Chapter 5

$\mathscr{L}$-fuzzy metric spaces

5.1 Introduction

The application of the fuzzy metric spaces is widespread. Various authors generalized the classical concepts of topology and functional analysis to fuzzy metric spaces. The $\mathscr{L}$-FMSs provided more general framework in this context for generalizing the classical concepts to fuzzy setting. In this chapter we study important topological concepts such as metrizability, precompactness, separability in $\mathscr{L}$-FMSs. We also prove a generalized form of the Lebesgue's covering lemma. It may be noted that the choice of triangular norm $\mathcal{T}$ is significant while discussing the results. The results of Section 5.2 and 5.3 are inspired in [30].

5.2 Metrizability of $\mathscr{L}$-fuzzy metric

Definition 5.2.1. *A topological space (X, τ) is said to admit a compatible $\mathscr{L}$-fuzzy metric if there exists an $\mathscr{L}$-fuzzy metric $\mathcal{M}$ such that $\tau = \tau_{\mathscr{L}}$. In such a case it is called $\mathscr{L}$-fuzzy metrizable.*

Remark 5.2.2. Let (X, d) be a metric space and $(X, \mathcal{M}_d, \mathcal{T})$ be the induced $\mathscr{L}$-FMS. Consider an open set A in (X, d). Then for each $x \in A$, there exists an open

[3] The contents of this chapter have appeared in the form of an article in *Annals of Fuzzy Mathematics and Informatics* (2014).

ball $B(x,r)$ such that $B(x,r) \subset A$, for some $r \in \mathbb{R} - \{0\}$.

Then,

$$
\begin{aligned}
& d(x,y) \leq r, \ \textit{for some } y \in A \\
\Leftrightarrow \quad & t + d(x,y) \leq t + r, \ \textit{for } t > 0 \\
\Leftrightarrow \quad & \frac{t}{t + d(x,y)} \geq \frac{t}{t+r} \\
\Leftrightarrow \quad & \mathcal{M}(x,y,t_\circ) \geq 1 - r = \mathcal{N}(r), \ \textit{where } t_\circ = 1 - r
\end{aligned}
$$

Therefore, A is open in the $\mathscr{L}$-FMS $(X, \mathcal{M}_d, \mathcal{T})$. Hence the topology generated by d coincides with the topology $\tau_{\mathcal{M}_d}$ generated by the induced $\mathscr{L}$-fuzzy metric $\mathcal{M}_d$.

The following result allows us to connect an $\mathscr{L}$-fuzzy metric to a topological space.

Theorem 5.2.3. *Every metrizable topological space admits a compatible $\mathscr{L}$-fuzzy metric.*

Proof. Suppose (X,τ) is a metrizable topological space. Let d be the metric on X compatible with τ. Since the topologies induced by the $\mathscr{L}$-fuzzy metric $\mathcal{M}_d$ and d are the same (Remark 5.2.2), $\mathcal{M}_d$ is compatible with τ . □

A classical result in the theory of metrizable topological spaces is the Kelley metrization lemma, which is stated as follows.

Lemma 5.2.4. [40] *A T_1 topological space (X,τ) is metrizable if and only if it admits a uniformity with a countable base.*

Theorem 5.2.5. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathscr{L}$-FMS. Then $(X, \tau_{\mathcal{M}})$ is a metrizable topological space.*

Proof. For each $n \in \mathbb{N}$, we define $U_n = \{(x,y) \in X \times X : \mathcal{M}(x,y,\frac{1}{n}) >_L \mathcal{N}(\frac{1}{n}))\}$. We shall prove that $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathscr{U}$ on X whose induced topology coincides with $\tau_{\mathcal{M}}$. For each $n \in \mathbb{N}$, we have:

$$\{(x,x) : x \in X\} \subseteq U_n, U_{n+1} \subseteq U_n \text{ and } U_n = U_n^{-1}$$

By the continuity of $\mathcal{T}$ for each $n \in \mathbb{N}$, $\exists$ an $m \in \mathbb{N}$ such that for $m > 2n$. Let (x, y) and $(y, z) \in U_m$. Since $\mathcal{M}(x, y, .)$ is nondecreasing, we get $\mathcal{M}(x, z, \frac{1}{n}) >_L \mathcal{M}(x, z, \frac{2}{m})$. Thus,

$$\begin{aligned}\mathcal{M}(x, z, \frac{1}{n}) &>_L \mathcal{M}(x, z, \frac{2}{m}) \\ &>_L \mathcal{T}(\mathcal{M}(x, y, \frac{1}{m}), \mathcal{M}(y, z, \frac{1}{m})) \\ &>_L \mathcal{T}(\mathcal{N}(\frac{1}{m}), \mathcal{N}(\frac{1}{m})) >_L \mathcal{N}(\frac{1}{n})\end{aligned}$$

Therefore $(x, z) \in U_n$. Further,

$$\mathcal{T}(\mathcal{N}(\frac{1}{m}), \mathcal{N}(\frac{1}{m})) >_L \mathcal{N}(\frac{1}{n})$$

Hence, $U_m \circ U_m \subseteq U_n$. therefore, $\{U_n : n \in \mathbb{N}\}$ is a countable base for a uniformity U on X. Since for each $x \in X$ and each $n \in \mathbb{N}$,

$$U_n(x) = \{y : \mathcal{M}(x, y, \frac{1}{n}) >_L \mathcal{N}(\frac{1}{n})\} = B(x, \frac{1}{n}, \frac{1}{n})$$

Therefore the topology induced by $\mathscr{U}$ coincides with $\tau_{\mathcal{M}}$. Hence, by Lemma 5.2.4 $(X, \tau_{\mathcal{M}})$ is a metrizable topological space. □

Corollary 5.2.6. *A topological space is metrizable if and only if it admits a compatible $\mathscr{L}$-fuzzy metric.*

Proof. The proof follows from Theorems 5.2.3 and 5.2.5. □

If X is a separable $\mathscr{L}$-FMS, then $(X, \tau_{\mathcal{M}})$ is a separable metrizable space. Therefore X is second countable [15]. This is in accordance with the following result stated by Efe.

Theorem 5.2.7. [14] *Every separable $\mathscr{L}$-FMS is second countable.*

Efe [15] called a metrizable topological space (X, τ) to be completely metrizable if it admits a complete metric. We prove the following result.

Theorem 5.2.8. *Let $(X, \mathcal{M}, \mathcal{T})$ be a complete $\mathscr{L}$-FMS. Then $(X, \tau_{\mathcal{M}})$ is completely metrizable.*

Proof. For every $n \in \mathbb{N}$, let us define $U_n = \{(x, y) : \mathcal{M}(x, y, \frac{1}{n}) >_L \mathcal{N}(\frac{1}{n})\}$. From Theorem 5.2.5, it follows that $\{U_n : n \in \mathbb{N}\}$ is a base for the uniformity $\mathscr{U}$ in X compatible with $\tau_{\mathcal{M}}$. Then, there exists a metric d with its induced uniformity coinciding with $\mathscr{U}$. We shall show that d is complete.

Let us consider a Cauchy sequence $\{x_n\}$ in (X, d). We show $\{x_n)\}$ is a Cauchy sequence in $(X, \mathcal{M}, \mathcal{T})$. Fix r, t, with $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$. We can find $k \in \mathbb{N}$ such that $\frac{1}{k} < t$ and $\frac{1}{k} <_L r$. Then there exists $n_\circ \in \mathbb{N}$ such that $(x_n, x_m) \in U_k$, for every $n, m \geq n_\circ$. Consequently, for every $n, m \geq n_\circ$,

$$\mathcal{M}(x_n, x_m, t) \geq_L \mathcal{M}(x_n, x_m, \frac{1}{k}) >_L \mathcal{N}(\frac{1}{k}) >_L \mathcal{N}(r)$$

Thus, $\{x_n)\}$ is a Cauchy sequence in the complete $\mathscr{L}$-FMS $(X, \mathcal{M}, \mathcal{T})$. Therefore $\{x_n)\}$ is convergent with respect to $\tau_{\mathcal{M}}$. Hence d is a complete metric on X. Thus, $(X, \tau_{\mathcal{M}})$ is completely metrizable. □

5.3 Compactness of $\mathscr{L}$-fuzzy metric

In this section we study compactness of an $\mathscr{L}$-fuzzy metric.

Since metrizability is a hereditary property, in an $\mathscr{L}$-FNS $(X, \mathcal{M}, \mathcal{T})$ we can state the following results:

Theorem 5.3.1. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathscr{L}$-FMS and $A \subseteq X$. Then the following assertions are equivalent:*

(1) *A is compact.*

(2) *A is countably compact.*

(3) *A is sequentially compact.*

Proof. Since the $\mathscr{L}$-FMS X is metrizable, therefore, the result follows straightforward (Engelking [15], Theorems 4.1.13, 4.1.17 and 5.1.3). □

5.3.1 Precompact $\mathscr{L}$-FMS

Let us prove an equivalent result for precompactness of an $\mathscr{L}$-fuzzy metric using the notion of sequences.

Lemma 5.3.2. *An $\mathscr{L}$-FMS is precompact if and only if every sequence has a Cauchy subsequence.*

Proof. Let $(X, \mathcal{M}, \mathcal{T})$ be a precompact $\mathscr{L}$-FMS and let $\{x_n\}$ be a sequence in X. Since X is precompact, for each $m \in \mathbb{N}$ there is a finite set A_m of X such that

$$X = \bigcup_{a \in A_m} B(a, \frac{1}{m}, \frac{1}{m})$$

Particularly, for $m = 1$, there exists an $a_1 \in A_1$ and a subsequence $\{x_{n_1}\}$ of $\{x_n\}$ such that $x_{n_1} \in B(a_1, 1, 1)$. Similarly, there exists an $a_2 \in A_2$ and a subsequence $\{x_{n_2}\}$ of $\{x_{n_1}\}$ such that $x_{n_2} \in B(a_2, \frac{1}{2}, \frac{1}{2})$. Proceeding inductively, for $m \in \mathbb{N}$, $m > 1$, there is an $a_m \in A_m$ and a subsequence $\{x_{n_m}\}$ of $\{x_{n_{m-1}}\}$ such that $x_{n_m} \in B(a_m, \frac{1}{m}, \frac{1}{m})$. Now consider the subsequence $\{x_{n_n}\}$ of $\{x_n\}$. For $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$, there exists an $n_\circ \in \mathbb{N}$ such that

$$\mathcal{T}(\mathcal{N}(\frac{1}{n_\circ}), \mathcal{N}(\frac{1}{n_\circ})) >_L \mathcal{N}(r) \text{ and } \frac{2}{n_\circ} < t$$

Then, for every $k, m \geq n_\circ$, we have:

$$\begin{aligned} \mathcal{M}(x_{k(k)}, x_{m(m)}, t) &\geq_L \mathcal{M}(x_{k(k)}, x_{m(m)}, \frac{2}{n_\circ}) \\ &\geq_L \mathcal{T}(\mathcal{M}(x_{k(k), a_{n_\circ}, \frac{1}{n_\circ}}), \mathcal{M}(x_{m(m)}, a_{n_\circ}, \frac{1}{n_\circ})) \\ &>_L \mathcal{T}(\mathcal{N}(\frac{1}{n_\circ}), \mathcal{N}(\frac{1}{n_\circ})) >_L \mathcal{N}(r) \end{aligned}$$

Hence, $\{x_{n_n}\}$ is a Cauchy subsequence of $\{x_n\}$ in X.

Conversely, assume $(X, \mathcal{M}, \mathcal{T})$ is not a precompact $\mathscr{L}$-FMS. Then, there exists an $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$ such that for each finite subset A of X,

$$X \neq \bigcup_{a \in A} B(a, r, t)$$

Fix $x_1 \in X$. Then there exists $x_2 \in X \backslash B(x_1, r, t)$. Proceeding similarly, we find $x_3 \in X \backslash \bigcup_{k=1}^{2} B(x_k, r, t)$. Proceeding inductively we have a sequence $\{x_{n_n}\}$ of distinct points in X such that

$$x_{n+1} \notin \bigcup_{k=1}^{n} B(x_k, r, t), \text{ for } n \in \mathbb{N}$$

Therefore, $\{x_{n_n}\}$ has no Cauchy subsequence. □

Theorem 5.3.3. *An $\mathscr{L}$-FMS $(X, \mathcal{M}, \mathcal{T})$ is separable if and only if $(X, \tau_{\mathcal{M}})$ admits a precompact $\mathscr{L}$-fuzzy metric.*

Proof. Suppose $(X, \mathcal{M}, \mathcal{T})$ is a separable $\mathscr{L}$-FMS. Then $(X, \tau_{\mathcal{M}})$ is a separable metrizable space. Therefore, $\tau_{\mathcal{M}}$ admits a compatible precompact metric d (Theorem 4.3 in [15]). We show that the induced $\mathscr{L}$-fuzzy metric $\mathcal{M}_d$ is precompact.

Let $\{x_n\}$ be a sequence in X. By the precompactness of d, $\{x_n\}$ has a Cauchy subsequence $\{x_{n_k}\}$ in (X, d). Fix $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$. Let us choose ε such that $\frac{t}{t+\varepsilon} >_L \mathcal{N}(r)$. Then, there exists $n_\circ \in \mathbb{N}$ such that $d(x_{k(n)}, x_{k(m)}) < \varepsilon$, for $n, m \geq n_\circ$. Therefore,

$$\mathcal{M}_d(x_{k(n)}, x_{k(m)}, t) >_L \frac{t}{t+\varepsilon} >_L \mathcal{N}(r), \text{ for all } n, m \geq n_\circ$$

So, $\{x_n\}$ is a Cauchy sequence in the $\mathscr{L}$-FMS $(X, \mathcal{M}_d, \mathcal{T})$. By Lemma 5.3.2, $(X, \mathcal{M}_d, \mathcal{T})$ is precompact.

Conversely, suppose $(X, \tau_{\mathcal{M}})$ admits a compatible precompact $\mathscr{L}$-fuzzy metric $\mathcal{M}'$. For each $n \in \mathbb{N}$, thus, there exists a finite subset A_n of X such that $X = \bigcup_{a \in A_n} B(a, \frac{1}{n}, \frac{1}{n})$.

Put $A = \bigcup_{n=1}^{\infty} A_n$. Then, A is countable. For each $x \in X$, consider a basic neighborhood $B(x, \frac{1}{m}, \frac{1}{m})$ of x. Then, there exists a in A_m such that $x \in B(a, \frac{1}{m}, \frac{1}{m})$ and hence, A is dense in X. Therefore, $(X, \mathcal{M}', \mathcal{T})$ is separable, i.e., $(X, \tau_{\mathcal{M}})$ is separable. □

Corollary 5.3.4. *Every precompact $\mathscr{L}$-FMS is second countable.*

Proof. Proof follows from Theorems 5.2.7 and 5.3.3. □

Definition 5.3.5. *Let $(X, \mathcal{M}, \mathcal{T})$ be a $\mathscr{L}$-FMS and $\{x_n\} \subseteq X$. A point $x \in X$ is said to be a cluster point of $\{x_n\}$ if every neighbourhood of x contains infinitely many points of $\{x_n\}$.*

In other words, we can say that $\{x_n\}$ has a subsequence converging to x in $\tau_{\mathcal{M}}$.

Lemma 5.3.6. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathscr{L}$-FMS. If a Cauchy sequence clusters to a point $x \in X$, then the sequence converges to x.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $(X, \mathcal{M}, \mathcal{T})$ having a cluster point $x \in X$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{n_k \to \infty} x_{n_k} = x$ with respect to $\tau_{\mathcal{M}}$. This implies, given $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$, $\exists n_\circ \in \mathbb{N}$ such that:

$$\mathcal{M}(x, x_{n_k}), \frac{t}{2}) >_L \mathcal{N}(s), \text{ for each } n \geq n_\circ \tag{5.3.1}$$

where $s \in L \backslash \{0_{\mathscr{L}}\}$ satisfies $\mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r)$. Also, $\exists n_1 \geq k(n_\circ)$ such that:

$$\mathcal{M}(x_n, x_m, \frac{t}{2}) >_L \mathcal{N}(s), \text{ for each } n \geq n_1 \tag{5.3.2}$$

Therefore, from (5.3.1) and (5.3.2) we have for $n \geq \max\{n_\circ, n_1\}$,

$$\begin{aligned}\mathcal{M}(x_n, x, t) &\geq_L \mathcal{T}(\mathcal{M}(x_n, x_{n_k}, \frac{t}{2}), \mathcal{M}(x_{n_k}, x, \frac{t}{2})) \\ &\geq_L \mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r), \text{ for any } r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\} \text{ and } t > 0.\end{aligned}$$

Hence, the Cauchy sequence $\{x_n\}$ converges to x. This completes the proof. □

Theorem 5.3.7. *An $\mathscr{L}$-FMS $(X, \mathcal{M}, \mathcal{T})$ is compact if and only if it is precompact and complete.*

Proof. Suppose $(X, \mathcal{M}, \mathcal{T})$ is a compact $\mathscr{L}$-FMS. Then for each $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$, the open cover $\{B(x, r, t) : x \in X\}$ of X has a finite subcover. Hence, X is precompact. Also, using Theorem 5.3.1, each Cauchy sequence $\{x_n\}$ in X has a cluster point $x \in X$. By Lemma 5.3.6 $\{x_n\}$ converges to x. Therefore, $(X, \mathcal{M}, \mathcal{T})$ is complete.

Conversely, let X be precompact and complete. Consider a sequence $\{x_n\}$ in X. By Lemma 5.3.2, $\{x_n\}$ has a Cauchy subsequence, say $\{x_{n_k}\}$. As X is complete, the Cauchy subsequence $\{x_{n_k}\}$ is convergent and thus, $\{x_n\}$ has a cluster point. Therefore, X is sequentially compact. Since $(X, \tau_{\mathcal{M}})$ is metrizable (Theorem 5.2.5) and every sequentially compact metrizable space is compact (using Theorem 5.3.1), therefore, $(X, \mathcal{M}, \mathcal{T})$ is compact. □

Remark 5.3.8. Theorems 5.3.3 and 5.3.7 show that every compact $\mathscr{L}$-FMS is separable. Also, Theorem 5.3.1 gives that every sequentially compact $\mathscr{L}$-FMS is precompact.

5.3.2 $\mathscr{L}F$- strongly bounded set

Definition 5.3.9. [36] *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathscr{L}$-FMS and $A \subseteq X$. The $\mathscr{L}$-fuzzy diameter of a set A is defined by:*

$$\delta_A = \sup_{t>0} \inf_{x,y \in A} \sup_{\varepsilon < t} \mathcal{M}(x, y, \varepsilon)$$

If $\delta_A = 1_{\mathscr{L}}$ then A is said to be $\mathscr{L}F$-strongly bounded.

Lemma 5.3.10. [36] *The set $A \subseteq X$ is $\mathscr{L}F$-strongly bounded if an only if for arbitrary negation $\mathcal{N}(r)$ and each $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ there exists $t > 0$ such that $\mathcal{M}(x, y, t) >_L \mathcal{N}(r)$, for all $x, y \in A$.*

Theorem 5.3.11. [14] *Every compact subset A of an $\mathscr{L}$-FNS X is $\mathscr{L}F$-strongly bounded.*

Clearly every compact set $A \subseteq X$ is closed and $\mathscr{L}F$-strongly bounded.
We then prove the following result:

Theorem 5.3.12. *Every precompact subset A of an $\mathscr{L}$-FMS $(X, \mathcal{M}, \mathcal{T})$ is $\mathscr{L}F$-strongly bounded.*

Proof. Let A be a precompact subset of X. Fix $t > 0$ and $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$. As A is precompact, there exists a finite subset $S \subseteq X$ such that

$$A \subseteq \bigcup_{a \in S} B(a, r, t)$$

Let $x, y \in A$. Then $x \in B(x_i, r, t)$ and $y \in B(x_j, r, t)$, for some i, j. It gives:

$$\mathcal{M}(x, x_i, t) >_L \mathcal{N}(r) \text{ and } \mathcal{M}(y, x_j, t) >_L \mathcal{N}(r) \tag{5.3.3}$$

Let $\alpha = \min \mathcal{M}(x_i, x_j, t) : 1 \leq i, j \leq n$. Clearly $\alpha \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$. Also $\exists s \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ (Remark 1.3.37) such that:

$$\mathcal{T}^2(\mathcal{N}(r), \mathcal{N}(r), \alpha) >_L \mathcal{N}(s) \tag{5.3.4}$$

Thus, (5.3.3), (5.3.4) and (5.3.5) give:

$$\begin{aligned} \mathcal{M}(x, y, 3t) &\geq_L \mathcal{T}^2(\mathcal{M}(x, x_i, t), \mathcal{M}(x_i, x_j, t), \mathcal{M}(x_j, y, t)) \\ &\geq_L \mathcal{T}^2(\mathcal{N}(r), \mathcal{N}(r), \alpha) >_L \mathcal{N}(s) \end{aligned}$$

for any $x, y \in A$. Hence A is $\mathscr{L}F$-strongly bounded. □

Remark 5.3.13. Theorems 5.3.1-5.3.12 establish some relationships among the topological concepts of compactness, completeness, boundedness, pecompactness, separability. These properties are fundamental and provide a basic tool to study the theory of space structure of $\mathscr{L}$-FMS.

5.4 Covering factor of an $\mathscr{L}$-fuzzy metric

In this part, we introduce a new concept called "covering factor" in an $\mathscr{L}$-FMS.

Definition 5.4.1. *An element $\varepsilon \in L\backslash\{0_{\mathscr{L}}\}$ is called a covering factor for an open cover $\mathscr{G} = \{G_i\}_{i\in\bigwedge}$ of a $\mathscr{L}$-FMS $(X, \mathcal{M}, \mathcal{T})$ if for every set A in X with fuzzy diameter $\delta_A >_L \mathcal{N}(\varepsilon)$ is contained in any G_i in $\mathscr{G}$.*

Lemma 5.4.2. *Let $B(x, r, t)$ be an open ball of an $\mathscr{L}$-FMS $(X, \mathcal{M}, \mathcal{T})$ with $r \in L\backslash\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$. Let A be a subset of X such that $\delta_A >_L \mathcal{N}(s)$ where $s \in L\backslash\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ satisfying $\mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r)$. If A intersects $B(x, s, \frac{t}{2})$, then $A \subseteq B(x, r, t)$.*

Lemma 5.4.3. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathscr{L}$-FMS. If $t > 0$ and $r, s \in L\backslash\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ such that $\mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r)$, then $B(x, s, \frac{t}{2}) \subset B(x, r, t)$.*

The above two results are straight forward, so we omit the proofs here. The following result is a generalized form of the Lebesgue's covering lemma.

Theorem 5.4.4. *In a sequentially compact $\mathscr{L}$-FMS with involutive negation, every open cover has a covering factor.*

Proof. Let $(X, \mathcal{M}, \mathcal{T})$ be a sequentially compact $\mathscr{L}$-FMS with a involutive negation $\mathcal{N}$. Also, let $\mathscr{G} = \{G_\alpha : \alpha \in \bigwedge\}$ be an open cover of X. Assume that there exist sets in X which are not contained in any G_α. Otherwise any $\varepsilon \in L\backslash\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ will work as an covering factor and the result is established.

Let us refer to these sets as "big sets". Consider such a set $A \subseteq X$. Let $\delta' = \inf\{\mathcal{N}(\delta_A) : \delta_A \text{ is fuzzy diameter of the big set A }\}$. Then, the following cases arise:

Case I: Let $\delta' = 1_{\mathscr{L}}$. It gives: for $A \subset X$ with $\mathcal{N}(\delta_A) <_L 1_{\mathscr{L}}$ i.e. $\delta_A >_L 0_{\mathscr{L}}$ is a subset of G_α for some $\alpha \in \bigwedge$. Hence each $\delta \in L\backslash\{0_{\mathscr{L}}\}$ is a covering factor.

Case II: Let $\delta' = 0_{\mathscr{L}}$. By definition of δ', for a given $n \in \mathbb{N}$ there exists a big set

B_n such that $\mathcal{N}(\delta_{B_n}) <_L \frac{1}{n}$. It gives:

$$\delta_{B_n} >_L \mathcal{N}(\frac{1}{n})$$

As the big set B_n is not contained in any G_α, B_n must contain atleast two elements. Therefore, $1_{\mathscr{L}} >_L \delta_{B_n} >_L \mathcal{N}(\frac{1}{n})$.

Construct a sequence $\{x_n\}$ where $x_n \in B_n$, for $n \in \mathbb{N}$. Since X is sequentially compact, $\{x_n\}$ has a convergent subsequence converging to some $x \in X$. As $X = \bigcup_{\alpha\in\bigwedge} G_\alpha$, we have:

$$x \in G_\beta, \text{ for } \beta \in \bigwedge \tag{5.4.1}$$

As G_β is open in X, there exists $r \in L\backslash\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$ such that

$$B(x, r, t) \subseteq G_\beta, r \in L\backslash\{0_{\mathscr{L}}, 1_{\mathscr{L}}\} \text{ and } t > 0 \tag{5.4.2}$$

Choose $s \in L\backslash\{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ such that $\mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r)$. By Lemma 5.4.3, we get $B(x, s, \frac{t}{2}) \subset B(x, r, t)$. Since $\{x_n\}$ has a subsequence converging to x, $x_n \in B(x, s, \frac{t}{2})$ for infinitely many n. Thus, $\exists N$ such that $x_N \in B(x, s, \frac{t}{2})$. Therefore,

$$\frac{1}{N} <_L s \Rightarrow \mathcal{N}(\frac{1}{N}) >_L \mathcal{N}(s) \tag{5.4.3}$$

However, $x_N \in B_N$ and $\delta_{B_N} >_L \mathcal{N}(\frac{1}{N})$. This gives $\delta_{B_N} >_L \mathcal{N}(s)$. By (5.4.2), (5.4.3) and Lemma 5.4.2, $B_N \subseteq B(x, r, t) \subseteq G_\beta$, a contradiction to the assumption. Thus $\delta' \neq 0_{\mathscr{L}}$.

Therefore, $0_{\mathscr{L}} <_L \delta' <_L 1_{\mathscr{L}}$ and the element $\delta = \mathcal{N}(\delta')$ will be the required covering factor. This completes the proof. □

Thus, in view of Theorems 5.3.1, 5.3.7, 5.3.12 and 5.4.4 and Remark 5.3.8, various compactness criteria of an $\mathscr{L}$-FMS can be interpreted with the help of a covering factor.