiplication is a linear space. □

Define the zero element θ^* of $\mathcal{C}$ as a equivalence class containing the sequence $\{\theta, \theta, \theta, ., ., .\}$. In other words, $\lim_{n\to\infty} x_n = \theta$ for $\{x_n\} \in \theta^*$. Now we define a fuzzy norm $\| \,.\, \|^*$ on $\mathcal{C}$.

Consider $x^* \in \mathcal{C}$ and $\{x_n\} \in x^*$. Since every sequence $\{x_n\}$ in the equivalent class x^* converges to the same limit, define $\| x^* \|^*$ as:

$$\| x^* \|^* = \| \lim_{n\to\infty} x_n \|$$

The norm so defined is independent of the choice of $\{x_n\}$.

We can write: $[\| x^* \|^*]_\alpha = [\| x \|]_\alpha$, where $\lim_{n\to\infty} x_n = x$ for $x(\neq \theta) \in X$ and $\alpha \in (0, 1]$. To show $\| x^* \|^*$ is a fuzzy norm:

For $x^* = \theta^* \Rightarrow \{x_n\} \sim \{\theta\} \Rightarrow \lim_{n\to\infty} \| x_n \|_\alpha^2 = 0 = \lim_{n\to\infty} \| x_n \|_\alpha^1$ for any $\alpha \in (0, 1]$. It gives $[\| x^* \|^*]_\alpha = \{0\} \Rightarrow \| x^* \|^* = \overline{0}$. Conversely, $\| x^* \|^* = \overline{0} \Rightarrow [\| x^* \|]_\alpha = \{0\}$ for all $\alpha \in (0, 1] \Rightarrow \lim_{n\to\infty} \| x_n \|_\alpha^2 = 0 = \lim_{n\to\infty} \| x_n \|_\alpha^1 \Rightarrow \{x_n\} \sim \{\theta\}$, i.e., $x^* = \theta^*$. It can be easily shown that $\| rx^* \|^* = | r | \| x^* \|^*$.

Next for $\alpha \in (0, 1]$ and $\{x_n\} \in x^*$ and $\{y_n\} \in y^*$,

$\| x_n + y_n \|_\alpha^2 \leq \| x_n \|_\alpha^2 + \| y_n \|_\alpha^2$ and $\| x_n + y_n \|_\alpha^1 \leq \| x_n \|_\alpha^1 + \| y_n \|_\alpha^1$. Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \| x_n + y_n \|_\alpha^2 \leq \lim_{n\to\infty} \| x_n \|_\alpha^2 + \lim_{n\to\infty} \| y_n \|_\alpha^2$$

and

$$\lim_{n\to\infty} \| x_n + y_n \|_\alpha^1 \leq \lim_{n\to\infty} \| x_n \|_\alpha^1 + \lim_{n\to\infty} \| y_n \|_\alpha^1$$

It gives $\| x^* + y^* \|^* \preceq \| x^* \|^* \oplus \| y^* \|^*$. Thus $\| \,.\, \|^*$ is a fuzzy norm on $\mathcal{C}$ and this leads to the following result:

Theorem 4.4.6. *The quadruple* $(\mathcal{C}, \| \,.\, \|^*, \min, \max)$ *is an FNLS with the fuzzy norm* $\| \,.\, \|^*$ *defined by*

$$\| x^* \|^* = \| x \|$$

where $\{x_n\} \in x^*$ *and* $\lim_{n\to\infty} x_n = x$.

Theorem 4.4.7. *If* $(X, \|\ .\ \|, L, R)$ *is a complete FNLS, then* $(\mathcal{C}, \|\ .\ \|^*, \min, \max)$ *is also a complete FNLS.*

Proof. Let us define a mapping $\phi : X \to \mathcal{C}$ by setting $\phi(x)$ as the equivalence class x^* containing the sequence $\{x, x, ...\}$. It is easy to show that ϕ is a one-to-one mapping from X onto $\mathcal{C}$. Also,

$$\begin{aligned} \| \phi(x) - \phi(y) \|^* &= \| x^* - y^* \|^* \\ &= \| \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n \| \\ &= \| x - y \| \end{aligned}$$

Thus, ϕ is an isometry of X onto $\mathcal{C}$. Since $(X, \|\ .\ \|, L, R)$ is complete, by Lemma 4.4.4, $(\mathcal{C}, \|\ .\ \|^*)$ is a complete FNLS. □