Chapter 3

Fuzzy compact operators in fuzzy normed linear space

3.1 Introduction

The application of compact operators on normed linear spaces such as Hilbert spaces is well known. Compact operators are particularly useful for dealing with integral equations and solving various problems of mathematical physics due to their close resemblance with the operators on finite dimensional spaces [42]. Application of spectral theory of compact operators have been widespread; these are found in statistical mechanics, partial differential equations, fluid mechanics, kinetic theory (Kramers-Fokker-Planck Operator) [32] and so on. Since fuzzy mathematics, fuzzy physics and related fields are constantly developing on the lines of their classical counterparts, the study of fuzzy compact operators on FNLS, therefore, carries high significance.

The rapid advances in linear theory of fuzzy normed spaces and fuzzy bounded linear operators naturally leads to the development of fuzzy compact operator theory. In 2004, Xiao and Zhu defined a fuzzy compact operator as follows:

Definition 3.1.1. [67] *Let* $(X, \| \, . \, \|, L, R)$ *and* $(Y, \| \, . \, \|, L, R)$ *be FNLSs and* $T : X \to Y$ *be a linear operator. Then* T *is said to be a fuzzy compact operator if* T *is*

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a fuzzy norm continuous operator and for each fuzzy bounded set $A \subset X$, $\overline{T(A)}$ is a fuzzy compact set of Y.

Sadeqi and Salehi [60] argued that the requirement of continuity in the above definition is not necessary. For, if $\lim\limits_{a\to 0^+} R(a,a) = 0$, then for a fuzzy compact operator T, the set $\overline{T(A)}$ is a fuzzy bounded set in Y (refer to Theorem 1.3.27 and Theorem 1.3.28). Hence, T maps a fuzzy bounded set A in X to a fuzzy bounded set $T(A)$ in Y. Therefore, the fuzzy compact operator T is XZ-bounded. As every XZ-bounded operator is also fuzzy norm continuous (Theorem 2.2.14), the fuzzy compact operator T is itself fuzzy norm continuous.

Let us give some examples of fuzzy compact operators in FNLSs.

Example 3.1.2. Consider two normed linear spaces $(X, \parallel . \parallel)$ and $(Y, \parallel . \parallel^{\sim})$ and let $T : X \to Y$ be a compact operator. Let $\parallel . \parallel_N$ and $\parallel . \parallel_N^{\sim}$ be the induced fuzzy norms on X and Y which are defined as follows:

$$\parallel x \parallel_N (t) = \begin{cases} 1, & t = \parallel x \parallel, \\ 0, & t \neq \parallel x \parallel; \end{cases}$$

and

$$\parallel y \parallel_N^{\sim} (t) = \begin{cases} 1, & t = \parallel y \parallel^{\sim}, \\ 0, & t \neq \parallel y \parallel^{\sim}. \end{cases}$$

Then $T : (X, \parallel . \parallel_N) \to (Y, \parallel . \parallel_N^{\sim})$ is a fuzzy compact operator.

Example 3.1.3. Let $C[0,1]$ be the set of all real valued functions on $[0,1]$ with the fuzzy norm:

$$\parallel f \parallel (t) = \begin{cases} 1, & t = \parallel f \parallel_{\sup}; \\ 0, & \text{otherwise.} \end{cases}$$

where $\parallel f \parallel_{\sup} = \sup_{x\in(0,1]} \mid f(x) \mid$, for any $f \in C[0,1]$.

If $K(x,y)$ is a real valued continuous function for $x, y \in (0,1]$, then the operator

$T : C[0,1] \to C[0,1]$ defined by:

$$(Tf)(x) = \int_0^1 K(x,y)f(y)dx,$$

where $f \in C[0,1]$ is a fuzzy compact operator.

Xiao and Zhu [67], Sadeqi and Salehi [60] have studied topological degree theory for fuzzy compact operators. In 2006, Lael and Nourouzi [43] studied some fundamental properties of fuzzy compact operators in FNLSs given by Bag and Samanta [3]. However, a careful examination of the development of fuzzy compact operators in fuzzy spaces indicates several theoretical gaps unlike their classical counterparts. In this chapter, we provide several new properties of fuzzy compact operators which are useful for the study of fuzzy compact operators.

3.2 Fuzzy compact operators and fuzzy boundedness

One of the basic properties of compact operators in classical space is its boundedness. Every compact operator is bounded in a normed linear space [42]. But in case of FNLSs, an adequate analogue of this result with respect to different notions of fuzzy boundedness do not appear to be very straightforward. In this section, we examine the relation between fuzzy compactness and fuzzy boundedness of linear operators on FNLSs.

Definition 3.2.1. *A sequence $\{x_n\}$ in an FNLS $(X, \| \, . \, \|, L, R)$ is said to be fuzzy bounded if $\{x_n\}$ is a fuzzy bounded subset in X.*

Theorem 3.2.2. *Let $(X, \| \, . \, \|, L, R)$ and $(Y, \| \, . \, \|, L, R)$ be FNLSs with $\lim_{a \to 0^+} R(a,a) = 0$. Let T be a linear operator from X to Y. Then T is fuzzy compact if and only if it maps every fuzzy bounded sequence $\{x_n\}$ in X to a sequence $\{Tx_n\}$ in Y which has a fuzzy convergent subsequence.*

Proof. Consider T to be a fuzzy compact operator and $\{x_n\}$ be a fuzzy bounded sequence in X. Then, the fuzzy closure $\overline{\{T(x_n)\}}$ is fuzzy compact. Therefore, $\{T(x_n)\}$ is sequentially compact (by Theorem 1.3.27) and so the sequence $\{T(x_n)\}$ has a fuzzy convergent subsequence.

Conversely, let $A \subset X$ be a fuzzy bounded set. Let $\{x_n\}$ be a sequence in $\overline{T(A)}$. Then there exists a sequence $\{y_n\}$ in $T(A)$ such that $\lim_{n\to\infty} \parallel x_n - y_n \parallel = \bar{0}$. Let $y_n = Tz_n$ for each n, where $z_n \in A$. Since A is fuzzy bounded, therefore $\{z_n\}$ is a fuzzy bounded sequence. By the hypothesis, $\{Tz_n\}$, i.e., $\{y_n\}$ has a fuzzy convergent subsequence, say $\{y_{n_k}\}$. Let $\lim_{n_k\to\infty} y_{n_k} = y$ for some $y \in Y$.

Now, using Lemma 1.3.19, for any $\alpha \in (0,1]$ there exists $\beta \in (0,\alpha]$ such that,

$$\parallel x_{n_k} - y \parallel_\alpha^2 = \parallel x_{n_k} - y_{n_k} + y_{n_k} - y \parallel_\alpha^2 \leq \parallel x_{n_k} - y_{n_k} \parallel_\beta^2 + \parallel y_{n_k} - y \parallel_\beta^2$$

Since $\lim_{n_k\to\infty} \parallel x_{n_k} - y_{n_k} \parallel_\beta^2 = \lim_{n_k\to\infty} \parallel y_{n_k} - y \parallel_\beta^2 = 0$, it gives
$\lim_{n_k\to\infty} \parallel x_{n_k} - y \parallel_\alpha^2 = 0$, i.e., $\lim_{n_k\to\infty} x_{n_k} = y$.

Thus $\{x_{n_k}\}$ is a fuzzy convergent subsequence of $\{x_n\}$. Therefore $\overline{T(A)}$ is fuzzy compact (Theorem 1.3.27). Hence T is a fuzzy compact operator. □

Lemma 3.2.3. *In an FNLS X with $R \leq \max$, every convergent sequence is fuzzy bounded.*

Proof. Let $\{x_n\}$ be a convergent sequence in X and let $\lim_{n\to\infty} x_n = x$ for $x(\neq \theta) \in X$. Then given any $\varepsilon > 0$ and $\alpha \in (0,1]$, there exists $n_\circ(\alpha)$ such that $\parallel x_n - x \parallel_\alpha^2 < \varepsilon$, $\forall n \geq n_\circ$. This gives

$\parallel x_n \parallel_\alpha^2 \leq \parallel x_n - x \parallel_\alpha^2 + \parallel x \parallel_\alpha^2 < \varepsilon + \parallel x \parallel_\alpha^2$, $\forall \alpha \in (0,1]$ and $n \geq n_\circ$.

Let $M_\alpha = \max\{\varepsilon + \parallel x \parallel_\alpha^2, \parallel x_1 \parallel_\alpha^2, \parallel x_2 \parallel_\alpha^2, ..., \parallel x_{n_\circ} \parallel_\alpha^2\}$. Then, $\parallel x_n \parallel_\alpha^2 \leq M_\alpha$, for all n. Thus $\{x_n\} \subseteq N(M_\alpha, \alpha)$ and so $\{x_n\}$ is a fuzzy bounded subset of X. □

We now have the the following result:

Theorem 3.2.4. *Let* $(X, \| \,.\, \|, L, R)$ *and* $(Y, \| \,.\, \|, L, R)$ *be FNLSs with* $R \leq \max$. *Then a linear operator* T *from* X *to* Y *is fuzzy compact if and only if it maps every convergent sequence* $\{x_n\}$ *in* X *to a sequence* $\{Tx_n\}$ *in* Y *which has a fuzzy convergent subsequence.*

Proof. Follows from Theorem 3.2.2 and Lemma 3.2.3. □

Thus, the fuzzy compactness of a linear operator can be interpreted with the help of fuzzy bounded sequence or convergent sequence.
Saheli et al. showed that a fuzzy compact operator may not be fuzzy bounded in general ([62], Example 6.1). However, since a fuzzy compact operator T is always XZ-bounded, we have the following results.

Theorem 3.2.5. *Let* $(X, \| \,.\, \|, L, R)$ *and* $(Y, \| \,.\, \|, L, R)$ *be FNLSs with* $\lim_{a \to 0^+} R(a, a) = 0$ *and let* $T : X \to Y$ *be fuzzy compact. If for each* $\alpha \in (0, 1]$ *there exists* $M_\alpha > 0$ *such that* $\frac{\|x\|_\alpha^2}{\|x\|_\alpha^1} \leq M_\alpha$ *and* $\sup_{\alpha \in (0,1]} \| x \|_\alpha^2 < +\infty$, $\sup_{\alpha \in (0,1]} \| y \|_\alpha^2 < +\infty$, *for all* $x \in X$ *and* $y \in Y$. *Then* T *is weakly fuzzy bounded.*

Theorem 3.2.6. *Let* $(X, \| \,.\, \|, L, R)$ *and* $(Y, \| \,.\, \|, L, R)$ *be FNLSs with* $\lim_{a \to 0^+} R(a, a) = 0$ *and* $T : X \to Y$ *be fuzzy compact. If for each* $\alpha \in (0, 1]$ *there exists* $M_\alpha > 0$ *such that* $\frac{\|x\|_\alpha^2}{\|x\|_\alpha^1} \leq M_\alpha$ *and* $\sup_{\alpha \in (0,1]} \| x \|_\alpha^2 < +\infty$, $\sup_{\alpha \in (0,1]} \| y \|_\alpha^2 < +\infty$, *for all* $x \in X$ *and* $y \in Y$. *Then* T *is fuzzy bounded.*

Since a fuzzy compact operator is XZ-bounded, proofs of the above results follow directly from Theorems 2.2.27 and 2.2.28. Thus, under certain conditions on the fuzzy norm, a fuzzy compact operator can be made bounded.

Theorem 3.2.7. *Let* $(X, \| \,.\, \|, L, R)$ *and* $(Y, \| \,.\, \|, L, R)$ *be FNLSs with* $\lim_{a \to 0^+} R(a, a) = 0$. *Let* $T : X \to Y$ *be a* XZ*-bounded operator. If* Y *is finite dimensional then* T *is fuzzy compact.*

Proof. Consider a fuzzy bounded subset A of X. Since T is XZ-bounded, therefore $T(A)$ is fuzzy bounded. It implies that the set $\overline{T(A)}$ is fuzzy bounded as well as fuzzy closed. Since Y is finite dimensional, therefore the set $\overline{T(A)}$ is fuzzy compact (using Theorem 1.3.29). Hence T is fuzzy compact. □

Corollary 3.2.8. *Let $T : X \to Y$ be strongly fuzzy bounded linear operator. If Y is finite dimensional then T is fuzzy compact.*

Proof. Since a strongly fuzzy bounded operator is XZ-bounded (Corollary 2.2.20), therefore the proof follows from Theorem 3.2.7. □

Similarly, we have the following corollaries:

Corollary 3.2.9. *Let $T : X \to Y$ be a weakly fuzzy bounded linear operator. If Y is finite dimensional then T is fuzzy compact.*

Corollary 3.2.10. *Let $T : X \to Y$ be a fuzzy bounded linear operator. If Y is finite dimensional then T is fuzzy compact.*

Bag and Samanta [4] proved that every linear operator T from a finite dimensional FNLS X to an FNLS Y is always weakly fuzzy bounded (weakly fuzzy continuous).

Corollary 3.2.11. *Let $T : X \to Y$ be a linear operator. If X is finite dimensional then T is fuzzy compact.*

Proof. Since X is finite dimensional, therefore T is weakly fuzzy bounded. However, as $\dim T(X) < \dim X$, therefore $T(X)$ is also finite dimensional. Thus, by Corollary 3.2.9, T is a fuzzy compact operator. Hence a linear operator T form a finite dimensional FNLS X to Y is fuzzy compact. □

Theorem 3.2.12. *Let $(X, \| \, . \, \|, L, R)$ be a FNLS with $\lim_{a \to 0^+} R(a, a) = 0$. Let $T : X \to X$ be a fuzzy compact operator and $S : X \to X$ be a strongly fuzzy bounded linear operator. Then ST and TS are fuzzy compact operators.*

Proof. Let $\{x_n\}$ be a fuzzy bounded sequence in X. As T is a fuzzy compact operator, $\{Tx_n\}$ has a fuzzy convergent subsequence $\{Tx_{n_k}\}$ (using Theorem 3.2.2).
Let $\lim_{n_k \to \infty} Tx_{n_k} = y$. Since S is strongly fuzzy bounded, therefore, S is also fuzzy norm continuous (using Lemma 2.2.19). Hence, $\lim_{n_k \to \infty} S(Tx_{n_k}) = Sy$. Thus, $(ST)\{x_n\}$ has a fuzzy convergent subsequence and so, ST is fuzzy compact.
Again, for a fuzzy bounded sequence $\{x_n\} \in X$, for each $\alpha \in (0, 1]$, there exists $M_\alpha > 0$ such that $\| x_n \|_\alpha^2 < M_\alpha$, for all n. Since S is strongly fuzzy bounded, there exists $k > 0$ such that $\| Sx_n \|_\alpha^2 \leq k \| x_n \|_\alpha^1$ for all n and for $\alpha \in (0, 1]$.
Thus, we have

$$\| Sx_n \|_\alpha^2 \leq k \| x_n \|_\alpha^1 \leq k \| x_n \|_\alpha^2 < kM_\alpha$$

which implies $\{Sx_n\}$ is fuzzy bounded sequence. As T is a fuzzy compact operator, therefore, $\{T(Sx_n)\}$ has a fuzzy convergent subsequence. This implies that $(TS)\{x_n\}$ has a fuzzy convergent subsequence and hence TS is a fuzzy compact operator. □

Remark 3.2.13. Since every weakly fuzzy bounded or fuzzy bounded or XZ-bounded operator is also fuzzy norm continuous, Theorem 3.2.12 also holds for a weakly fuzzy bounded or a fuzzy bounded or an XZ-bounded operator.

3.3 Fuzzy compact operators and finite dimensional FNLSs

Let us study fuzzy compact operators in the setting of finite dimensional FNLSs.

3.3.1 Riesz's Lemma

In functional analysis, Riesz's Lemma is a source of many interesting results [42]. The following result is a generalized form of Riesz's Lemma to the FNLSs with the general right norm.

Lemma 3.3.1. *Let* $(X, \parallel . \parallel, L, R)$ *with* $\lim_{a\to 0^+} R(a,a) = 0$ *be an FNLS and* Y, Z *be two subspaces of* X*. If* Y *is a fuzzy closed proper subset of* Z*, then for every real number* $\theta \in (0,1)$ *and each* $\alpha \in (0,1]$*, there exists* $z \in Z$ *such that* $\parallel z \parallel_\alpha^2 < 1$ *and* $\parallel z - y \parallel_\alpha^2 \geq \theta$ *for all* $y \in Y$*.*

Proof. Since Y is a proper subset of Z, there exists an element $v \in Z - Y$. Consider $d = \inf_{y\in Y} \sup_{\alpha\in(0,1]} \parallel v - y \parallel_\alpha^2$. Obviously $d \geq 0$ (using Remark 1.3.7). We show that $d > 0$.

For, if $d = 0$, i.e., $\inf_{y\in Y} \sup_{\alpha\in(0,1]} \parallel v - y \parallel_\alpha^2 = 0$.

$\Rightarrow$ for $\alpha \in (0,1], \exists y \in Y$ such that $\sup_{\alpha\in(0,1]} \parallel v - y \parallel_\alpha^2 < \alpha$

$\Rightarrow \parallel v - y \parallel_\alpha^2 < \alpha$, for each $\alpha \in (0,1]$

$\Rightarrow \parallel v - y \parallel_\alpha^2 = \mid -1 \mid \parallel y - v \parallel_\alpha^2 < \alpha$, i.e., $y - v \in N(\alpha, \alpha)$,

which gives $y \in v + N(\alpha,\alpha)$, for any $\alpha \in (0,1]$. Thus, $\{v + N(\alpha,\alpha)\} \cap Y \neq \phi$ and so v is a point of closure of Y, implies $v \in \overline{Y}$. Since Y is fuzzy closed, therefore, $\overline{Y} = Y$. Hence, $v \in Y$, which is a contradiction. Therefore $d > 0$.

Consider $\theta \in (0,1)$. There exists $y_\circ \in Y$ such that

$$d \leq \sup_{\alpha\in(0,1]} \parallel v - y_\circ \parallel_\alpha^2 < k' \leq \frac{d}{\theta}$$

Therefore, for any $\alpha \in (0,1]$

$$d \leq \parallel v - y_\circ \parallel_\alpha^2 < k' \leq \frac{d}{\theta} \tag{3.3.1}$$

Consider $z \in Z$ such that $z = \frac{v - y_\circ}{k'}$. Therefore, $\parallel z \parallel_\alpha^2 = \frac{1}{k'} \parallel v - y_\circ \parallel_\alpha^2 < 1$, using (3.3.1) Now for any $y \in Y$, we have

$$\parallel z - y \parallel_\alpha^2 = \parallel \frac{v - y_\circ}{k'} - y \parallel_\alpha^2 = \frac{1}{k'} \parallel v - y_\circ - k'y \parallel_\alpha^2 = \frac{1}{k'} \parallel v - y_1 \parallel_\alpha^2$$

where $y_1 = y_\circ + k'y \in Y$. Therefore $\parallel v - y_1 \parallel_\alpha^2 \geq d$. Then,

$$\parallel z - y \parallel_\alpha^2 = \frac{1}{k'} \parallel v - y_1 \parallel_\alpha^2 \geq \frac{d}{k'} \geq \theta$$

Hence, $\| z - y \|_\alpha^2 \geq \theta$, for any $y \in Y$. $\square$

Using Lemma 3.3.1, we prove some properties of fuzzy compact operators in a finite dimensional FNLS.

Let $(X, \| \,.\, \|, L, R)$ be an FNLS and $R \leq \max$. Consider the set $\underline{N}(1, \alpha) = \{x \in X : \| x \|_\alpha^2 \leq 1\}$, for $\alpha \in (0, 1]$. Fang and Song [64] proved the following:

Lemma 3.3.2. *Let $(X, \| \,.\, \|, L, R)$ be an FNLS with $R \leq \max$. Then, for each $\alpha \in (0, 1]$ $\underline{N}(1, \alpha)$ is a fuzzy closed, convex and absorbing set.*

Remark 3.3.3. Xiao and Zhu [65] showed that a finite n-dimensional FNLS $(X, \| \,.\, \|, L, R)$ is linearly homomorphic to the n-dimensional FNLS $(E^n, \| \,.\, \|, L, R)$ induced by a classical n-dimensional normed linear space $(E^n, \| \,.\, \|^*)$. In this case, a set A in X is fuzzy bounded, fuzzy closed, precompact in $(E^n, \| \,.\, \|, L, R)$ if and only if A is bounded, closed, precompact in $(E^n, \| \,.\, \|^*)$.

In the finite dimensional normed linear space $(E^n, \| \,.\, \|^*)$, the closed unit ball $= \{x \in X : \| x \|^* \leq 1\}$ is bounded. In the FNLS $(E^n, \| \,.\, \|, L, R)$, for each $\alpha \in (0, 1]$,

$$\underline{N}(1, \alpha) = \underline{N}(1, 1) = \{x \in X : \| x \|^* \leq 1\}$$

Thus, the set $\underline{N}(1, \alpha)$ is fuzzy bounded $(E^n, \| \,.\, \|, L, R)$.

This leads to the natural question: whether or not the set $\underline{N}(1, \alpha)$ in $(X, \| \,.\, \|, L, R)$ is fuzzy bounded. We have the following:

Lemma 3.3.4. *Let $(X, \| \,.\, \|, L, R)$ be an FNLS with $R \leq \max$. If $\lim\limits_{\alpha \to 0^+} \| x \|_\alpha^2 < +\infty$ for $x\ (\neq \theta) \in X$, then $\underline{N}(1, \alpha)$ is fuzzy bounded for each $\alpha \in (0, 1]$.*

Proof. Consider the set $\underline{N}(1, \alpha)$ for a fixed $\alpha \in (0, 1]$. Let $\beta \in (0, 1]$ be arbitrary. For $\alpha \leq \beta$, we have $\| x \|_\beta^2 \leq \| x \|_\alpha^2 \leq 1$. Then $\underline{N}(1, \alpha) \subseteq N(1, \beta)$.

Assume $\beta < \alpha$. Since $\lim\limits_{\alpha \to 0^+} \| x \|_\alpha^2 < +\infty$, for any $x \in X$, For any β, $\exists \alpha_\beta \in (0, \beta]$ such that $\| x \|_{\alpha_\beta}^2 < +\infty$.

It gives: $\| x \|_{\alpha_\beta}^2 < M_{\alpha_\beta}$, for some $M_{\alpha_\beta} > 0$. Thus, we have

$$\| x \|_\beta^2 \leq \| x \|_{\alpha_\beta}^2 < M_{\alpha_\beta}$$

and, hence $x \in N(M_{\alpha_\beta}, \beta)$. Therefore, $\underline{N}(1,\alpha) \subseteq N(M_{\alpha_\beta}, \beta)$, for any $\beta < \alpha$. Hence, $\underline{N}(1,\alpha)$ is fuzzy bounded, for each $\alpha \in (0,1]$. □

Theorem 3.3.5. *Let $(X, \| \,.\, \|, L, R)$ be a finite dimensional FNLS with $R \leq \max$. If $\lim_{\alpha \to 0^+} \| x \|_\alpha^2 < +\infty$ for $x(\neq \theta) \in X$, then $\underline{N}(1,\alpha)$ is fuzzy compact set.*

Proof. Since the set $\underline{N}(1,\alpha)$ is fuzzy bounded (by Lemma 3.3.4) and fuzzy closed (using Lemma 3.3.2), therefore $\underline{N}(1,\alpha)$ is fuzzy compact whenever X is finite dimensional. □

Theorem 3.3.6. *If the set $\underline{N}(1,\alpha)$ is fuzzy compact, for any $\alpha \in (0,1]$ in an FNLS $(X, \| \,.\, \|, L, R)$, with $R \leq \max$, then X is finite dimensional.*

Proof. Let the set $\underline{N}(1,\alpha_\circ)$ be fuzzy compact for some $\alpha_\circ \in (0,1]$. Let, if possible X be infinite dimensional. Let $x_1 \in X$ such that $\| x_1 \|_{\alpha_\circ}^2 \leq 1$. Consider the subspace X_1 of X generated by x_1. Since $\dim X_1 = 1$ and as every finite dimensional FNLS is fuzzy closed (Felbin [20], Theorem 4.1), so X_1 is fuzzy closed proper subspace of X.

Therefore, using Lemma 3.3.1, there exists $x_2 \in X$ such that

$\| x_2 \|_{\alpha_\circ}^2 \leq 1$ and $\| x_2 - x_1 \|_{\alpha_\circ}^2 \geq \frac{1}{2}$.

Then, following the argument as in the classical case [42] and proceeding inductively, we obtain a sequence $\{x_n\}$ in $\underline{N}(1,\alpha_\circ)$ such that for a $\alpha_\circ \in (0,1]$, $\| x_m - x_n \|_{\alpha_\circ}^2 \geq \frac{1}{2}$. So $\{x_n\}$ cannot have a convergent subsequence. It contradicts the compactness of $\underline{N}(1,\alpha_\circ)$. Hence the FNLS X must be finite dimensional. □

Theorem 3.3.7. *Let $(X, \| \,.\, \|, L, R)$ be an FNLS, $R \leq \max$. If $\lim_{\alpha \to 0^+} \| x \|_\alpha^2 < +\infty$ for $x(\neq \theta) \in X$, then X is finite dimensional if and only if the set $\underline{N}(1,\alpha)$ is fuzzy compact, for any $\alpha \in (0,1)$.*

Proof. The proof directly follows from Theorems 3.3.5 and 3.3.6. □

Remark 3.3.8. This result is important in its own way as it characterizes the finite dimensionality of FNLS in terms of the set $\underline{N}(1,\alpha)$.

Theorem 3.3.9. *In an infinite dimensional FNLS* $(X, \| \, . \, \|$, *with* $R \leq \max$, *for which* $\lim_{\alpha \to 0^+} \| x \|_\alpha^2 < +\infty$, *for all* $x (\neq \theta) \in X$ *and* $\alpha \in (0,1]$, *the identity operator* $T : X \to X$ *is not fuzzy compact.*

Proof. The identity operator T maps the fuzzy bounded set $\underline{N}(1,\alpha)$ into itself. Assume T to be fuzzy compact. Therefore, $\overline{T\underline{N}(1,\alpha)} = \overline{\underline{N}(1,\alpha)}$ is fuzzy compact. Since $\underline{N}(1,\alpha)$ is fuzzy closed, we have $\overline{\underline{N}(1,\alpha)} = \underline{N}(1,\alpha)$. Thus, $\underline{N}(1,\alpha)$ is fuzzy compact. Hence X is finite dimensional (using Theorem 3.3.7), which is a contradiction. Therefore, the identity operator T is not a fuzzy compact operator. □

3.3.2 Range of a fuzzy compact operator

Xiao and Zhu [65] proved that in an FNLS X, with $\lim_{a \to 0^+} R(a,a) = 0$, if $A \subseteq X$ is precompact, then A is separable. Therefore, $A \subseteq X$ is compact $\Rightarrow$ A is sequentially compact (using Theorem 1.3.27) $\Rightarrow$ A is precompact (using Theorem 1.3.28) $\Rightarrow$ A is separable. Thus, every compact set A in X is also separable.

The range of a compact operator in a normed space is always separable. In case of FNLSs, we prove the following result.

Theorem 3.3.10. *Let* $(X, \| \, . \, \|, L, R)$ *be a FNLS with* $R \leq \max$ *and* $\lim_{\alpha \to 0^+} \| x \|_\alpha^2 < +\infty$, *for* $x \in X$ *and* $\alpha \in (0,1]$. *Then, the range* $R(T)$ *of a fuzzy compact operator* $T : X \to X$ *is separable.*

Proof. Consider the set $V_{\alpha n} = \{x \in X : \| x \|_\alpha^2 \leq n\}$, where $\alpha \in (0,1]$ and $n \in \mathbb{N}$. By $(F2)$, $V_{\alpha n}$=n $\underline{N}(1,\alpha)$. Therefore, $V_{\alpha n}$ is fuzzy bounded for each $\alpha \in (0,1]$ and $n \in \mathbb{N}$. Since T is fuzzy compact, therefore, $T(V_{\alpha n})$ is relatively compact.

Hence, each $T(V_{\alpha n})$ is separable.

Now, let $x(\neq \theta) \in X$. For each $0 < \alpha \leq 1$, we can find an $n \in \mathbb{N}$, sufficiently large, such that $\frac{1}{n} \leq \alpha$. As $\| \,.\, \|_\alpha^2$ is monotonic decreasing for $\alpha \in (0,1]$ and $\lim\limits_{\alpha \to 0^+} \| x \|_\alpha^2 < +\infty$; we can obtain that $\| x \|_\alpha^2 \leq \| x \|_{\frac{1}{n}}^2 \leq n$. Hence, $x \in V_{\frac{1}{n}n}$. It gives that $X = \bigcup\limits_{n=1}^{\infty} V_{\frac{1}{n}n}$. Hence $T(X) = \bigcup\limits_{n=1}^{\infty} T(V_{\frac{1}{n}n})$.

As each $T(V_{\frac{1}{n}n})$ is separable, it has a countable dense subset, say, D_n. Let $D = \bigcup\limits_{n=1}^{\infty} D_n$. Clearly D is countable. Since each $T(V_{\frac{1}{n}n}) \subseteq \overline{D_n}$, therefore,

$$\bigcup_{n=1}^{\infty} T(V_{\frac{1}{n}n}) \subseteq \bigcup_{n=1}^{\infty} \overline{D_n}$$

Thus $T(X) = R(T) \subseteq \overline{D}$. Hence, D is dense in $R(T)$ and $R(T)$ is separable. □

This result allows us to deduce a very interesting result in a finite dimensional FNLS.

Theorem 3.3.11. *Let $(X, \| \,.\, \|, L, R)$ be a finite dimensional FNLS with $R \leq \max$ and $\lim\limits_{\alpha \to 0^+} \| x \|_\alpha^2 < +\infty$, for $x \in X$ and $\alpha \in (0,1]$. Then, the range $R(T)$ of a linear operator $T : X \to X$ is separable.*

Proof. As X is finite dimensional, therefore, by Corollary 3.2.11, the linear operator T is fuzzy compact. The result now follows directly from Theorem 3.3.10. □

3.4 Space of all fuzzy compact operators

In Chapter 2, we described the spaces of different bounded linear operators in the fuzzy setting. In this section, we study the space of all fuzzy compact operators with respect to the spaces of XZ-bounded, weakly fuzzy bounded and fuzzy bounded operators. The choice of right norm R may vary according to the notion of fuzzy boundedness.

Let $C(X, Y)$ be the set of all fuzzy compact operators from an FNLS $(X, \| \,.\, \|, L_1, R_2)$ to an FNLS $(Y, \| \,.\, \|, L_2, R_2)$.

Theorem 3.4.1. *Let X and Y be FNLSs such that $\lim_{a\to 0^+} R_i(a,a) = 0$ for $i = 1, 2$. Then the set $C(X,Y)$ is a linear subspace of $B^b(X,Y)$, the space of all XZ-bounded linear operators from X to Y.*

Proof. Consider the linear space $B^b(X,Y)$ of all XZ-bounded linear operators from X to Y. The linear operations on $B^b(X,Y)$ are defined by:

$(T_1 + T_2)x = T_1x + T_2x$ and $(rT)x = r(Tx)$

where $T_1, T_2 \in B^b(X,Y)$, $r \in (-\infty, +\infty)$, $x \in X$. Since every fuzzy compact operator is XZ-bounded, therefore, $C(X,Y) \subset B^b(X,Y)$.

Let T_1 and $T_2 \in C(X,Y)$ and $\{x_n\}$ be a fuzzy bounded sequence in X. Since T_1 is fuzzy compact, $\{T_1x_n\}$ has a fuzzy convergent subsequence $\{T_1x_{n_k}\}$. Then the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is also fuzzy bounded. As T_2 is fuzzy compact, the sequence $\{T_2x_{n_k}\}$ has a fuzzy convergent subsequence $\{T_2z_n\}$.

Hence, (T_1z_n) and (T_2z_n) are fuzzy convergent subsequences.

Let $\lim_{n\to\infty} T_1z_n = u$ and $\lim_{n\to\infty} T_2z_n = v$, where $u, v \in Y$. Using Lemma 1.3.20, for any $\alpha \in (0,1]$, there is $\beta \in (0,\alpha]$ such that

$$\| (T_1+T_2)(z_n)-(u+v) \|_\alpha^2 = \| (T_1z_n-u)+(T_2z_n-v) \|_\alpha^2 \leq \| T_1z_n-u \|_\beta^2 + \| T_2z_n-v \|_\beta^2$$

Letting $n \to \infty$, we get:

$\| (T_1 + T_2)(z_n) - (u + v) \|_\alpha^2 \to 0.$

It gives $\lim_{n\to\infty}(T_1 + T_2)(z_n) = u + v$. Hence, $\{(T_1 + T_2)(x_n)\}$ has a fuzzy convergent subsequence $\{(T_1 + T_2)(z_n)\}$, where $\{x_n\}$ is a fuzzy bounded sequence in X. Thus, $T_1 + T_2$ is a fuzzy compact operator and $T_1 + T_2 \in C(X,Y)$.

Next, for $T \in C(X,Y)$ and fuzzy bounded sequence $\{x_n\}$, $\{Tx_n\}$ has a fuzzy convergent subsequence $\{Tx_{n_k}\}$. Let $\lim_{n_k\to\infty} Tx_{n_k} = y$, for some $y \in Y$. Using $(F2)$ (Definition 1.3.11), we get:

$\| (tT)x_{n_k} - ty \|_\alpha^2 = | t | \| Tx_{n_k} - y \|_\alpha^2$, for any $t > 0$

Letting $n_k \to \infty$, we have, $\lim_{n_k\to\infty}(tT)x_{n_k} = ty$. Hence, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$

so that $tT(x_{n_k})$ converges. Thus, $tT \in C(X,Y)$ for any $T \in C(X,Y)$ and $t > 0$. Therefore, $C(X,Y)$ is a linear subspace of $B^b(X,Y)$. □

Xiao and Zhu gave the fuzzy norm for an XZ-bounded operator as follows:

Definition 3.4.2. [66]. *Let $T \in B^b(X,Y)$ with $\lim_{a\to 0^+} R_i(a,a) = 0$ for $i = 1,2$. The fuzzy norm $\| T \|: \mathbb{R} \to [0,1]$ is a function defined as:*

$$\| T \| (t) = \lim_{\alpha\to 0^+} \sup_{\|x\|_\alpha^2=1} \| Tx \| (t)$$

For any $\beta \in (0,1]$, $[\| T \|]_\beta = [\| T \|_\beta^1, \| T \|_\beta^2]$ where $\| T \|_\beta^1 = \lim_{\alpha\to 0^+} \inf_{\|x\|_\alpha^2=1} \| Tx \|_\beta^1$ and $\| T \|_\beta^2 = \lim_{\alpha\to 0^+} \sup_{\|x\|_\alpha^2=1} \| Tx \|_\beta^2$.

Theorem 3.4.3. [66] *The FNLS $B^b(X,Y)$ is complete when Y is complete with $\lim_{a\to 0^+} R_1(a,a) = 0$ and $R_2 \leq \max$.*

Theorem 3.4.4. *The linear space $C(X,Y)$ is an FNLS with the operator norm defined as in Definition 3.4.2. Further $C(X,Y)$ is complete when Y is complete with $\lim_{a\to 0^+} R_1(a,a) = 0$ and $R_2 \leq \max$.*

Proof. The proof follows from Theorems 3.4.1 and 3.4.3. □

As a fuzzy compact operator is also fuzzy bounded and weakly fuzzy bounded, Theorems 3.4.1 and 3.4.4 allow us to state the following results which are straightforward.

Theorem 3.4.5. *Let X and Y be FNLSs with the conditions as in Theorem 2.2.27, where $R_i \leq \max$ for $i = 1,2$. Then $C(X,Y)$ is a linear subspace of $B'(X,Y)$, the FNLS of all weakly fuzzy bounded linear operators from X to Y. In fact, $C(X,Y)$ is a FNLS with the operator norm in $B'(X,Y)$ as defined by Bag and Samanta* [4]. *Further $C(X,Y)$ is complete when Y is complete.*

Theorem 3.4.6. *Let X and Y be FNLSs with the conditions as defined in Theorem 2.2.28, where $R_i \leq \max$ for $i = 1,2$. Then $C(X,Y)$ is a subspace of $B(X,Y)$, the*

FNLS of all fuzzy bounded linear operators from X to Y with the operator norm in $B(X,Y)$ as defined by Hasankhani et. al [31]. *Further $C(X,Y)$ is complete when Y is complete.*

In spite of the above results, a fuzzy compact operator T of an infinite dimensional FNLS X to Y fails to be invertible in any of $B^b(X,Y)$ or $B(X,Y)$ or $B'(X,Y)$. Suppose, if possible, T has an inverse S in $B(X,Y)$ (or in $B^b(X,Y)$ or $B'(X,Y)$). Then, by Theorem 3.2.12, $TS = ST = I$, where I is the identity operator on X, is also fuzzy compact. However, if I is fuzzy compact, then X is finite dimensional (using Theorem 3.3.9), which is a contradiction. Thus T in not invertible.

Theorem 3.4.7. *Let $(X, \| \,.\, \|, L, R)$ and $(Y, \| \,.\, \|, L, R)$ be FNLSs with $\lim_{a\to 0^+} R(a,a) = 0$. Let $\{T_n\}$ be a sequence of fuzzy compact operators from X to Y, where Y is complete. If $\{T_n\}$ converges to $T : X \to Y$ with respect to the operator norm in $C(X,Y)$, that is $\| T_n - T \| \to \bar{0}$, then the limit operator T is fuzzy compact.*

Proof. Consider a fuzzy bounded sequence $\{x_m\}$ in X. Then, using the similar argument as in its classical counterparts [42], we can construct a subsequence $\{y_m\} = \{x_{m,m}\}$ such that for every $n \in \mathbb{N}$, the sequence $\{T_n y_m\}$ is Cauchy.
Then, for $\varepsilon > 0$, there exists N such that

$$\| T_n y_j - T_n y_k \|_\alpha^2 < \frac{\varepsilon}{3}, \forall j,k > N \text{ and for each } \alpha \in (0,1] \tag{3.4.1}$$

Choose $\alpha \in (0,1]$. Using Lemma 1.3.20, there exists $\beta \in (0,\alpha]$ such that $\forall j,k > N$,

$$\| Ty_j - Ty_k \|_\alpha^2 \leq \| Ty_j - T_n y_j \|_\beta^2 + \| T_n y_j - T_n y_k \|_\beta^2 + \| T_n y_k - Ty_k \|_\beta^2 \tag{3.4.2}$$

Since $\| T_n - T \| \to \bar{0}$, therefore $\lim_{n\to\infty} \| T_n - T \|_\beta^1 = \lim_{n\to\infty} \| T_n - T \|_\beta^2 = 0$.
Hence,

$$\lim_{n\to\infty} \lim_{\gamma\to 0^+} \sup_{\|y_j\|_\gamma^2 = 1} \| T_n y_j - Ty_j \|_\beta^2 = 0 \tag{3.4.3}$$

Since $\{y_m\}$ is fuzzy bounded, for each $\gamma \in (0, \beta]$ we can find an M_γ such that $\{y_m\} \subseteq N(M_\gamma, \gamma)$. That is, $\| y_m \|_\gamma^2 < M_\gamma$, for all m. Therefore, for each $\gamma \in (0, \beta]$ there exists $n_\circ \in \mathbb{N}$ such that (3.4.3) gives:

$$\| T_{n_\circ} y_j - T y_j \|_\beta^2 < \frac{\varepsilon}{3M_\gamma} \| y_j \|_\gamma^2 < \frac{\varepsilon}{3} \tag{3.4.4}$$

Using (3.4.1) and (3.4.4) in (3.4.2), we get $\| Ty_j - Ty_k \|_\alpha^2 < \varepsilon$ for all $j, k > N$ (considering $n = n_\circ$). Therefore, $\{Ty_m\}$ is a Cauchy sequence in Y. Since Y is complete, therefore $\{Ty_m\}$ converges. Thus, T is fuzzy compact. □

Remark 3.4.8. The above result also holds when the FNLS $C(X, Y)$ is considered with the operator norms as in $B'(X, Y)$ and $B(X, Y)$. Thus the operator limit of a sequence of fuzzy compact operators in $C(X, Y)$ is always a fuzzy compact operator. Proofs are similar to Theorem 3.4.7.