Chapter 2

Linear operators in fuzzy normed linear spaces

2.1 Introduction

Linear operators are of great importance in classical functional analysis. In FNLSs too, linear operators have evolved during the last few decades. Many authors have contributed to the study of operators in FNLSs and their applications. We refer to the work of Sadeqi and Kia [59] and the references listed there for details. The authors used the notion of operators between the FNLS $\mathcal{C}[a,b]$ of all bounded sequences on the closed unit ball $[a,b]$ and the n-dimensional FNLS $\mathbb{R}^n$ in an image processing application.

In this chapter, we explore linear operators and establish several results in an FNLS as described in the following subsections.

2.2 Boundedness of linear operators in FNLSs

One of the fundamental aspects of a linear operator in classical functional analysis is its boundedness. In the classical case, a linear operator T from a normed linear space

[1]The results of Section 2.2 and 2.3 of this chapter have been accepted for publication in *New Mathematics and Natural Computation*, World Scientific (2019).
The results of Section 2.4 of this chapter have been published in the form of an article in *Annals of Fuzzy Mathematics and Informatics* (2017) .

X into a normed linear space Y is bounded if and only if it maps every bounded subset in X to a bounded set in Y. Fuzzy boundedness of linear operators, on the other hand, in the fuzzy setting does not follow similar equivalence in general. Usually these equivalent conditions give rise to different notions of fuzzy bounded linear operators. As a result, various notions of fuzzy bounded linear operators are found in literature.

2.2.1 Different notions of fuzzy boundedness and fuzzy continuity

In this section, we provide the different notions of fuzzy bounded linear operators and fuzzy continuous linear operators in FNLSs that are found in the literature.
One of the earlier definitions of fuzzy bounded linear operators in FNLSs is given by Felbin in 1999.
Let $(X, \parallel . \parallel, L, R)$ and $(Y, \parallel . \parallel, L, R)$ be two FNLSs.

Definition 2.2.1. [22] *The linear operator $T : X \to Y$ is said to be fuzzy bounded if there exists a fuzzy real number $\eta \in \mathcal{F}^+, \eta \succeq \bar{0}, b_\alpha < \infty$ for every $\alpha \in (0, 1]$, where $[\eta]_\alpha = [a_\alpha, b_\alpha]$ such that, $\forall x \in X$:*

$$\parallel Tx \parallel \preceq \eta \odot \parallel x \parallel \tag{2.2.1}$$

Earlier, in 1998, Itoh and Chō gave the following definition.

Definition 2.2.2. [34] *The linear operator $T : X \to Y$ is called a fuzzy bounded operator if there exists a fuzzy number $K \succeq \bar{0}$ ($K \in \mathcal{F}^+$) with $\sup\{K_\alpha^2 | \alpha \in (0, 1]\} < \infty$, such that*

(a) *whenever $s \leq \parallel x \parallel_1^1, t \leq K_1^1$ and $st \leq \parallel Tx \parallel_1^1$,*
$\parallel Tx \parallel (st) \geq \min(\parallel x \parallel (s), K(t))$ *and*

(b) *whenever $s \geq \parallel x \parallel_1^1, t \geq K_1^1$ and $st \geq \parallel Tx \parallel_1^1$,*
$\parallel Tx \parallel (st) \leq \max(\parallel x \parallel (s), K(t))$

where $[K]_\alpha = [K^1_\alpha, K^2_\alpha]$ *and* $[\| Tx \|]_\alpha = [\| Tx \|^1_\alpha, \| Tx \|^2_\alpha]$

However, in the same paper, the authors proved that the inequalities (a) and (b) in Definition 2.2.2 are equivalent to the inequality:

$$\| Tx \| \preceq K \odot \| x \|, \forall x (\neq \theta) \in X. \tag{2.2.2}$$

This is similar to (2.1.1). Therefore definition by Felbin (Definition 2.2.1) and definition by Itoh and Chō (Definition 2.2.2) are the same. In both these cases the right norm R and left norm L are taken as $R = \max$ and $L = \min$.

In 2003, Xiao and Zhu defined another type of fuzzy bounded linear operators, using general right and left norms instead of the standard right and left norms. We refer to this notion as XZ-bounded linear operators.

Definition 2.2.3. [66] *The linear operator* $T : X \to Y$ *is called* XZ*-bounded if* T *maps fuzzy bounded subsets of* X *into fuzzy bounded subsets of* Y.

Later, in 2008, Bag and Samanta defined two types of fuzzy boundedness of a linear operator, viz., strongly fuzzy bounded linear operator and weakly fuzzy bounded linear operator.

Definition 2.2.4. [4] *A linear operator* $T : X \to Y$ *is said to be strongly fuzzy bounded if there exists* $k > 0$ *such that:*

$$\| Tx \| \oslash \| x \| \preceq \bar{k}, \forall x (\neq \theta) \in X. \tag{2.2.3}$$

Definition 2.2.5. [4] *A linear operator* $T : X \to Y$ *is said to be weakly fuzzy bounded if there exists a fuzzy interval* $\eta \in \mathcal{F}^+, \eta > \bar{0}$ *such that:*

$$\| Tx \| \oslash \| x \| \preceq \eta, \forall x (\neq \theta) \in X. \tag{2.2.4}$$

Remark 2.2.6. Bag and Samanta proved that every strongly fuzzy bounded linear operator is weakly fuzzy bounded. They provided an example to show that converse does not hold ([4], Example 4.2).

In literature, one can find various other notions of boundedness of linear operators in fuzzy setting. However most of the other notions fall into one of the notions mentioned above.

Similar to fuzzy boundedness, there are also various notions of fuzzy continuous linear operator in FNLSs. Xiao and Zhu and Itoh and Chō gave similar definitions of fuzzy continuous linear operators with right norm R satisfying $\lim_{a\to 0^+} R(a,a) = 0$.

Definition 2.2.7. [34, 65] *A linear operator $T : X \to Y$ is said to be fuzzy norm continuous if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} Tx_n = Tx$, for arbitrary sequence $\{x_n\} \in X$.*

Bag and Samanta defined two types of continuity, which are strongly fuzzy continuity and weakly fuzzy continuity.

Definition 2.2.8. [4] *The linear operator $T : X \to Y$ is said to be strongly fuzzy continuous at $x \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\| Tx - Ty \|_\alpha^2 < \varepsilon, whenever \| x - y \|_\alpha^1 < \delta, \forall \alpha \in (0,1].$$

If T is strongly fuzzy continuous at all points of X, then T is said to be strongly fuzzy continuous on X.

Definition 2.2.9. [4] *The linear operator $T : X \to Y$ is said to be weakly fuzzy continuous at $x \in X$ if for a given $\varepsilon > 0$, there exists $\delta \succ \bar{0}$ such that for $[\delta]_\alpha = [\delta_\alpha^1, \delta_\alpha^2]$ and $\alpha \in (0,1]$*

$\| Tx - Tx_\circ \|_\alpha^1 < \varepsilon$ whenever $\| x - x_\circ \|_\alpha^2 < \delta_\alpha^2$, and

$\| Tx - Tx_\circ \|_\alpha^2 < \varepsilon$ whenever $\| x - x_\circ \|_\alpha^1 < \delta_\alpha^1$.

Bag and Samanta further proved the following:

Theorem 2.2.10. [4] *A linear operator $T : X \to Y$ is strongly(weakly) fuzzy bounded if and only if T is strongly (weakly) fuzzy continuous.*

Hence every strongly fuzzy continuous operator T is weakly fuzzy continuous. In 2010, Hasankhani et al. proposed another notion of continuous operators, namely strongly continuous linear operators.

Definition 2.2.11. [31] *A linear operator* $T : X \to Y$ *is said to be strongly continuous if for* $\varepsilon > 0$*, there exists* $\delta \succ \bar{0}$ *such that if* $\| x \|_\alpha^2 < \delta_\alpha^1$, $\| Tx \|_\alpha^2 < \varepsilon$ *and if* $\| x \|_\alpha^1 < \delta_\alpha^2$, $\| Tx \|_\alpha^1 < \varepsilon$*, for* $\alpha \in (0, 1]$ *and* $x \in X$*.*

Hasankhani et al. obtained the following equivalent conditions.

Theorem 2.2.12. [31] *A linear operator* T *is fuzzy bounded if and only if* T *strongly continuous.*

Theorem 2.2.13. [31] *If* $T : X \to Y$ *is fuzzy bounded then,* T *is fuzzy norm continuous.*

With a general right norm R satisfying $\lim_{a \to 0^+} R(a, a) = 0$, Xiao and Zhu proved the following result:

Theorem 2.2.14. [66] *A linear operator* $T : X \to Y$ *is* XZ*-bounded if and only if* T *is fuzzy norm continuous.*

Combining Theorems 2.2.13 and 2.2.14 we get the following result:

Theorem 2.2.15. *If* $T : X \to Y$ *is fuzzy bounded, then* T *is* XZ*-bounded.*

Remark 2.2.16. Fuzzy boundedness is not equivalent to fuzzy norm continuity. We refer to Example 6.2 in [31] which shows that fuzzy norm continuity does not imply fuzzy boundedness.

In view of Theorems 2.2.13 and 2.2.14 and Remark 2.2.16, we can conclude that XZ- boundedness (refer to Definition 2.2.3) is, therefore, not equivalent to fuzzy boundedness (refer to Definition 2.2.1) of a linear operator.

Remark 2.2.17. The right norm and left norm, in the above discussion, satisfy $R \leq \max$ and $L \geq \min$.

Remark 2.2.18. Despite the use of the same notation $\| \, . \, \|$ for the fuzzy norms on X as well as Y, it may be mentioned that these are different in general.

2.2.2 Main results

In the sequel, we undertake a comparative study on the boundedness and continuity of linear operators in FNLSs. We shall use the right norm $R \leq \max$ and the left norm $L \geq \min$, unless otherwise specified.

Theorem 2.2.19. *Let $(X, \| \, . \, \|, L, R)$ and $(Y, \| \, . \, \|, L, R)$ be FNLSs and $T : X \to Y$ be a linear operator. If T is strongly fuzzy bounded, then T is fuzzy norm continuous.*

Proof. Let $\lim_{n\to\infty} x_n = x$. Therefore, $\lim_{n\to\infty} \| x_n - x \|_\alpha^1 = \lim_{n\to\infty} \| x_n - x \|_\alpha^2 = 0$. As T is strongly fuzzy bounded, there exists $k > 0$ such that $\| Tx_n - Tx \| \oslash \| x_n - x \| \preceq \bar{k}$. Hence:

$$\| Tx_n - Tx \|_\alpha^2 \leq k \| x_n - x \|_\alpha^1 \text{ and } \| Tx_n - Tx \|_\alpha^1 \leq k \| x_n - x \|_\alpha^2$$

Thus, we have $\lim_{n\to\infty} \| Tx_n - Tx \|_\alpha^1 = \lim_{n\to\infty} \| Tx_n - Tx \|_\alpha^2 = 0 \Rightarrow \lim_{n\to\infty} Tx_n = Tx$. Therefore, T is fuzzy norm continuous. □

Corollary 2.2.20. *A strongly fuzzy bounded linear operator T from X to Y, is XZ-bounded.*

Proof. Proof follows directly from Theorems 2.2.14 and 2.2.19. □

Corollary 2.2.21. *A weakly fuzzy bounded linear operator T from X to Y, is fuzzy norm continuous.*

Proof. Proof follows directly from Remark 2.2.6 and Theorem 2.2.19. □

In reference to Lemma 2.2.19 and Corollary 2.2.21, we now show that the converse is not true in general, i.e., a fuzzy norm continuous linear operator need not be weakly fuzzy bounded and hence, by Remark 2.2.6, need not be strongly fuzzy bounded.

Example 2.2.22. Let X be a vector space over $\mathbb{R}$ and $B = \{e_i\}_{i=1}^\infty$ a basis for X ($\dim X = \infty$). With $L = \min$ and $R = \max$, let us define the fuzzy norms on X as follows:

$$\| x \|_\circ (t) = \begin{cases} 1, & \text{if } t = \sum_{i=1}^{n} | a_i |; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\| x \| (t) = \begin{cases} \sum_{i=1}^{n} \frac{| a_i |}{t^i}, & \text{if } \sum_{i=1}^{n} | a_i | \leq t; \\ 1, & \text{if } t = \sum_{i=1}^{n} | a_i | = 0; \\ 0, & \text{otherwise.} \end{cases}$$

where, $x = \sum_{i=1}^{n} a_i e_i$. Then the α-level sets of $\| . \|_\circ$ and $\| . \|$ are:

$$[\| x \|_\circ]_\alpha = [\sum_{i=1}^{n} | a_i |, \sum_{i=1}^{n} | a_i |], \text{ and } [\| x \|]_\alpha = [\sum_{i=1}^{n} | a_i |, \sum_{i=1}^{n} \frac{| a_i |}{\alpha^i}]$$

Let $T : (X, \| . \|) \to (X, \| . \|_\circ)$ be a linear operator and for each $e_n \in B$, let us define as: $Te_n = ne_n$.

The linear T is fuzzy norm continuous:

Let us consider a sequence $\{x_n\} \subseteq X$ such that $\lim_{n\to\infty} x_n = 0$. Then, $\lim_{n\to\infty} \| x_n \|_\alpha^1 = \lim_{n\to\infty} \| x_n \|_\alpha^2 = 0$.

Let $x_n = \sum_{i=1}^{k_n} a_{n_i} e_i$. Then, $Tx_n = \sum_{i=1}^{k_n} i a_{n_i} e_i$. It gives: $\| Tx_n \|_{\circ\alpha}^2 = \sum_{i=1}^{k_n} | a_{n_i} i |$, for all $\alpha \in (0, 1]$. Then, we have:

$$\| Tx_n \|_{\circ\alpha}^2 = \sum_{i=1}^{k_n} | a_{n_i} i | = \sum_{i=1}^{k_n} i | a_{n_i} | \leq \sum_{i=1}^{k_n} \frac{1}{\alpha^i} | a_{n_i} | = \| x_n \|_\alpha^2$$

Letting $n \to \infty$, we get $\| Tx_n \|_{\circ\alpha}^2 \to 0$ as $\| x_n \|_\alpha^2 \to 0$, for any $\alpha \in (0, 1]$. Thus $\lim_{n\to\infty} Tx_n = 0$ and hence T is fuzzy norm continuous.

The linear T is not weakly fuzzy bounded:

Let T be weakly fuzzy bounded. There exists a fuzzy real number $\eta \succ \bar{0}$ and $\eta \in \mathcal{F}^+$, such that, $\| Tx \|_\circ \oslash \| x \| \preceq \eta$, for all $x \in X$. Therefore:

$$\| Tx \|_{\circ 1}^2 = \| ne_n \|_{\circ 1}^2 = | n | = n \leq \eta_1^2 \| e_n \|_1^2 = \eta_1^2$$

Hence $n \leq \eta_1^2$, for all $n \in \mathbb{N}$, which is a contradiction. Hence T is not weakly fuzzy bounded.

Next, we proceed to compare fuzzy bounded and weakly fuzzy bounded linear operators.

Theorem 2.2.23. *Let* $(X, \| \, . \, \|, L, R)$ *and* $(Y, \| \, . \, \|, L, R)$ *be FNLSs and let* $T : X \to Y$ *be a linear operator. If for each* $\alpha \in (0,1]$, $\sup_{\substack{x \in X \\ x \neq \theta}} \| x \|_\alpha^2 < +\infty$, *then* T *is fuzzy bounded* $\Leftrightarrow$ T *is weakly fuzzy bounded.*

Proof. Suppose $T : X \to Y$ is fuzzy bounded. Then there exists $\eta \succ \bar{0}$ such that $\| Tx \| \preceq \eta \odot \| x \|$ for all $x (\neq \theta) \in X$. That gives $\| Tx \|_\alpha^1 \leq \eta_\alpha^1 \| x \|_\alpha^1$ and $\| Tx \|_\alpha^2 \leq \eta_\alpha^2 \| x \|_\alpha^2$. Set $m_\alpha = \inf_{\substack{x \in X \\ x \neq \theta}} \| x \|_\alpha^1$ and $M_\alpha = \sup_{\substack{x \in X \\ x \neq \theta}} \| x \|_\alpha^2$, for any $\alpha \in (0,]$. Then,

$$\frac{\| Tx \|_\alpha^1}{\| x \|_\alpha^2} \leq \frac{\eta_\alpha^1 \| x \|_\alpha^1}{\| x \|_\alpha^2} \leq \eta_\alpha^1 \text{ and, } \frac{\| Tx \|_\alpha^2}{\| x \|_\alpha^1} \leq \frac{\eta_\alpha^2 \| x \|_\alpha^2}{\| x \|_\alpha^1} \leq \eta_\alpha^2 \frac{M_\alpha}{m_\alpha}$$

Let $\mathcal{K}$ be the fuzzy real number generated by the family of nested bounded closed intervals $\{[\eta_\alpha^1, \eta_\alpha^2 \frac{M_\alpha}{m_\alpha}] : \alpha \in (0,1]\}$. Obviously $\mathcal{K} \succ \bar{0}$ and $\mathcal{K} \in \mathcal{F}^+$ (by Proposition 1.3.10). Then, we have $\| Tx \| \oslash \| x \| \preceq \mathcal{K}$, for all $x (\neq \theta) \in X$. Hence T is weakly fuzzy bounded.

Conversely, suppose T is a weakly fuzzy bounded linear operator. Therefore, there exists $\eta \in \mathcal{F}, \eta \succ \bar{0}$ such that $\| Tx \| \oslash \| x \| \preceq \eta$. Using similar argument, we have:

$$\frac{\| Tx \|_\alpha^1}{\| x \|_\alpha^1} \leq \frac{\eta_\alpha^1 \| x \|_\alpha^2}{\| x \|_\alpha^1} \leq \eta_\alpha^1 \frac{M_\alpha}{m_\alpha} \text{ and, } \frac{\| Tx \|_\alpha^2}{\| x \|_\alpha^2} \leq \frac{\eta_\alpha^2 \| x \|_\alpha^1}{\| x \|_\alpha^2} \leq \eta_\alpha^2$$

Then, there exists a fuzzy real number $\mathcal{K} \in \mathcal{F}^+$ and $\mathcal{K} \succ \bar{0}$ generated by the family of nested bounded closed intervals $\{[\eta_\alpha^1 \frac{M_\alpha}{m_\alpha}, \eta_\alpha^2] : \alpha \in (0,1]\}$ and $\| Tx \| \preceq \mathcal{K} \odot \| x \|$. Thus T is fuzzy bounded. □

Analogues to Lemma 5.8 by Hasankhani et al. [31], we obtain the following necessary condition for a linear operator to be weakly fuzzy bounded.

Lemma 2.2.24. *Let $(X, \| \, . \, \|)$ and $(Y, \| \, . \, \|)$ be FNLSs and let $T : X \to Y$ be a linear operator. If for all $\alpha \in (0,1]$, $\sup_{\|x\|_\alpha^1 \le 1} \| Tx \|_\alpha^2 < \infty$ and $\sup_{\alpha \in (0,1]} \sup_{\|x\|_\alpha^2 \le 1} \| Tx \|_\alpha^1 < \infty$, then T is weakly fuzzy bounded.*

Proof. Let $S_\alpha = \sup_{\|x\|_\alpha^1 \le 1} \| Tx \|_\alpha^2$, for all $\alpha \in (0,1]$. If $\alpha \le \gamma$, then $\| x \|_\alpha^1 \le \| x \|_\gamma^1$ and $\| Tx \|_\gamma^2 \le \| Tx \|_\alpha^2$. This implies that $S_\gamma \le S_\alpha$. Let us assume that $I_\alpha = \inf_{\beta < \alpha} S_\beta$. For $\alpha < \gamma$, we have $S_\beta < S_\alpha < S_\gamma$ and hence

$$I_\gamma \quad \le \quad I_\alpha \tag{2.2.5}$$

Let $\{\alpha_k\}$ be an increasing sequence in $(0,1]$ converging to α. Since $\alpha_k \le \alpha$ for each k, therefore $I_\alpha \le I_{\alpha_k}$ and hence

$$I_\alpha \quad \le \quad \inf_k I_{\alpha_k}. \tag{2.2.6}$$

Using definition of infimum, there is a $\beta_\circ < \alpha$ such that $S_{\beta_\circ} \le I_\alpha + \epsilon$. Since α_k is increasing and converges to α, there exists $k_\circ > 0$ such that $\beta_\circ < \alpha_{k_\circ}$. Then $\inf_k I_{\alpha_k} \le I_{\alpha_{k_\circ}} \le S_{\beta_\circ}$ and hence $\inf_k I_{\alpha_k} \le I_\alpha + \epsilon$.
Letting $\epsilon \to 0$, we have

$$\inf_k I_{\alpha_k} \quad \le \quad I_\alpha. \tag{2.2.7}$$

Hence by (2.2.6) and (2.2.7), $\inf_k I_{\alpha_k} = I_\alpha$ implies

$$\inf_k I_{\alpha_k} \quad = \quad I_\alpha = \lim_{k \to \infty} I_{\alpha_k}. \tag{2.2.8}$$

Now, we have $S_\alpha \le S_\beta$, for all $\beta < \alpha$, thus $S_\alpha \le \inf_{\beta < \alpha} S_\beta = I_\alpha$. Then, for any $x (\neq 0) \in X$

$$\| Tx \|_\alpha^2 \quad \le \quad S_\alpha \| x \|_\alpha^1 \le I_\alpha \| x \|_\alpha^1 \, . \tag{2.2.9}$$

Next, let $I = \sup_{\alpha \in (0,1]} \sup_{\|x\|_\alpha^2 \le 1} \| Tx \|_\alpha^1$. Then, for any $x (\neq 0) \in X$

$$\| Tx \|_\alpha^1 \le I \| x \|_\alpha^2 \le (I + I_\alpha) \| x \|_\alpha^2 \, . \tag{2.2.10}$$

By Proposition 1.3.10, we obtain that $\{[\frac{\|Tx\|_\alpha^1}{\|x\|_\alpha^2}, \frac{\|Tx\|_\alpha^2}{\|x\|_\alpha^1}] : \alpha \in (0,1]\}$ generates the fuzzy real number $\| Tx \| \oslash \| x \|$ for all $x(\neq 0) \in X$. Define $[\eta]_\alpha = [I, I + I_\alpha]$. Then, the inequalities (2.2.5) and (2.2.8) imply that the family $\{[\eta]_\alpha : \alpha \in (0,1]\}$ represents the α-level sets of a fuzzy real number η. Using inequalities (2.2.9) and (2.2.10), we obtain that $\| Tx \| \oslash \| x \| \preceq \eta$.

Hence T is a weakly fuzzy bounded linear operator. □

Remark 2.2.25. If the FNLS X is finite dimensional, then, for all $\alpha \in (0,1]$: $\sup\limits_{\|x\|_\alpha^1 \leq 1} \| Tx \|_\alpha^2 < \infty$ and $\sup\limits_{\alpha\in(0,1]} \sup\limits_{\|x\|_\alpha^2 \leq 1} \| Tx \|_\alpha^1 < \infty$. We thus have:

Corollary 2.2.26. *A linear operator T from a finite dimensional FNLS X to an FNLS Y is weakly fuzzy bounded.*

In view of Theorem 2.2.14 and Example 2.2.22, we can conclude that an XZ-bounded operator is not weakly fuzzy bounded, in general. However, we obtain a condition for a XZ-bounded operator to be weakly fuzzy bounded.

Theorem 2.2.27. *Let $(X, \| \, . \, \|)$ and $(Y, \| \, . \, \|)$ be FNLSs and $T : X \to Y$ be XZ-bounded. If for each $\alpha \in (0,1]$ there exists $M_\alpha > 0$ such that $\frac{\|x\|_\alpha^2}{\|x\|_\alpha^1} \leq M_\alpha$ and $\sup\limits_{\alpha\in(0,1]} \| x \|_\alpha^2 < +\infty$, $\sup\limits_{\alpha\in(0,1]} \| y \|_\alpha^2 < +\infty$, for all $x \in X$ and $y \in Y$, then T is weakly fuzzy bounded.*

Proof. Let $T : X \to Y$ be XZ-bounded operator, so T maps a fuzzy bounded subset of X into a fuzzy bounded subset of Y. Choose any $\alpha \in (0,1]$. Consider the set $N_1 = \{x \in X : \| x \|_\alpha^1 \leq 1\}$. For $\beta \in (0, \alpha]$,

$$\| x \|_\alpha^2 \leq \| x \|_\beta^2 \leq M_\beta \| x \|_\beta^1 \leq M_\beta \| x \|_\alpha^1 \leq M_\beta \tag{2.2.11}$$

and, for $\beta \in (\alpha, 1]$,

$$\| x \|_\beta^2 \leq \| x \|_\alpha^2 \leq M_\alpha \| x \|_\alpha^1 \leq M_\alpha \tag{2.2.12}$$

Therefore, for any $\beta \in (0,1]$ we can find $M'_\beta = \max(M_\beta, M_\alpha)$, such that (2.2.11) and (2.2.12) imply $N_1 \subseteq N(M'_\beta, \beta)$. Thus, N_1 is a fuzzy bounded set in X and so TN_1 is a fuzzy bounded subset in Y. Thus for any $\beta \in (0,1]$, using Lemma 1.3.25,

$$\sup_{x\in N_1} \| Tx \|_\beta^2 < +\infty, \text{i.e.}, \sup_{\|x\|_\alpha^1 \le 1} \| Tx \|_\beta^2 < +\infty$$

Since $\beta \in (0,1]$ is arbitrary, we have

$$\sup_{\|x\|_\alpha^1 \le 1} \| Tx \|_\alpha^2 < +\infty \tag{2.2.13}$$

Also, for any $\beta \in (0,1]$, $\{x : \| x \|_\beta^2 \le 1\} \subseteq \{x : \| x \|_\beta^1 \le 1\}$ and $\| Tx \|_\beta^1 \le \| Tx \|_\beta^2$. Therefore

$$\sup_{\|x\|_\beta^2 \le 1} \| Tx \|_\beta^1 \le \sup_{\|x\|_\beta^2 \le 1} \| Tx \|_\beta^2 \le \sup_{\|x\|_\beta^1 \le 1} \| Tx \|_\beta^2 < +\infty \tag{2.2.14}$$

Thus for each $\beta \in (0,1]$, (2.2.14) gives:

$$\sup_{\|x\|_\beta^2 \le 1} \| Tx \|_\beta^1 < +\infty$$

Since $\sup_{\beta\in(0,1]} \| x \|_\beta^2 < +\infty$, $\sup_{\beta\in(0,1]} \| y \|_\beta^2 < +\infty$, for all $x \in X$ and $y \in Y$, therefore

$$\sup_{\beta\in(0,1]} \sup_{\|x\|_\beta^2 \le 1} \| Tx \|_\beta^1 < +\infty \tag{2.2.15}$$

From (2.1.13), (2.1.15) and Lemma 2.2.24, T is weakly fuzzy bounded. □

Theorem 2.2.28. *Let $(X, \| \ . \ \|)$ and $(Y, \| \ . \ \|)$ be FNLSs and $T : X \to Y$ be XZ-bounded. If for each $\alpha \in (0,1]$ there exists $M_\alpha > 0$ such that $\frac{\|x\|_\alpha^2}{\|x\|_\alpha^1} \le M_\alpha$ and $\sup_{\alpha\in(0,1]} \| x \|_\alpha^2 < +\infty$, $\sup_{\alpha\in(0,1]} \| y \|_\alpha^2 < +\infty$, for all $x \in X$ and $y \in Y$, then T is fuzzy bounded.*

Proof. Using similar argument as in Theorem 2.2.27, we get, for all $\alpha \in (0,1]$

$$\sup_{\|x\|_\alpha^1 \le 1} \| Tx \|_\alpha^2 < +\infty \tag{2.2.16}$$

Since for any $\beta \in (0,1]$, $\| Tx \|_\beta^1 \leq \| Tx \|_\beta^2$, therefore $\sup_{\|x\|_\beta^1 \leq 1} \| Tx \|_\beta^1 \leq \sup_{\|x\|_\beta^1 \leq 1} \| Tx \|_\beta^2 < +\infty$, using (2.2.16). Hence,

$$\sup_{\beta \in (0,1]} \sup_{\|x\|_\beta^1 \leq 1} \| Tx \|_\beta^1 < +\infty \tag{2.2.17}$$

From inequalities (2.1.16) and (2.1.17), we have T is fuzzy bounded (Hasankhani et. al [31], Lemma 5.8). □

A weakly fuzzy bounded linear operator T may not be strongly fuzzy bounded (Remark 2.2.6). We, however, have the following result:

Theorem 2.2.29. *Let T be a weakly fuzzy bounded linear operator from a FNLS $(X, \| \,.\, \|)$ to a FNLS $(Y, \| \,.\, \|)$. If $\sup_{\alpha \in (0,1]} \eta_\alpha^2 < +\infty$ where, $\| Tx \| \oslash \| x \| \preceq \eta$ and $\eta \succ \bar{0} \in \mathcal{F}^+$. Then T is strongly fuzzy bounded as well.*

Proof. Let $T : X \to Y$ be weakly fuzzy bounded. Therefore $\exists \eta \succ \bar{0} \in \mathcal{F}^+$ so that $\| Tx \| \oslash \| x \| \preceq \eta$. It gives:

$$\| Tx \|_\alpha^1 \leq \eta_\alpha^1 \| x \|_\alpha^2 \text{ and } \| Tx \|_\alpha^2 \leq \eta_\alpha^2 \| x \|_\alpha^1 \tag{2.2.18}$$

for each $\alpha \in (0,1]$. We have $0 < \eta_\alpha^1 \leq \eta_\alpha^2 < +\infty$. Let $\eta_\alpha = \max(\eta_\alpha^1, \eta_\alpha^2)$ and $k = \sup_{\alpha \in (0,1]} \eta_\alpha$. Since $\sup_{\alpha \in (0,1]} \eta_\alpha^2 < +\infty$, we have $0 < k < +\infty$. Hence, using (2.2.18), we get

$$\| Tx \|_\alpha^1 \leq k \| x \|_\alpha^2 \quad \text{and} \quad \| Tx \|_\alpha^2 \leq k \| x \|_\alpha^1$$

Thus, $\exists k > 0$ such that $\| Tx \| \oslash \| x \| \preceq \bar{k}$ and hence T is strongly fuzzy bounded. □

The interrelationships amongst various boundedness notions help us in obtaining a fair analysis of the existing theorems of linear operators in FNLSs.

2.3 Space of strongly fuzzy bounded operators

Different notions of fuzzy boundedness of linear operators apparently constitute various types of fuzzy norms. As mentioned earlier, various authors discussed the FNLSs of linear operators. In this section, we study the completeness of the FNLS of strongly fuzzy bounded operators with respect to the FNLS of weakly fuzzy bounded operators.

Before going to the main results, we prove the following general property of an FNLS.

Lemma 2.3.1. *Let $(X, \| \,.\, \|$ be a complete FNLS with $\lim_{a \to 0^+} R(a, a) = 0$. If A is a fuzzy closed subset of X, then A is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in A. Then $\{x_n\}$ is a Cauchy sequence in X also. Since X is complete, therefore, $\{x_n\}$ converges in X such that $\lim_{n\to\infty} x_n = x$ where $x \in X$. As A is fuzzy closed, using Lemma 1.3.26, we have $x \in A$. Thus $\{x_n\}$ converges in A and so A is complete. □

Let $(X, \| \,.\, \|, L, R)$ and $(Y, \| \,.\, \|)$ be two FNLSs, with $R \leq \max$.

Bag and Samanta [4] defined the fuzzy norm $\| \,.\, \|^*$ of a linear operator $T : X \to Y$ as a function $\| T \|^* : \mathbb{R} \to [0, 1]$ such that:

$$\| T \|^* (t) = \bigvee \{\alpha \in (0, 1] : t \in [\| T \|^{*1}_\alpha, \| T \|^{*2}_\alpha]\},$$

where

$$\| T \|^{*1}_\alpha = \sup_{\substack{x \in X \\ x \neq \theta}} \frac{\| Tx \|^1_\alpha}{\| x \|^2_\alpha}, \text{ and } \| T \|^{*2}_\alpha = \sup_{\substack{x \in X \\ x \neq \theta}} \frac{\| Tx \|^2_\alpha}{\| x \|^1_\alpha}$$

The fuzzy norm $\| T \|^*$ is, thus, generated by the family of nested bounded closed intervals $\{[\| T \|^{*1}_\alpha, \| T \|^{*2}_\alpha] : \alpha \in (0, 1]\}$.

Let us denote the space of all strongly fuzzy bounded operators from X to Y by $B^s(X, Y)$ and the space of all weakly fuzzy bounded operators by $B(X, Y)$. Ji et al. [37] stated that the spaces $B^s(X, Y)$ and $B(X, Y)$ are quite different spaces. However, it is already been proved that every strongly fuzzy bounded linear operator is weakly fuzzy bounded, so we have $B^s(X,Y) \subset B(X,Y)$. We can, in fact, prove the following:

Theorem 2.3.2. $B^s(X,Y)$ *is a fuzzy closed proper subspace of* $B(X,Y)$.

Proof. Let $\{T_n\}$ be a sequence of strongly fuzzy bounded linear operators in $B^s(X,Y)$ and let $\lim_{n\to\infty} T_n = T$, for some $T \in B(X,Y)$. Fix any $x \in X$ and $\alpha \in (0,1]$. For $\varepsilon > 0$ there exists n_ε (depending on ε and α) such that $\| T_n - T \|_\alpha^{*2} < \varepsilon$ for all $n \geq n_\varepsilon$.

That is, $\sup_{\substack{x\in X\\ x\neq\theta}} \dfrac{\| T_n x - Tx \|_\alpha^2}{\| x \|_\alpha^1} < \varepsilon$ for all $n \geq n_\varepsilon$, i.e., $\| T_n x - Tx \|_\alpha^2 < \varepsilon \| x \|_\alpha^1$, for all $n \geq n_\varepsilon$ and $x \in X$

In particular, for any $x \in X$,

$$\| T_{n_\varepsilon} x - Tx \|_\alpha^2 < \varepsilon \| x \|_\alpha^1, \tag{2.3.1}$$

As T_{n_ε} is strongly fuzzy bounded, there exists $K_{n_\varepsilon} > 0$ such that $\| T_{n_\varepsilon} x \| \oslash \| x \| \preceq K_{n_\varepsilon}, \forall x \in X$. Therefore:

$$\| T_{n_\varepsilon} x \|_\alpha^2 < K_{n_\varepsilon} \| x \|_\alpha^1 \;\text{ and }\; \| T_{n_\varepsilon} x \|_\alpha^1 < K_{n_\varepsilon} \| x \|_\alpha^2 \tag{2.3.2}$$

Since $R \leq \max$, for any $x (\neq \theta) \in X$, from (2.3.1) and (2.3.2):

$$\| Tx \|_\alpha^2 \leq \| Tx - T_{n_\varepsilon} x \|_\alpha^2 + \| T_{n_\varepsilon} x \|_\alpha^2 \leq \varepsilon \| x \|_\alpha^1 + K_{n_\varepsilon} \| x \|_\alpha^1$$

i.e.,

$$\| Tx \|_\alpha^2 \leq \| x \|_\alpha^1 (\varepsilon + K_{n_\varepsilon}) \tag{2.3.3}$$

Thus, $\sup_{\substack{x\in X\\ x\neq\theta}} \dfrac{\| Tx \|_\alpha^2}{\| x \|_\alpha^1} < (\varepsilon + K_{n_\varepsilon})$.

Similarly, $\sup_{\substack{x\in X\\ x\neq\theta}} \dfrac{\| Tx \|_\alpha^1}{\| x \|_\alpha^2} < (\varepsilon + K_{n_\varepsilon})$. Therefore, $\| Tx \| \oslash \| x \| \preceq \overline{(\varepsilon + K_{n_\varepsilon})}, \forall x \in X$ and hence, T is strongly fuzzy bounded, i.e., $T \in B^s(X,Y)$.

Therefore $B^s(X,Y)$ is fuzzy closed subset of $B(X,Y)$. □

Theorem 2.3.3. *If $B(X,Y)$ is complete then $B^s(X,Y)$ is also complete.*

Proof. As $B^s(X,Y)$ is a fuzzy closed subset of $B(X,Y)$, the proof follows directly from Lemmas 2.3.1 and 2.3.2. □

Ji et al. [37] proved that the space $B(X,Y)$ is complete if Y is complete. Thus, completeness of Y implies completeness of $B^s(X,Y)$. However, $B^s(X,Y)$ may be complete even if Y is not complete ([37], Example 3.3). Therefore, while completeness of $B(X,Y)$ implies the completeness of $B^s(X,Y)$, the converse is not true in general.

2.4 Few results on linear operators

In this section, we obtain several results on linear operators in FNLSs. The results are obtained using the general right and left norms respectively.

2.4.1 Extension of a linear operator

We first introduce an extension of a linear operator in FNLSs and then prove a boundedness result for the extension operator. The definitions of range space and null space of a linear operator are considered as in Kreiszig [42].

Definition 2.4.1. *Let $T : (X, \| \,.\, \|, L, R) \to (Y, \| \,.\, \|, L, R)$ be a linear operator. The restriction of the operator T to a set $A \subseteq X$, denoted by $T\mid_A$, is defined as:*

$T\mid_A : A \to Y,\ T\mid_A(x) = Tx, \forall x \in A.$

Definition 2.4.2. *An extension of T from a set $M \subset X$ to X is the operator $\widetilde{T} : X \to Y$ such that: $\widetilde{T}\mid_M = T, i.e., \widetilde{T}(x) = T(x),\ \forall x \in M.$*

Theorem 2.4.3. *Let $(X, \| \,.\, \|, L, R)$ and $(Y, \| \,.\, \|, L, R)$ be FNLSs with $\lim_{a\to 0^+} R(a,a) = 0$ where Y is complete. Let $T : \mathcal{D} \to Y$ be a strongly fuzzy bounded linear operator, where $\mathcal{D} \subset X$. Then T has an extension $\widetilde{T} : \overline{\mathcal{D}} \to Y$ such that $\widetilde{T}$ is strongly fuzzy bounded linear operator of fuzzy norm $\| \widetilde{T} \|^* = \| T \|^*$.*

Proof. Consider any $x \in \overline{\mathcal{D}}$. Then, there exists a sequence $\{x_n\}$ in $\mathcal{D}$ such that $\lim_{n\to\infty} x_n = x$ (by Lemma 1.3.26). Since T is a strongly fuzzy bounded linear operator, $\exists k \in \mathbb{R}^+$ such that, for all $x \in X$, $\| Tx \| \oslash \| x \| \preceq \bar{k}$. Thus, we have $\| Tx_m - Tx_n \| \oslash \| x_m - x_n \| \preceq \bar{k}$. This gives:

$$\frac{\| Tx_m - Tx_n \|_\alpha^1}{\| x_m - x_n \|_\alpha^2} \leq k, \text{ and } \frac{\| Tx_m - Tx_n \|_\alpha^2}{\| x_m - x_n \|_\alpha^1} \leq k \tag{2.4.1}$$

As $\{x_n\}$ converges, $\{x_n\}$ is a Cauchy sequence. Therefore, $\lim_{n,m\to\infty} \| x_m - x_n \|_\alpha^2 = \lim_{n,m\to\infty} \| x_m - x_n \|_\alpha^1 = 0$. Thus, (2.4.1) gives, $\lim_{n,m\to\infty} \| Tx_m - Tx_n \|_\alpha^2 = 0$, and hence, $\{Tx_n\}$ is a Cauchy sequence in Y. Since Y is complete, $\{Tx_n\}$ converges. Let $\lim_n Tx_n = y$, where $y \in Y$.

Let us define a function $\widetilde{T} : \overline{\mathcal{D}} \to Y$ as $\widetilde{T}x = y$ for $x \in \overline{\mathcal{D}}$. As $\{x_n\}$ is a sequence in $\mathcal{D} \subseteq X$, for all n, we have $\| Tx_n \|_\alpha^1 \leq k \| x_n \|_\alpha^2$.

Now,

$$\begin{aligned} \| \widetilde{T}x \|_\alpha^1 = \| y \|_\alpha^1 = \| \lim_{n\to\infty} Tx_n \|_\alpha^1 = \lim_{n\to\infty} \| Tx_n \|_\alpha^1 \\ \leq k \lim_{n\to\infty} \| x_n \|_\alpha^2 = k \| \lim_{n\to\infty} x_n \|_\alpha^2 \\ = k \| x \|_\alpha^2 \end{aligned}$$

Thus, $\| \widetilde{T}x \|_\alpha^1 \leq k \| x \|_\alpha^2$, i.e., $\frac{\|\widetilde{T}x\|_\alpha^1}{\|x\|_\alpha^2} \leq k$.

In the same way, it can be proved that $\frac{\|\widetilde{T}x\|_\alpha^2}{\|x\|_\alpha^1} \leq k$. Combining these, we get $\| \widetilde{T}x \| \oslash \| x \| \preceq \bar{k}$ and hence $\widetilde{T}$ is strongly fuzzy bounded.

We now show that $\| T \|^* = \| \widetilde{T} \|^*$. For $x \in \mathcal{D}$, $Tx = \widetilde{T}x$ gives $\| T \|^* = \| \widetilde{T} \|^*$.

For $x \in \overline{\mathcal{D}} - \mathcal{D}$: There exists a sequence (x_n) in $\mathcal{D}$ such that $\lim_{n\to\infty} x_n = x$. For all such $x \in \overline{\mathcal{D}} - \mathcal{D}$,

$$\| \widetilde{T} \|_\alpha^{*1} = \sup_x \frac{\| \widetilde{T}x \|_\alpha^1}{\| x \|_\alpha^2} = \sup_x \frac{\| \lim_{n\to\infty} Tx_n \|_\alpha^1}{\| x \|_\alpha^2} = \sup_x \frac{\| T(\lim_{n\to\infty} x_n) \|_\alpha^1}{\| x \|_\alpha^2} = \sup_x \frac{\| Tx \|_\alpha^1}{\| x \|_\alpha^2},$$

which gives $\| \widetilde{T} \|_\alpha^{*1} = \| T \|_\alpha^{*1}$. In the similar way, we can show that: $\| \widetilde{T} \|_\alpha^{*2} = \| T \|_\alpha^{*2}$.

Hence the two families $\{[\| T \|_\alpha^{*1}, \| T \|_\alpha^{*2}] : \alpha \in (0,1]\}$ and $\{[\| \widetilde{T} \|_\alpha^{*1}, \| \widetilde{T} \|_\alpha^{*2}$

$] : \alpha \in (0,1]\}$ generate the same fuzzy number (using Proposition 1.3.10). Hence $\| T \|^* = \| \widetilde{T} \|^*$. □

Example 2.4.4. Consider $X = Y = \mathbb{R}$, the linear space of all real numbers and $L = \min$ and $R = \max$. Define the fuzzy norms $\| \,.\, \|$ and $\| \,.\, \|^\sim$ on $\mathbb{R}$ as follows:

$$\| x \| (t) = \begin{cases} \frac{|x|}{t} & \text{if } | x | < t; \\ 1 & \text{if } | x | = t = 0; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\| x \|^\sim (t) = \begin{cases} 1 & \text{if } | x | = t; \\ 0 & \text{otherwise.} \end{cases}$$

The α-level sets of $\| x \|$ and $\| x \|^\sim$ are given by: $[\| x \|]_\alpha = [| x |, \frac{|x|}{\alpha}]$ and $[\| x \|]^\sim_\alpha = [| x |, | x |]$.

Define a mapping $T : \mathcal{D} \to X$, $\mathcal{D} = (-1, 1)$ as $Tx = x, \forall x \in \mathcal{D}$. Clearly T is linear and strongly fuzzy bounded. For

$$\frac{\| Tx \|^{\sim 1}_\alpha}{\| x \|^2_\alpha} = \frac{| x |}{\frac{|x|}{\alpha}} = \alpha \leq 1, \forall x \neq \theta \in \mathcal{D} \tag{2.4.2}$$

$$\frac{\| Tx \|^{\sim 2}_\alpha}{\| x \|^1_\alpha} = \frac{| x |}{| x |} = 1, \forall x (\neq \theta) \in \mathcal{D} \tag{2.4.3}$$

From (2.4.2) and (2.4.3), $\| Tx \|^\sim \oslash \| x \| \preceq \bar{1}$, $\forall x (\neq \theta) \in \mathcal{D}$. Hence T is strongly fuzzy bounded. As $R = \max \Rightarrow \lim_{a \to 0^+} R(a, a) = 0$, using Lemma 1.3.26 we get $\overline{\mathcal{D}} = [-1, 1]$. Now define a function $\widetilde{T} : \overline{\mathcal{D}} \to X$ as follows:

$$\widetilde{T}x = \begin{cases} -1 & \text{if } x = -1; \\ 1 & \text{if } x = 1; \\ Tx & \text{if } x (\neq -1, 1) \in \overline{\mathcal{D}}. \end{cases}$$

Then, clearly $\widetilde{T} |_{\mathcal{D}} = T$ and hence $\widetilde{T}$ is an extension of T. Also,

$$\| T \|^{*1}_\alpha = \sup_{x(\neq \theta) \in \mathcal{D}} \frac{\| Tx \|^{\sim 1}_\alpha}{\| x \|^2_\alpha} = \sup_{x(\neq \theta) \in \mathcal{D}} \frac{| x |}{\frac{|x|}{\alpha}} = \alpha \leq 1.$$

$$\| T \|_\alpha^{*2} = \sup_{x(\neq\theta)\in\mathcal{D}} \frac{\| Tx \|_\alpha^{\sim 2}}{\| x \|_\alpha^1} = \sup_{x(\neq\theta)\in\mathcal{D}} \frac{| x |}{| x |} = 1.$$

$$\| \widetilde{T} \|_\alpha^{*1} = \sup_{x(\neq\theta)\in\overline{\mathcal{D}}} \frac{\| \widetilde{T}x \|_\alpha^{\sim 1}}{\| x \|_\alpha^2} = \max\{\frac{\| \widetilde{T}(-1) \|_\alpha^{\sim 1}}{\| (-1) \|_\alpha^2}, \frac{\| \widetilde{T}(1) \|_\alpha^{\sim 1}}{\| (1) \|_\alpha^2}, \sup_{x\in\overline{\mathcal{D}}-\{-1,1\}} \frac{\| \widetilde{T}x \|_\alpha^{\sim 1}}{\| x \|_\alpha^2}\}$$
$$= \max\{\frac{| (-1) |}{\frac{(-1)}{\alpha}}, \frac{| 1 |}{\frac{|1|}{\alpha}}, \| T \|_\alpha^{*1}\} = \max\{\alpha, \alpha, \alpha\} = \alpha \leq 1.$$

$$\| \widetilde{T} \|_\alpha^{*2} = \sup_{x(\neq\theta)\in\overline{\mathcal{D}}} \frac{\| \widetilde{T}x \|_\alpha^{\sim 2}}{\| x \|_\alpha^1} = \max\{\frac{\| \widetilde{T}(-1) \|_\alpha^{\sim 2}}{\| (-1) \|_\alpha^1}, \frac{\| \widetilde{T}(1) \|_\alpha^{\sim 2}}{\| (1) \|_\alpha^1}, \sup_{x\in\overline{\mathcal{D}}-\{-1,1\}} \frac{\| \widetilde{T}x \|_\alpha^{\sim 2}}{\| x \|_\alpha^1}\}$$
$$= \max\{\frac{| (-1) |}{| (-1) |}, \frac{| 1 |}{| 1 |}, \| T \|_\alpha^{*2}\} = \max\{1, 1, 1\} = 1.$$

Hence, $\| T \|_\alpha^{*1} = \| \widetilde{T} \|_\alpha^{*1}$ and $\| T \|_\alpha^{*2} = \| \widetilde{T} \|_\alpha^{*2}$.

Therefore, the fuzzy real numbers $\| T \|^*$ and $\| \widetilde{T} \|^*$ generated by the families of closed intervals $\{[\| T \|_\alpha^{*1}, \| T \|_\alpha^{*2}], \alpha \in (0,1]\}$ and $\{[\| \widetilde{T} \|_\alpha^{*1}, \| \widetilde{T} \|_\alpha^{*2}], \alpha \in (0,1]\}$ respectively are the same, i.e., $\| T \|^* = \| \widetilde{T} \|^*$.

2.4.2 Inverse of a linear operator

In this section, we define the inverse of a linear operator on FNLSs and establish an existence theorem.

Definition 2.4.5. *Let $(X, \| \, . \, \|, L, R)$ and $(Y, \| \, . \, \|, L, R)$ be FNLSs and $T : X \to Y$ be a linear operator. Let $\mathcal{R}(T)$ and $\mathcal{N}(T)$ be the range space and null space of T respectively (definitions are same as their classical counterparts).*
If T is bijective, then inverse of T^{-1} exists and for $A \subseteq Y$, let us define

$$T^{-1}[A \cap \mathcal{R}(T)] = \{x \in X : Tx \in A\}.$$

First we prove the following lemma.

Lemma 2.4.6. *Let T be a XZ-bounded linear operator from an FNLS $(X, \| \, . \, \|, L, R)$ to an FNLS $(Y, \| \, . \, \|, L, R)$ with $\lim_{a\to 0^+} R(a, a) = 0$. Then, the null space $\mathcal{N}(T)$ of T is fuzzy closed.*

Proof. Let $x \in \overline{\mathcal{N}(T)}$. Then, there exists a sequence $\{x_n\}$ in $\mathcal{N}(T)$ such that $\lim_{n\to\infty} x_n = x$. Since T is XZ-bounded, therefore, T is fuzzy norm continuous. Thus, we get $\lim_{n\to\infty} Tx_n = Tx$. Again, for each n, $Tx_n = \theta$ as $x_n \in \mathcal{N}(T)$. Therefore, $Tx = 0$ and hence $x \in \mathcal{N}(T)$. Hence, $\mathcal{N}(T)$ is fuzzy closed. □

Since a strongly fuzzy bounded or weakly fuzzy bounded or fuzzy bounded linear operator is also XZ-bounded, Lemma 2.4.6 is valid for this type of linear operators as well.

Theorem 2.4.7. *Let T be a strongly fuzzy bounded linear operator from a complete FNLS $(X, \| \,.\, \|, L, R)$ to an FNLS $(Y, \| \,.\, \|, L, R)$ with $\lim_{a\to 0^+} R(a,a) = 0$. Suppose that for some $c > 0$, $\| Tx \| \oslash \| x \| \succeq \bar{c}$, $\forall x \in X$. Then the range $\mathcal{R}(T)$ of T is fuzzy closed.*
Further if $\mathcal{R}(T) = Y$, then T is invertible and $\| T^{-1} \|^{*2}_{\alpha} \leq \frac{1}{c}$.

Proof. Let $\{y_n\}$ be a convergent sequence in $\mathcal{R}(T)$ and let $\lim_{n\to\infty} y_n = y$. Corresponding to each y_n, there exists x_n in X such that $Tx_n = y_n$. Therefore, $\lim_{n\to\infty} Tx_n = y$. As

$$\| Tx_n \| \oslash \| x_n \| \succeq \bar{c} \tag{2.4.4}$$

we have:

$$\| Tx_n \|^2_{\alpha} \geq c \| x_n \|^1_{\alpha} \text{ and } \| Tx_n \|^1_{\alpha} \geq c \| x_n \|^2_{\alpha} \tag{2.4.5}$$

As $\{Tx_n\}$ converges, it is a Cauchy sequence. Using the inequality (2.4.5) we get:

$$c \| x_m - x_n \|^2_{\alpha} \leq \| Tx_m - Tx_n \|^{\sim 1}_{\alpha} \leq \| Tx_m - Tx_n \|^{\sim 2}_{\alpha} \tag{2.4.6}$$

This implies that $\lim_{m,n\to\infty} \| x_m - x_n \|^2_{\alpha} = 0$. Therefore $\{x_n\}$ is Cauchy. As X is complete, (x_n) converges in X. Let $\lim_{n\to\infty} x_n = x$, for $x \in X$.
Since T is strongly fuzzy bounded, therefore T is fuzzy norm continuous (Lemma

2.2.19) and we have $\lim\limits_{n\to\infty} Tx_n = Tx$. Hence, $Tx = y$.

Therefore, $y \in \mathcal{R}(T)$ and thus $\mathcal{R}(T)$ is fuzzy closed.

Next, let $\mathcal{R}(T) = Y$, then T is onto. Consider $x_1, x_2 \in X$. Using (2.4.4), we get $\| Tx_1 \|_\alpha^1 \neq \| Tx_2 \|_\alpha^1$ and $\| Tx_1 \|_\alpha^2 \neq \| Tx_2 \|_\alpha^2$. Thus $Tx_1 \neq Tx_2$ and so T is one-one. Hence T is invertible. Let T^{-1} be the inverse of T. We have

$$\| T^{-1} \|_\alpha^{2*} = \sup_{x\in Y, x\neq\theta} \frac{\| T^{-1}x \|_\alpha^2}{\| x \|_\alpha^1} \tag{2.4.7}$$

For $x(\neq \theta) \in Y$, let $T^{-1}x = y \in X \Rightarrow Ty = x$. Using $\| Ty \| \oslash \| y \| \succeq \bar{c}$ (from 2.4.4), we obtain for any $x \in Y$:

$$\frac{\| Ty \|_\alpha^1}{\| y \|_\alpha^2} \geq c \Rightarrow \| y \|_\alpha^2 \leq c^{-1} \| Ty \|_\alpha^1 \Rightarrow \| T^{-1}x \|_\alpha^2 \leq c^{-1} \| x \|_\alpha^{\sim 1}$$

Therefore, we get: $\sup\limits_{x\in Y, x\neq\theta} \dfrac{\| T^{-1}x \|_\alpha^2}{\| x \|_\alpha^1} \leq c^{-1}$ and hence $\| T^{-1} \|_\alpha^{2*} \leq c^{-1}$. □

Corollary 2.4.8. *Let T be a weakly fuzzy bounded linear operator from a complete FNLS $(X, \| \,.\, \|, L, R)$ to an FNLS $(Y, \| \,.\, \|, L, R)$ with $\lim\limits_{a\to 0^+} R(a,a) = 0$. Suppose there is a fuzzy real number $\eta > \bar{0}$ such that $\| Tx \| \oslash \| x \| \succeq \eta$. Then the range $\mathcal{R}(T)$ of T is fuzzy closed. Also if $\mathcal{R}(T) = Y$, then T is invertible and $\| T^{-1} \|_\alpha^{*2} \leq \frac{1}{\eta_\alpha^1}$.*

Proof. Proceeding as in Theorem 2.4.7, for any convergent sequence $\{y_n\}$ in $\mathcal{R}(T)$, there exists a sequence $x_n \in X$ such that $Tx_n = y_n$. Therefore, $\lim\limits_{n\to\infty} Tx_n = y$, for $y \in Y$. Using similar argument as above, $\{x_n\}$ is Cauchy. As X is complete, $\{x_n\}$ converges in X. Let $\lim\limits_{n\to\infty} x_n = x$, for $x \in X$.

Since T is weakly fuzzy bounded, T is fuzzy norm continuous (Corollary 2.2.21) and therefore $\lim\limits_{n\to\infty} Tx_n = Tx$. It gives $Tx = y$ and hence $y \in \mathcal{R}(T)$.

Thus, $\mathcal{R}(T)$ is fuzzy closed.

Next, let $\mathcal{R}(T) = Y$, then T is onto and one one, similar to Theorem 2.4.7. Hence T is invertible. Consider T^{-1} be the inverse of T. We have

$$\| T^{-1} \|_\alpha^{2*} = \sup_{y\in Y, y\neq\theta} \frac{\| T^{-1}y \|_\alpha^2}{\| y \|_\alpha^1} \tag{2.4.8}$$

For $y \in Y$, let $T^{-1}y = x \in X \Rightarrow Tx = y$. Using

$$\parallel Tx \parallel \oslash \parallel x \parallel \succeq \eta, \tag{2.4.9}$$

For any $y \in Y$ we obtain:

$$\frac{\parallel Tx \parallel_\alpha^1}{\parallel x \parallel_\alpha^2} \geq \eta_\alpha^1 \Rightarrow \parallel x \parallel_\alpha^2 \leq \frac{1}{\eta_\alpha^1} \parallel Tx \parallel_\alpha^1 \Rightarrow \parallel T^{-1}y \parallel_\alpha^2 \leq \frac{1}{\eta_\alpha^1} \parallel y \parallel_\alpha^{\sim 1}, \tag{2.4.10}$$

Therefore, $\sup\limits_{y \in Y, y \neq \theta} \dfrac{\parallel T^{-1}y \parallel_\alpha^2}{\parallel y \parallel_\alpha^1} \leq \dfrac{1}{\eta_\alpha^1}$ and hence $\parallel T^{-1} \parallel_\alpha^{2*} \leq \frac{1}{\eta_\alpha^1}$. □

Similar results can be proved for fuzzy bounded linear operators. The proofs are straightforward and follows from Theorem 2.4.7 and Corollary 2.4.8.

Corollary 2.4.9. *Let T be a fuzzy bounded linear operator from a complete FNLS $(X, \parallel . \parallel, L, R)$ to an FNLS $(Y, \parallel . \parallel, L, R)$ with $\lim\limits_{a \to 0^+} R(a, a) = 0$. Suppose there is a fuzzy real number $\eta > \bar{0}$ such that $\parallel Tx \parallel \succeq \eta \odot \parallel x \parallel$. Then the range $\mathcal{R}(T)$ of T is fuzzy closed. Also if $\mathcal{R}(T) = Y$, then T is invertible and $\parallel T^{-1} \parallel_\alpha^{*2} \leq \frac{1}{\eta_\alpha^2}$.*