Chapter 1

Introduction

The thesis comprises of five chapters including the introductory chapter. In this chapter, we have provided a general introduction of the field of study followed by a brief background of the development of the subject. An overview of the basic concepts, notions and notations, definitions and results that are relevant to the work in this thesis is also provided here. A chapter-wise organization for the rest of the thesis is provided at the end of the chapter.

1.1 General introduction

1.1.1 Sets to Fuzzy sets

The notion of a set is one of the most important one, used frequently in every day life, as well as in mathematics. In 1666, Leibnitz defined a notion of a set as "any number of things whatever may be taken simultaneously and yet treated as whole". Continued attempts were made to improve the theory of sets.

In crisp set theory, the membership of elements in a set is assessed in binary terms according to a bivalent conditions: an element either belongs (members) or does not belong (non-members) to the set. A sharp unambiguous distinction exists between the members and non-members of the set. However, in many real life problems one comes across sets that do not exhibit this characteristic. The world surrounding us is full of uncertainty, the information we obtain from the environment, the notions

we use and the data resulting from our observation or measurement are, in general, vague or imprecise. So every formal description of the real world or some of its aspects is only an approximation and an idealization of the actual state.
In order to deal with the same, a new theory, was introduced by Professor Lotfi A. Zadeh in 1965 whose objects are called fuzzy sets: sets with boundaries that are not precise.
As its name implies, the theory of fuzzy sets is basically a theory of graded concepts. The elements of a fuzzy set have the membership grade which is not a matter of affirmation or denial but rather a matter of degree to which that individual is similar or compatible with the concept represented by the fuzzy set.

1.1.2 Fuzzy sets to fuzzy mathematics

Zadeh introduced the theory of fuzzy sets with a view to reconcile mathematical modeling and human knowledge in various application sciences. A great amount of literature has appeared around the concept of fuzzy sets in a very wide spectrum of areas ranging from mathematics and logic to advanced engineering methodologies. Many applications can be found in various contexts, from medicine to finance, from human behavior analysis to consumer products, from machine control to computational linguistics. At the same time, fuzzy sets gained significance in the contemporary studies concerning the logical and set-theoretical foundations of mathematics. There has been extensive research to develop fuzzy analogues of the theories of category theory, topology, functional analysis, algebra, analysis, graph theory, theory of generalized measure and integrals etc.
General topology was one of the earliest branches of mathematics which applied fuzzy set theory systematically. The combination and synthesis of ideas, notions and methods of fuzzy set theory with general topology has resulted in "fuzzy topology" as a new branch of mathematics. It was in 1968 that C.L. Chang [6] grafted the notion of a fuzzy set into general topology and attempted to develop the basic topological notions for such spaces.

However due to the presence of stratum structure, it is often difficult to establish a theory in fuzzy topology or similar branches of fuzzy mathematics. One single notion or concept in classical setting usually leads to several counterparts in the fuzzy setting. One of such important problems in fuzzy topology which eluded mathematicians for some time is to obtain an appropriate concept of fuzzy metric and fuzzy norm.

1.1.3 Fuzzy metric and fuzzy norm

The theory of metric spaces and of normed linear spaces is of fundamental importance in mathematics, physics, computer science, statistics etc. Many problems can be solved by finding an appropriate metric or norm for making some measurements. But it is also a well-known fact that, in practice, assigning a fixed number to the distance between two points is not a precise idea. In many situations the average of several measurements or an interval is assigned to model the inexactness. To deal with uncertainties arising in such physical situations, the concept of fuzzy metric or fuzzy norm may be more suitable than the crisp concepts. It may be noted that even before the inception of fuzzy set theory, in 1942, M. Menger introduced the concept of a generalized metric space applying probability distribution function to the distance between two points. Following him, this space is called a probabilistic (generalized) Menger space. Later, it was proved that Menger spaces are some special cases of fuzzy normed linear spaces or fuzzy metric spaces. During the last few decades, the study of fuzzy metric spaces and of fuzzy normed linear spaces received a significant attention. The objective of these investigations is to study fuzzy analogues of fundamental concepts and properties of topology and functional analysis. At the same time one may recall the fascinating applications of the notions of fuzzy metric and fuzzy norm in quantum particle physics especially in respect of both string and e^{∞} theory and also in color image processing techniques [51, 52, 59]. This variety of the fields of application indicates the importance and usefulness of fuzzy norm and fuzzy metric theory.

1.2 Background

1.2.1 Fuzzy normed linear spaces

It boggled the minds of researchers as how to define fuzzy norm on a linear space. In 1984, A. K. Katsaras [39] first introduced the notion of fuzzy seminorm and fuzzy norm on a vector space. In 1992, C. Felbin [20] offered an alternative definition of a fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space, using the treatment of fuzzy metric by O. Kaleva and S. Seikkala [38]. Later, in 2002, J. Xiao and X. Zhu [65] modified this definition by restricting the definition of fuzzy real numbers to gave a concise as well as general definition of a fuzzy normed linear space. With this notion of fuzzy norm, fuzzy analogues of several important concepts of normed linear spaces have been established. Another development along this line of inquiry took place when in 1994, where S. C. Cheng and J. N. Mordeson [7] considered a fuzzy norm on a linear space whose associated metric is of O. Kramosil and J. Michalek type [41]. Following Cheng and Mordeson, in 2003, T. Bag and S. K. Samanta [3] introduced another notion of fuzzy norm using the min triangular norm. The novelty of this definition is the validity of a decomposition theorem for this type of fuzzy norm into a family of crisp norms and using this decomposition theorem it has been possible to establish many important results of fuzzy functional analysis. This fuzzy norm was further generalized by I. Golet [26] in 2007 considering a general t-norm replacing the particular choice of "min" t-norm. Another generalization has also been made by following the definition of $\mathscr{L}$-fuzzy set by J. Goguen [25] to give the notion of $\mathscr{L}$-fuzzy normed linear space. This generalized notion is given by G. Deschrijver et. al [12].

Many authors studied the notions and results of classical functional analysis in fuzzy normed linear spaces. During the four decades of existence of fuzzy normed linear spaces, some of the extensive study has been made in the following areas:

- ∘ various topological properties determined by a fuzzy norm. For instance, completion of fuzzy normed linear spaces, completeness of a fuzzy norm, separability, completion, convergence of sequences, bases neighborhood, approximation properties, topological degree theory [2, 18, 20, 21, 44, 46, 59, 63–65, 67, 68, 72].
- ∘ fixed point theory in fuzzy normed linear spaces along with their applications [8, 27, 49].
- ∘ Hyers-Ulam-Rassias stability of functional equations in fuzzy normed linear spaces [28, 45, 58, 61].
- ∘ linear operators in fuzzy normed linear spaces; boundedness, continuity with different measures. fundamental theorems of functional analysis: open mapping theorem, closed graph theorem, Hahn-Banach theorem, bounded inverse theorems etc. in fuzzy normed linear spaces [4, 7, 34, 35, 37, 60, 62, 68–70].
- ∘ nonlinear operators in fuzzy normed linear spaces [17, 71].
- ∘ fuzzy topology induced by fuzzy normed linear spaces [9, 48, 70].

1.2.2 Fuzzy metric spaces

The notion of fuzzy metric spaces appeared in the literature even before the notion of fuzzy normed linear spaces. In 1975, O. Kramosil and J. Michalek [41] introduced the notion of fuzzy metric by extending the concept of the probabilistic metric space to the fuzzy situation. Later, in 1994, A. George and P. Veeramani [23] modified this definition by imposing some stronger conditions on the fuzzy metric to gave a consistent definition of a fuzzy metric. They were able to obtain a Hausdorff topology on these spaces and studied various important topological properties of the fuzzy metric spaces. In 1984, O. Kaleva and S. Seikkala [38] generalized the notion of the metric space by setting the distance between two points to be a non-negative fuzzy real number. Using ordering and addition in the set of fuzzy numbers they have obtained a triangle inequality which is analogous to the ordinary triangle inequality.

Kaleva and Seikkala also proved that each Menger probabilistic metric space could be considered as a fuzzy metric space. Some other definitions of fuzzy metric were given by M. A. Erceg [16], Z. K. Deng [10], M. Grabiek [29]. In 2007, R. Saadati, A. Razani and H. Adibi [53] gave a generalized notion of a fuzzy metric spaces due to George and Veeramani using $\mathscr{L}$-fuzzy sets. In $\mathscr{L}$-fuzzy metric spaces, perhaps the most important generalization is the consideration of an order structure beyond the unit interval $[0, 1]$. This way, an $\mathscr{L}$-fuzzy metric space offers a more general framework to generalize the topological concepts in fuzzy setting than the fuzzy metric spaces.

The theory of $\mathscr{L}$-fuzzy metric space is of relatively recent origin than the theory of fuzzy metric spaces. It is observed that most of the work on $\mathscr{L}$-fuzzy metric spaces hovered around the study of fixed point theorem and its applications [1, 33, 36, 53, 54, 57, 58]. Few attempts have also been made to study topological properties of $\mathscr{L}$-fuzzy metric spaces [14, 18, 53, 56].

1.3 Preliminaries

In this section we provide the definitions of the fuzzy normed linear spaces and $\mathscr{L}$-fuzzy metric spaces. For brevity, we shall use FNLS to denote a fuzzy normed linear space and $\mathscr{L}$-FMS to denote an $\mathscr{L}$-fuzzy metric space.

In the following subsections, we have stated the definitions and results used in the thesis.

Throughout the thesis, we denote the set of all real numbers by $\mathbb{R}$ and set of all positive real numbers by $\mathbb{R}^+$.

1.3.1 Fuzzy normed linear spaces

The fuzzy norms given by Felbin [20] and Xiao and Zhu [65] are based on non-negative fuzzy real numbers. While there are several versions of fuzzy real numbers, we consider those in the sense of Xiao and Zhu, which are as follows:

Definition 1.3.1. [65] *A mapping* $\eta : \mathbb{R} \to [0, 1]$ *is called a fuzzy real number (fuzzy interval), whose* α*-level set is denoted by* $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$*, if it satisfies the following:*

(N1) *There exists* $t_\circ \in \mathbb{R}$ *such that* $\eta(t_\circ) = 1$.

(N2) *for each* $\alpha \in (0, 1]$; $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$, *where* $-\infty < \eta^1_\alpha \leq \eta^2_\alpha < +\infty$.

Remark 1.3.2. This definition differs from the notion of fuzzy real number used by Felbin [20], which permits the cases $\eta^1_\alpha = -\infty$ and $\eta^2_\alpha = +\infty$.

The set of all fuzzy real numbers is denoted by $\mathcal{F}$. To each $r \in \mathbb{R}$, we can consider $\bar{r} \in \mathcal{F}$ defined by $\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$. Thus the set of reals $\mathbb{R}$ can be embedded in $\mathcal{F}$.
Further, η is called convex if $\eta(t) \geq \min(\eta(s), \eta(r))$ where $s \leq t \leq r$.
If there exists a $t_\circ \in \mathbb{R}$ such that $\eta(t_\circ) = 1$, then η is called normal.

Lemma 1.3.3. [65] $\eta \in \mathcal{F}$ *if and only if* $\eta : \mathbb{R} \to [0, 1]$ *satisfies the following:*

(1) η *is normal, convex and upper semi-continuous.*

(2) $\lim_{t \to -\infty} \eta(t) = \lim_{t \to +\infty} \eta(t) = 0$.

As α-level sets of a convex fuzzy real number is an interval, Dubois and Prade [13] argued to call this as a *fuzzy interval*.
A partial ordering " $\preceq$ " in $\mathcal{F}$ is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0, 1]$ where $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha]$ [20]. The strict inequality in $\mathcal{F}$ is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0, 1]$.

Definition 1.3.4. [65] *Let* $\eta \in \mathcal{F}$. *Then* η *is called a positive fuzzy real number if for all* $t < 0$, $\eta(t) = 0$. *The set of all positive fuzzy real numbers is denoted by* $\mathcal{F}^+$.

Remark 1.3.5. [65] As $\eta \in \mathcal{F}^+$ is upper semi-continuous, it follows that $\eta(t) = 0$, $\forall t \leq 0$.

Lemma 1.3.6. [65] *Let $\eta \in \mathcal{F}$. Then $\eta \in \mathcal{F}^+$ if and only if $\eta^1_\alpha \geq 0$ for each $\alpha \in (0,1]$.*

Remark 1.3.7. As $\eta^2_\alpha \geq \eta^1_\alpha$, thus Lemma 1.3.6 gives that $\eta^2_\alpha \geq 0$, for each $\alpha \in (0,1]$.

The arithmetic operations $\oplus$, $\ominus$ and $\odot$ in $\mathcal{F}$ are defined as by Mizumoto and Tanaka [50]. For $\eta, \delta \in \mathcal{F}$:

$$(\eta \oplus \delta)(t) = \sup_{s\in\mathbb{R}} \min\{\eta(s), \delta(t-s)\}, t \in \mathbb{R}.$$

$$(\eta \ominus \delta)(t) = \sup_{s\in\mathbb{R}} \min\{\eta(s), \delta(s-t)\}, t \in \mathbb{R}.$$

$$(\eta \odot \delta)(t) = \sup_{s\in\mathbb{R}, s\neq 0} \min\{\eta(s), \delta(\frac{t}{s})\}, t \in \mathbb{R}.$$

For $\eta \in \mathcal{F}$ and $\delta(> \bar{0}) \in \mathcal{F}^+$, Bag and Samanta [4] also defined an operation $\oslash$ as:

$$(\eta \oslash \delta)(t) = \sup_{s\in\mathbb{R}} \min\{\eta(st), \delta(s)\}, t \in \mathbb{R}.$$

Felbin proved the following result:

Proposition 1.3.8. [20] *Let $\eta, \delta \in \mathcal{F}$ and $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha], [\delta]_\alpha = [a^2_\alpha, b^2_\alpha], \alpha \in (0,1]$. Then:*

(a) $[\eta \oplus \delta]_\alpha = [a^1_\alpha + a^2_\alpha, b^1_\alpha + b^2_\alpha]$,

(b) $[\eta \ominus \delta]_\alpha = [a^1_\alpha - b^2_\alpha, b^1_\alpha - a^2_\alpha]$,

(c) $[\eta \odot \delta]_\alpha = [a^1_\alpha a^2_\alpha, b^1_\alpha b^2_\alpha]$,

(d) $[\bar{1} \oslash \delta]_\alpha = [\frac{1}{b^2_\alpha}, \frac{1}{a^2_\alpha}], a^2_\alpha > 0, \forall \alpha \in (0,1]$.

Kaleva and Seikkala proved the following result which allows us to generate a fuzzy real number from a family of closed intervals.

Lemma 1.3.9. [38] *Let $[a_\alpha, b_\alpha], 0 < \alpha \leq 1$, be a given family of nonempty intervals. If:*

(i) $[a_{\alpha_1}, b_{\alpha_1}] \supset [a_{\alpha_2}, b_{\alpha_2}]$, *for all* $0 < \alpha_1 \leq \alpha_2$,

(ii) $[\lim_{k\to\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$, *whenever* α_k *is an increasing sequence in* $(0, 1]$ *converging to* α,

then the family $[a_\alpha, b_\alpha]$ *represents* α*-level sets of a fuzzy number. Conversely, if* $[a_\alpha, b_\alpha], 0 < \alpha \leq 1$, *are the* α*-level sets of a fuzzy real number then the conditions* (i) *and* (ii) *are satisfied.*

Bag and Samanta proved the following result on fuzzy real numbers.

Proposition 1.3.10. [4] *Let* $\{[a_\alpha, b_\alpha]; \alpha \in (0, 1]\}$ *be a family of nested bounded closed intervals. Let the function* $\eta : \mathbb{R} \to [0, 1]$ *be defined by* $\eta(t) = \bigvee\{\alpha \in (0, 1] : t \in [a_\alpha, b_\alpha]\}$. *Then* η *is a fuzzy real number. The* α*- level sets of* η *are denoted by* $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha], \alpha \in (0, 1]$.

Here η is the fuzzy real number generated by the family of nested bounded closed intervals $\{[a_\alpha, b_\alpha]; \alpha \in (0, 1]\}$.
Following is the definition of a fuzzy norm on a linear space as given by Xiao and Zhu :

Definition 1.3.11. [65] *Let* X *be a vector space over* $\mathbb{R}$ *and the mappings* L, R *(respectively, left norm and right norm):* $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ *be symmetric, non decreasing in both arguments and satisfying* $L(0, 0) = 0$ *and* $R(1, 1) = 1$. *Let* $\parallel . \parallel$ *be a mapping from* X *into* $\mathcal{F}^+$.
Write: $[\parallel x \parallel]_\alpha = [\parallel x \parallel^1_\alpha, \parallel x \parallel^2_\alpha]$, *for* $x \in X, 0 < \alpha \leq 1$. *Then the quadruple* $(X, \parallel . \parallel, L, R)$ *is called a fuzzy normed linear space (briefly, FNLS) and* $\parallel . \parallel$ *is a fuzzy norm, if the following axioms are satisfied:*

(F1) $\parallel x \parallel = \bar{0}$ *if and only if* $x = \theta$, θ *is the zero element of* X,

(F2) $\parallel rx \parallel = | r | \parallel x \parallel, x \in X, r \in \mathbb{R}$, (F3) *for all* $x, y \in X$

(F3L) $\| x + y \| (s + t) \geq L(\| x \| (s), \| y \| (t))$

whenever $s \leq \| x \|_1^1, t \leq \| x \|_1^1$ *and* $s + t \leq \| x + y \|_1^1$,

(F3R) $\| x + y \| (s + t) \leq R(\| x \| (s), \| y \| (t))$

whenever $s \geq \| x \|_1^1, t \geq \| x \|_1^1$ *and* $s + t \geq \| x + y \|_1^1$.

Example 1.3.12. Let $(X, \| \,.\, \|_C)$ be an ordinary normed linear space. Then a fuzzy norm $\| \,.\, \|$ on X can be defined as

$$\| x \| (t) = \begin{cases} 0, & 0 \leq t \leq a \| x \|_C \;\; or \;\; t \geq b \| x \|_C, \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a}, & a \| x \|_C \leq t \leq \| x \|_C, \\ \frac{-t}{(b-1)\|x\|_C} + \frac{b}{b-1}, & \| x \|_C \leq t \leq b \| x \|_C \end{cases}$$

where $\| x \|_C$ is the norm of $x (\neq \theta)$, $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\| x \| = \bar{0}$. Hence $(X, \| \,.\, \|, L, R)$ is an FNLS with $R = \max$ and $L = \min$.

Remark 1.3.13. Using Lemma 1.3.6 and $F1$ (refer to Definition 1.3.11), in an FNLS $(X, \| \,.\, \|, L, R)$, we have for each $\alpha \in (0, 1]$, $\inf\limits_{\substack{x \in X \\ x \neq \theta}} \| x \|_\alpha^1 > 0$.

Felbin [20] proved that if $L = \min$ and $R = \max$, then the triangular inequality (F3) in Definition 1.3.11 is equivalent to

$$\| x + y \| \preceq \| x \| \oplus \| y \| .$$

In this case, the FNLS X is called standard FNLS. Definition 1.3.11 can, therefore, be restated as follows:

Definition 1.3.14. [4] *Let* X *be a linear space over* $\mathbb{R}$. *Let* $\| \,.\, \| : X \to \mathcal{F}^+$ *be a mapping satisfying:*

(i) $\| x \| = \bar{0}$ *if and only if* $x = \theta$

(ii) $\| rx \| = | r | \| x \|, x \in X, r \in \mathbb{R}$

(iii) *for all* $x, y \in X$, $\| x + y \| \preceq \| x \| \oplus \| y \|$

Remark 1.3.15. Clearly $\| \, . \, \|_\alpha^i$; $i = 1, 2$ are crisp norms on X for each $\alpha \in (0, 1]$. Thus every normed linear space can be considered as a particular case of an FNLS with $R = \max$ and $L = \min$.

Also, each classical norm $\| \, . \, \|$ on a linear space X induces a fuzzy norm $\| \, . \, \|_N$ with right norm and left norm as $R = \max$ and $L = \min$ respectively, as follows:

$$\| x \|_N (t) = \begin{cases} 1, & t = \| x \|, \\ 0, & t \neq \| x \|; \end{cases}$$

The α-level sets of $\| \, . \, \|_N$ are given by $[\| x \|_N] = [\| x \|, \| x \|]$, $x \in X$.

Lemma 1.3.16. [65] *Let $(X, \| \, . \, \|, L, R)$ be an FNLS. Then $R \leq \max \Rightarrow \lim_{a \to 0^+} R(a, b) \leq b$, for all $b \in (0, 1] \Rightarrow \lim_{a \to 0^+} R(a, a) = 0$.*

Lemma 1.3.17. [67] *Let $(X, \| \, . \, \|, L, R)$ be an FNLS such that R satisfies (R): for each $\alpha \in (0, 1]$, there exists $\beta \in (0, \alpha]$ such that $R(\beta, \gamma) < \alpha, \forall \gamma \in (0, \alpha)$, then $\| \, . \, \|_\alpha^2$ is continuous at each $x \in X$.*

Lemma 1.3.18. [67] *Let $(X, \| \, . \, \|, L, R)$ be an FNLS. Then $(R) \Leftrightarrow \lim_{a \to 0^+} R(a, b) \leq b$.*

Proof. Suppose (R) holds. Then for each $b \in (0, 1]$ there exists $\beta = b \in (0, b]$ such that $R(\beta, y) \leq b$ for $y = \beta/2$, i.e., $R(b, b/2) \leq b$. Thus $0 < R(a, b) \leq R(b, b/2) \leq b$ for all $a \in (0, b/2]$. Hence $\lim_{a \to 0^+} R(a, b) \leq b$ and thus, $(R) \Rightarrow \lim_{a \to 0^+} R(a, b) \leq b$. Conversely it is clear that $\lim_{a \to 0^+} R(a, b) \leq b \Rightarrow (R)$. □

Using Lemma 1.3.17 and Lemma 1.3.18, we have the following: if right norm R satisfies $\lim_{a \to 0^+} R(a, b) \leq b$, then $\| \, . \, \|_\alpha^2$ is continuous at each $x \in X$.

Lemma 1.3.19. [65] *Let $(X, \| \, . \, \|, L, R)$ be an FNLS.*

(1) *If $L \geq \min$, then (F3R) $\Rightarrow \| x + y \|_\alpha^1 \geq \| x \|_\alpha^1 + \| y \|_\alpha^1$ for $\alpha \in (0, 1]$ and $x, y \in X$.*

(2) *If $R \leq \max$, then (F3R) $\Rightarrow \| x + y \|_\alpha^2 \leq \| x \|_\alpha^2 + \| y \|_\alpha^2$ for $\alpha \in (0, 1]$ and $x, y \in X$.*

Lemma 1.3.20. [65] *Let* $(X, \| \, . \, \|, L, R)$ *be an FNLS,* $\lim_{a \to 0^+} R(a,a) = 0$. *Then (F3R)* $\Rightarrow$ *for each* $\alpha \in (0,1]$ *there is* $\alpha_\circ \in (0,\alpha]$ *such that* $\| x+y \|_\alpha^2 \leq \| x \|_{\alpha_\circ}^2 + \| y \|_{\alpha_\circ}^2$ *for each* $x, y \in X$.

In our work, we have used the definitions and results of Felbin [20], Bag and Samanta [4] and Xiao and Zhu [65] while maintaining their respective notations as far as possible.

Definition 1.3.21. [20] *Let* $(X, \| \, . \, \|, L, R)$ *be an FNLS. Let* $\{x_n\}$ *be a sequence in* X. *The sequence* $\{x_n\}$ *is said to be convergent to* $x \in X$, *denoted by* $\lim_{n \to \infty} x_n = x$, *if* $\lim_{n \to \infty} \| x_n - x \|_\alpha^1 = \lim_{n \to \infty} \| x_n - x \|_\alpha^2 = 0$ *for each* $\alpha \in (0,1]$.
The sequence $\{x_n\}$ *is called a Cauchy sequence if* $\lim_{m,n \to \infty} \| x_m - x_n \|_\alpha^1 = \lim_{m,n \to \infty} \| x_m - x_n \|_\alpha^2 = 0$ *for every* $\alpha \in (0,1]$. X *is said to be complete if every Cauchy sequence in* X *is convergent.*

Theorem 1.3.22. [20] *In an FNLS* $(X, \| \, . \, \|, L, R)$, *every convergent sequence is a Cauchy sequence.*

Theorem 1.3.23. [65] *An FNLS* $(X, \| \, . \, \|, L, R)$ *with* $\lim_{a \to 0^+} R(a,a) = 0$ *is a Hausdorff topological vector space whose neighborhood base of origin* θ *is* $\{N(\varepsilon, \alpha) : \varepsilon > 0, \alpha \in (0,1]\}$. *For each* $\varepsilon > 0, \alpha \in (0,1]$, *the set* $N(\varepsilon, \alpha) = \{x : \| x \|_\alpha^2 < \varepsilon\}$.

Xiao and Zhu gave the following definitions considering a right norm R satisfying $\lim_{a \to 0^+} R(a,a) = 0$.

Definition 1.3.24. [65] *Let* $(X, \| \, . \, \|, L, R)$ *be an FNLS. Consider a set* $A \subseteq X$ *and* $x_\circ \in X$.
The point $x_\circ$ *is called a point of closure of* A *if* $\{x_\circ + N(\alpha, \alpha)\} \cap A \neq \phi$ *for every* $\alpha \in (0,1]$. *Let* $\bar{A}$ *be the set of all points of closure of* A*;* A *is called fuzzy closed if* $\bar{A} = A$. *Clearly,* $\bar{A}$ *is fuzzy closed.*
The set A *is called a fuzzy bounded if for each* $\alpha \in (0,1]$ *there exists* $M = M(\alpha) > 0$

such that $A \subseteq N(M, \alpha)$.

The point $x_\circ$ *is called an interior point of* A *if there exists* $N(\varepsilon_\circ, \alpha_\circ)$ *such that* $x_\circ + N(\varepsilon_\circ, \alpha_\circ) \subseteq A$. *Int* A *denotes the set of all interior points of* A*;* A *is called fuzzy open if Int* A=A.

Lemma 1.3.25. [68] *Let* $(X, \| \,.\, \|, L, R)$ *be an FNLS and* $A \subseteq X$. *Then* A *is a fuzzy bounded set in* X *iff* $\sup_{x \in A} \| x \|_\alpha^2 < +\infty$, *for each* $\alpha \in (0, 1]$.

Lemma 1.3.26. [65] *Let* $(X, \| \,.\, \|, L, R)$ *be an FNLS with* $\lim_{a \to 0^+} R(a, a) = 0$. *Then for* $A \subseteq X$, $x \in \bar{A}$ *if and only if there exists* $\{x_n\} \subseteq A$ *such that* $\lim_{n \to \infty} x_n = x$.

Fang and Song [17] proved that a subset A of X is fuzzy bounded if and only if $\exists M > 0$, such that for each $\alpha \in (0, 1]$, $\sup_{x \in A} \| x \|_\alpha^2 < M$.
Xiao and Zhu proved the following equivalent condition:

Theorem 1.3.27. [65] *Let* $(X, \| \,.\, \|, L, R)$ *be an FNLS with* $\lim_{a \to 0^+} R(a, a) = 0$. *Then* $A \subseteq X$ *is perfectly normal and paracompact, and the following assertions are equivalent:*

(1) *A is compact (i.e., every fuzzy open cover of A has a finite subcover).*

(2) *A is countably compact (i.e., every countable fuzzy open cover of A has a finite subcover).*

(3) *A is sequentially compact (i.e., every sequence of points of A has subsequence converging to a point of A).*

Theorem 1.3.28. [65] *Let* $(X, \| \,.\, \|, L, R)$ *be an FNLS with* $\lim_{a \to 0^+} R(a, a) = 0$, $A \subseteq X$. *Then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3), *where*

(1) *A is sequentially compact.*

(2) *A is precompact.*

(3) *A is fuzzy bounded.*

Theorem 1.3.29. [65] *Let* $(X, \| \, . \, \|, L, R)$ *be a finite dimensional FNLS with* $\lim_{a \to 0^+} R(a, a) = 0$ *and* $A \subseteq X$*. Then,*

(1) *A is complete if and only if A is fuzzy closed.*

(2) *A is compact if and only if A is fuzzy bounded and fuzzy closed.*

From the above results, one can see that an FNLS which is also a topological vector space, is rich in topological properties according to the change from weak right norm R to strong right norm R.

Remark 1.3.30. The right norm R and left norm L have great significance in the theory of FNLS. In this context we refer to the work done by Xiao and Zhu [65]. Therefore while generalizing the results in FNLS, the choice of right and left norms need to be specific according to the context. In our study we have attempted to obtained the results with general right and left norms, whenever possible.

1.3.2 $\mathscr{L}$-fuzzy metric space

We shall assume all lattices $\mathscr{L} = (L, \leq_L)$ to be complete. A complete lattice L is a partially ordered set in which all subsets have both a supremum (join) and an infimum (meet). Let $0_{\mathscr{L}} = \inf L$ and $1_{\mathscr{L}} = \sup L$, for a lattice $\mathscr{L}$.
For example, the pair $([0, 1], \leq)$ is a complete lattice where $\leq$ stands for usual comparison of real numbers. Let us denote this pair as $(L', \leq_{L'})$, where $L' = [0, 1]$ denotes the set and $\leq_{L'}$ is the usual comparison.

Lemma 1.3.31. [11] *Consider the set* L^* *and operation* $\leq_{L^*}$ *on* L^* *defined by:*
$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2$ *and* $x_1 + x_2 \leq 1\}$*, and*
$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$*, and* $x_2 \geq y_2$*, for every* $(x_1, x_2), (y_1, y_2) \in L^*$*. Then* $(L^*, \leq_{L^*})$ *is a complete lattice.*

Definition 1.3.32. [25] *An $\mathscr{L}$-fuzzy set is defined as a mapping $\mathcal{A}: U \to L$, where U is a non empty set called a universe. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u satisfies $\mathcal{A}$.*

Triangular norms play a very important part in the theory of $\mathscr{L}$-fuzzy metric spaces; as the right norm and left norm in case of FNLSs. Classically, a triangular norm T on $([0,1], \leq)$ is a mapping $T : [0,1]^2 \to [0,1]$ which is increasing, commutative, associative and satisfies $T(x,1) = x$, for all $x \in [0,1]$, called the boundary condition. In case of any lattice $\mathscr{L}$, irrespective of its completeness, this definition can be generalized as follows.

Definition 1.3.33. [53] *A triangular norm (t-norm) on $\mathscr{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:*

(i) *boundary condition: $\mathcal{T}(x, 1_{\mathscr{L}}) = x$, $\forall x \in L$*

(ii) *commutativity: $\mathcal{T}(x,y) = \mathcal{T}(y,x)$, $\forall x, y \in L$*

(iii) *associativity: $\mathcal{T}(x, \mathcal{T}(y,z) = \mathcal{T}(\mathcal{T}(x,y), z)$, $\forall x, y, z \in L$*

(iv) *monotonicity: $x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x,y) \leq_L \mathcal{T}(x', y')$, $\forall x, x', y, y' \in L$*

Definition 1.3.34. [53] *A t-norm $\mathcal{T}$ on $\mathscr{L}$ is said to be continuous if for any $x, y \in L$ and any sequences $\{x_n\}$ and $\{y_n\}$ in L such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, we have: $\lim_{n\to\infty} \mathcal{T}(x_n, y_n) = \mathcal{T}(x,y)$.*

Example 1.3.35. (1) $\mathcal{T}(x,y) = \min(x,y)$ and (2) $\mathcal{T}(x,y) = xy$ are two continuous t-norms on $[0,1]$.

A t-norm can also be defined recursively as an $(n+1)$-ary operation ($n \in \mathbb{N}$) by $\mathcal{T}^1 = \mathcal{T}$ and $\mathcal{T}^n(x_1, ..., x_{n+1}) = \mathcal{T}(\mathcal{T}^{n-1}(x_1, ..., x_n), x_{n+1})$ for $n \geq 2$ and $x_i \in L$.

Definition 1.3.36. [53] *A negation on a lattice $\mathscr{L}$ is a order reversing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathscr{L}}) = 1_{\mathscr{L}}$ and $\mathcal{N}(1_{\mathscr{L}}) = 0_{\mathscr{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$, then $\mathcal{N}$ is called an involutive negation.*

The standard negation $\mathcal{N}_s$ on $([0,1], \leq)$ is defined as $\mathcal{N}_s(x) = 1 - x$, for $x \in [0,1]$.

Remark 1.3.37. In our study, we assume that $\mathcal{T}$ is a continuous t-norm on lattice $\mathscr{L}$ such that for every $\mu \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$, there is a $\lambda \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ such that

$$\mathcal{T}^{n-1}(\mathcal{N}(\lambda), ..., \mathcal{N}(\lambda)) >_l \mathcal{N}(\mu).$$

The following definition of an $\mathscr{L}$-FMS is due to Saadati et al.

Definition 1.3.38. [53] *The 3-tuple $(X, \mathcal{M}, \mathcal{T})$ is said to be an $\mathscr{L}$-fuzzy metric space (briefly, $\mathscr{L}$-FMS) if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-norm on $\mathscr{L}$ and $\mathcal{M}$ is an $\mathscr{L}$-fuzzy set on $X^2 \times (0, +\infty)$ satisfying the following conditions for every x, y, z in X and t, s in $(0, +\infty)$:*

(FM1) $\mathcal{M}(x, y, t) >_L 0_{\mathscr{L}}$,

(FM2) $\mathcal{M}(x, y, t) = 1_{\mathscr{L}}$ *for all* $t > 0$ *if and only if* $x = y$,

(FN3) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$,

(FM4) $\mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, t + s)$,

(FM5) $\mathcal{M}(x, y, .) : (0, +\infty) \to L$ *is continuous.*

Here $\mathcal{M}$ is called an $\mathscr{L}$-fuzzy metric. The fuzzy metric $\mathcal{M}(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t.

Example 1.3.39. [14] Let (X, d) be a metric space. Define $\mathcal{T}(a, b) = ab$, for all $a, b \in L'$ and let $\mathscr{L}$ be an $\mathscr{L}$-fuzzy set defined as: for all $t, h, m, n \in \mathbb{R}^+$

$$\mathscr{L}(x, y, t) = \frac{ht^n}{ht^n + md(x, y)} \tag{1.3.1}$$

Then $(X, \mathscr{L}, \mathcal{T})$ is an $\mathscr{L}$-FMS. If $h = m = n = 1$, then (1.3.1) gives:

$$\mathscr{L}(x, y, t) = \frac{t}{t + d(x, y)} \tag{1.3.2}$$

In this case $(X, \mathscr{L}, \mathcal{T})$ is called the standard $\mathscr{L}$-FMS [23]. Let us denote the $\mathscr{L}$-fuzzy metric by $\mathscr{L}_d$. Hence every metric induces an $\mathscr{L}$-fuzzy metric.

Lemma 1.3.40. [14] *Let $(X, \mathscr{L}, \mathcal{T})$ be an $\mathscr{L}$-FMS. Then, $\mathscr{L}(x, y, t)$ is nondecreasing with respect to t, for all $x, y \in x$.*

Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathscr{L}$-FMS. The following definitions are due to Saadati [56]. For $t \in (0, +\infty)$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ is defined as:

$$B(x, r, t) = \{y \in X : \mathcal{M}(x, y, t) >_L \mathcal{N}(r)\}$$

A subset $A \subseteq X$ is said to be open if for each $x \in A$, there exist $t > 0$ and $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{M}}$ denotes the family of all open subsets of X. Then $\tau_{\mathcal{M}}$ is a topology (in the classical sense) on X induced by the $\mathscr{L}$-fuzzy metric $\mathcal{M}$.

Proposition 1.3.41. [56] *In an $\mathscr{L}$-FMS $(X, \mathcal{M}, \mathcal{T})$, $\{B(x, \frac{1}{k}, \frac{1}{k}) : k \in \mathbb{N}\}$ is a countable local base for each $x \in X$. Hence the topology $\tau_{\mathcal{M}}$ is first countable.*

Definition 1.3.42. [53] *Let $(X, \mathscr{L}, \mathcal{T})$ be an $\mathscr{L}$-FMS. A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if for each $\varepsilon \in L \backslash \{0_{\mathscr{L}}\}$ and $t > 0$, there exist $n_0 \in \mathcal{N}$ such that: for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$*

$$\mathscr{L}(x_n, x_m, t) >_l \mathcal{N}(\varepsilon)$$

The sequence $\{x_n\}$ is said to be convergent to $x \in X$, denoted by $x_n \overset{M}{\to} x$, if $\mathscr{L}(x_n, x, t) = \mathscr{L}(x, x_n, t) \to 1_{\mathscr{L}}$ as $n \to \infty$ for every $t > 0$.
An $\mathscr{L}$-FMS is said to be complete iff every cauchy sequence is convergent.

Theorem 1.3.43. [14] *Every $\mathscr{L}$-FMS $(X, \mathcal{M}, \mathcal{T})$ is Hausdorff.*

Definition 1.3.44. [14] *An $\mathscr{L}$-FMS $(X, \mathcal{M}, \mathcal{T})$ is called precompact if for each $r \in L \backslash \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ and $t > 0$, $\exists$ a finite set $A \subset X$ such that $X = \cup_{a \in A} B(a, r, t)$. In* *this case, $\mathcal{M}$ is said to be precompact $\mathscr{L}$-fuzzy metric on X.*

An $\mathscr{L}$-fuzzy metric is called compact if $(X, \tau_{\mathcal{M}})$ is a compact topological space.

$A \subseteq X$ is called countably compact if every countable open cover of A has finite subcover.

$A \subseteq X$ is called sequentially compact if every sequence in A has a subsequence converging to a point of A.

Remark 1.3.45. For our study we have considered the fuzzy norm given by Felbin [20], where as the notion of the $\mathscr{L}$-fuzzy metric is in accordance to the fuzzy norm given by Bag and Samanta, an approach different from Felbin's approach. However, all these notions were proved to be interrelated [5]. In our work, we choose two different approaches of the two fundamental topological structures, i.e., norm and metric in the fuzzy setting. It is expected to provide a better understanding of the variation in the classical concepts of topology and of functional analysis in the fuzzy framework.

1.4 Organization of the thesis

There are five chapters in this thesis including the introductory chapter.
Chapter 2 is dedicated to the study linear operators in FNLSs. Linear operators play an important role in functional analysis. First we discuss and compare different approaches to fuzzy boundedness of linear operators. We then trace the interrelations between these approaches with an aim to provide a unified framework for the study of linear operators theory in fuzzy setting. The space of strongly fuzzy bounded operators is probed for its completeness. Further, we obtain few interesting results on extension and inverse of linear operators in the FNLSs.
Chapter 3 is devoted to the study of fuzzy compact operators in FNLSs. We study boundednes of fuzzy compact operators with respect to the different notions of fuzzy

boundedness. Fuzzy compact operators are also examined in the setting of finite dimensional FNLSs. A generalized form of Riesz's Lemma is obtained in the setting of FNLSs. Subject to specific condition, the completeness of the space of fuzzy compact operators is also established.

Chapter 4 contains several topological characteristics of FNLSs. We study the vector topology induced by a fuzzy norm. The notion of Schauder basis is developed in FNLSs. The completeness of the space of convergent sequences in a complete FNLS ia also examined in this chapter.

In Chapter 5, we study the $\mathscr{L}$-FMS to characterize the topology generated by an $\mathscr{L}$-fuzzy metric. Using the notions of precompact and sequentially compact $\mathscr{L}$-FMSs, a relationship between compact and complete $\mathscr{L}$-fuzzy metric in terms of precompactness is obtained. Subsequently, a generalized form of the Lebesgue covering lemma for the sequentially compact $\mathscr{L}$-fuzzy metric spaces is established.