Chapter 7
Probability of Erroneous Decoding for Integer Codes Correcting Asymmetric and Symmetric Errors

The contents of this chapter are based on the paper mentioned below:

- Pokhrel, N.K. and Das, P.K. Probability of erroneous decoding for integer codes correcting burst asymmetric/unidirectional/symmetric errors within a byte and up to double asymmetric errors between two bytes. *Kuwait Journal of Science*, 2022, doi: 10.48129/kjs.online.

Chapter 7

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7.1 Overview

We frequently need enough knowledge about the likelihood of error patterns occurring in order to use a code's error-correcting mechanism in a communication channel. Therefore, irrespective of the pattern of error, knowing an error-correcting code in terms of its decoding probability becomes important. In this chapter, we have derived the probability of erroneous decoding for the integer $(B_lEC)_b$ codes discussed in Definition 1.17-1.20. This class of integer code is capable of correcting symmetric burst errors within a b-bit byte. So it is suitable to study this class over a BSC (Figure 1.2). Other than this class of integer codes discussed over the BSC, we have considered the integer $SBEC$ and SEC-$(B_lAEC)_b$ codes discussed in Definition 1.27 and Definition 1.40-1.41 respectively. The $SBEC$ code is capable of correcting single, double (random) and triple adjacent errors within a b-bit byte, whereas the integer SEC-$(B_lAEC)_b$ code is capable of correcting single symmetric and asymmetric burst errors within a b-bit byte. Also, we have considered integer $IDAEC$ codes discussed in Result 1.23, which is defined over the Z-channel (Figure 1.1). This class of integer code is capable of correcting single and double asymmetric errors occurring within and between any two b-bit bytes.

7.2 Probability and BER

In this section, we discuss the probability of erroneous decoding in the integer codes discussed in the preceding section. It is followed by the BER and a graphical analysis of the probability and BER for different code rates. The next theorem determines the probability of erroneous decoding for integer $(B_lEC)_b$ codes.

Theorem 7.1. *The probability of erroneous decoding for a* $((k+1)b, kb)$ *integer* $(B_lEC)_b$ *code is*
$(k+1)\left[2b\epsilon(1-\epsilon)^{(k+1)b-1} + 4\epsilon^2(1-\epsilon)^{(k+1)b-2}\left(\frac{(1-\epsilon)b}{2\epsilon}\left\{\left(\frac{1+\epsilon}{1-\epsilon}\right)^{l-1} - 1\right\} - \left(\frac{1-\epsilon}{2\epsilon}\right)^2\left(1 + \frac{2\epsilon l(1+\epsilon)^{l-1}}{(1-\epsilon)^l} - \left(\frac{1+\epsilon}{1-\epsilon}\right)^l\right)\right)\right]$, *where* ϵ *is the crossover probability in the BSC.*

Proof. A received codeword will have $(k+1)b$ bits divided into $k+1$ b-bit bytes. In the case of a $((k+1)b, kb)$ integer $(B_lEC)_b$ code, there are $2(k+1)b$ symmetric bursts of length 1. In particular, these are $\underbrace{100\ldots0}_{(k+1)b-bits}$, $\underbrace{010\ldots0}_{(k+1)b-bits}$, ..., $\underbrace{000\ldots1}_{(k+1)b-bits}$. Thus the probability for $l=1$ will be $2(k+1)b\epsilon(1-\epsilon)^{(k+1)b-1}$. Similarly for $l=2$, there are $4(b-1)$ symmetric bursts of length 2. These are $\underbrace{110\ldots0}_{(k+1)b-bits}$, $\underbrace{-110\ldots0}_{(k+1)b-bits}$, $\underbrace{1-10\ldots0}_{(k+1)b-bits}$, $\underbrace{-1-10\ldots0}_{(k+1)b-bits}$ continued up to the bursts of the form $\underbrace{00\ldots11}_{(k+1)b-bits}$. So the probability here will be $4(k+1)(b-1)\epsilon^2(1-\epsilon)^{(k+1)b-2}$. In general for $l < b$, the probability will be $4(b-l+1)\left[\epsilon^2(1-\epsilon)^{(k+1)b-2} + \binom{l-2}{1}2\epsilon^3(1-\epsilon)^{(k+1)b-3} + \ldots + \binom{l-2}{l-2}2^{l-2}\epsilon^l(1-\epsilon)^{(k+1)b-l}\right]$.
Thus the probability of erroneous decoding for a $((k+1)b, kb)$ integer $(B_lEC)_b$ code correcting symmetric bursts of length up to l will be

$$2(k+1)b\epsilon(1-\epsilon)^{(k+1)b-1} + 4(k+1)(b-1)\epsilon^2(1-\epsilon)^{(k+1)b-2} +$$
$$4(k+1)(b-2)\left[\epsilon^2(1-\epsilon)^{(k+1)b-2} + \binom{2}{1}2^1\epsilon^3(1-\epsilon)^{(k+1)b-3}\right] + \ldots$$
$$+ 4(k+1)(b-l+1)\left[\epsilon^2(1-\epsilon)^{(k+1)b-2} + \binom{l-2}{1}2\epsilon^3(1-\epsilon)^{(k+1)b-3} + \ldots\right.$$
$$\left. + \binom{l-2}{l-2}2^{l-2}\epsilon^l(1-\epsilon)^{(k+1)b-l}\right]$$

$$=(k+1)\left[2b\epsilon(1-\epsilon)^{(k+1)b-1}+4\sum_{j=1}^{l-1}\sum_{i=0}^{j-1}(b-j)\binom{l-2}{i}(2\epsilon)^i(1-\epsilon)^{-i}\epsilon^2(1-\epsilon)^{(k+1)b-2}\right]$$

$$=(k+1)\left[2b\epsilon(1-\epsilon)^{(k+1)b-1}+4\epsilon^2(1-\epsilon)^{(k+1)b-2}\sum_{j=1}^{l-1}\sum_{i=0}^{j-1}(b-j)\binom{l-2}{i}\left(\frac{2\epsilon}{1-\epsilon}\right)^i\right]$$

$$=(k+1)\Bigg[2b\epsilon(1-\epsilon)^{(k+1)b-1}+$$

$$4\epsilon^2(1-\epsilon)^{(k+1)b-2}\sum_{j=1}^{l-1}(b-j)\Bigg(\left(\frac{2\epsilon}{1-\epsilon}\right)^0+\binom{l-2}{1}\left(\frac{2\epsilon}{1-\epsilon}\right)^1+\dots$$

$$+\binom{l-2}{l-2}\left(\frac{2\epsilon}{1-\epsilon}\right)^{j-1}\Bigg)\Bigg]$$

$$=(k+1)\left[2b\epsilon(1-\epsilon)^{(k+1)b-1}+4\epsilon^2(1-\epsilon)^{(k+1)b-2}\sum_{j=1}^{l-1}(b-j)\left(\frac{1+\epsilon}{1-\epsilon}\right)^{j-1}\right]$$

$$=(k+1)\Bigg[2b\epsilon(1-\epsilon)^{(k+1)b-1}+$$

$$4\epsilon^2(1-\epsilon)^{(k+1)b-2}\left(\sum_{j=1}^{l-1}b\left(\frac{1+\epsilon}{1-\epsilon}\right)^{j-1}-\sum_{j=1}^{l-1}j\left(\frac{1+\epsilon}{1-\epsilon}\right)^{j-1}\right)\Bigg]$$

$$=(k+1)\Bigg[2b\epsilon(1-\epsilon)^{(k+1)b-1}+4\epsilon^2(1-\epsilon)^{(k+1)b-2}\Bigg(\frac{(1-\epsilon)\,b}{2\epsilon}\left\{\left(\frac{1+\epsilon}{1-\epsilon}\right)^{l-1}-1\right\}-$$

$$\left(\frac{1-\epsilon}{2\epsilon}\right)^2\left(1+\frac{2\epsilon l(1+\epsilon)^{l-1}}{(1-\epsilon)^l}-\left(\frac{1+\epsilon}{1-\epsilon}\right)^l\right)\Bigg)\Bigg].$$

□

The next theorem determines the probability of erroneous decoding for integer $IDAEC$ codes.

Theorem 7.2. *The probability of erroneous decoding for a* $((k+1)b, kb)$ *integer* $IDAEC$ *code is* $(k+1)\frac{b}{2}\epsilon(1-\epsilon)^{n-2}\left(2+\epsilon((k+1)b-3)\right)$*, where* ϵ *is the crossover probability in the* Z*-channel.*

Proof. The integer $IDAEC$ codes are capable of correcting double asymmetric errors occurring in two b-bit bytes simultaneously in addition to the double and single errors occurring within a b-bit byte. The single asymmetric errors occurring within a b-bit byte are of the form

$$\underbrace{\underbrace{100\ldots0}_{b-bits}\underbrace{000\ldots0}_{b-bits}\ldots\underbrace{000\ldots0}_{b-bits}}_{(k+1)b-bits},\underbrace{\underbrace{000\ldots0}_{b-bits}\underbrace{001\ldots0}_{b-bits}\ldots\underbrace{000\ldots0}_{b-bits}}_{(k+1)b-bits},\ldots$$
$$\ldots,\underbrace{\underbrace{000\ldots0}_{b-bits}\underbrace{000\ldots0}_{b-bits}\ldots\underbrace{000\ldots1}_{b-bits}}_{(k+1)b-bits}.$$

So, the probability is $(k+1)b\epsilon(1-\epsilon)^{(k+1)b-1}$.

Similarly for double asymmetric errors occurring within a b-bit byte, the errors are of the form

$$\underbrace{\underbrace{110\ldots0}_{b-bits}\underbrace{000\ldots0}_{b-bits}\ldots\underbrace{000\ldots0}_{b-bits}}_{(k+1)b-bits},\underbrace{\underbrace{000\ldots0}_{b-bits}\underbrace{110\ldots0}_{b-bits}\ldots\underbrace{000\ldots0}_{b-bits}}_{(k+1)b-bits},\ldots$$
$$\ldots,\underbrace{\underbrace{000\ldots0}_{b-bits}\underbrace{000\ldots0}_{b-bits}\ldots\underbrace{000\ldots11}_{b-bits}}_{(k+1)b-bits}.$$

Thus, the probability is $(k+1)\left(\binom{b}{2}\epsilon^2(1-\epsilon)^{(k+1)b-2}\right) = (k+1)\frac{b}{2}(b-1)\epsilon^2(1-\epsilon)^{(k+1)b-2}$.

Double asymmetric errors occurring between two b-bit bytes are of the form

$$\underbrace{\underbrace{001\ldots0}_{b-bits}\underbrace{100\ldots0}_{b-bits}\ldots\underbrace{000\ldots0}_{b-bits}}_{(k+1)b-bits},\underbrace{\underbrace{000\ldots0}_{b-bits}\underbrace{100\ldots0}_{b-bits}\ldots\underbrace{001\ldots0}_{b-bits}}_{(k+1)b-bits},\ldots$$
$$\ldots,\underbrace{\underbrace{001\ldots0}_{b-bits}\underbrace{000\ldots0}_{b-bits}\ldots\underbrace{000\ldots1}_{b-bits}}_{(k+1)b-bits}.$$

Thus the probability is $\binom{k+1}{2}b^2\epsilon^2(1-\epsilon)^{(k+1)b-2} = \frac{b^2k(k+1)}{2}\epsilon^2(1-\epsilon)^{(k+1)b-2}$.

Since a $((k+1)b, kb)$ integer $IDAEC$ code is capable of correcting any one of the mentioned types of errors at a time, therefore the probability will be

$$\begin{aligned}
&(k+1)b\epsilon(1-\epsilon)^{(k+1)b-1} + (k+1)\frac{b}{2}(b-1)\epsilon^2(1-\epsilon)^{(k+1)b-2} + \\
&\frac{b^2k(k+1)}{2}\epsilon^2(1-\epsilon)^{(k+1)b-2} \\
=&(k+1)\left[\epsilon(1-\epsilon)^{(k+1)b-1} + \epsilon^2(1-\epsilon)^{(k+1)b-2}\left(\frac{b(b-1)}{2} + \frac{b^2k}{2}\right)\right] \\
=&(k+1)\left[\epsilon(1-\epsilon)^{(k+1)b-1} + \epsilon^2(1-\epsilon)^{(k+1)b-2}\left((k+1)\frac{b^2}{2} - \frac{b}{2}\right)\right] \\
=&(k+1)\frac{b}{2}\epsilon(1-\epsilon)^{(k+1)b-2}\left(2+\epsilon\left((k+1)b-3\right)\right).
\end{aligned}$$

□

The next two theorems determine the probability of erroneous decoding for integer $SBEC$ and SEC-$(B_lAEC)_b$ codes.

Theorem 7.3. *The probability of erroneous decoding for a* $((k+1)b, kb)$ *integer* $SBEC$ *code is*
$2(k+1)\epsilon(1-\epsilon)^{(k+1)b-1}\left[b+b(b-1)\epsilon(1-\epsilon)^{(k+1)b-1}+4(b-2)\epsilon^2(1-\epsilon)^{(k+1)b-2}\right]$, *where* ϵ *is the crossover probability in the BSC.*

Proof. The code discussed here is capable of correcting symmetric errors in the form of single, double (random) and triple adjacent occurring within a b-bit byte. The single symmetric errors are of the form

$$\underbrace{\underbrace{10\dots0}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots0}_{b-bits}}_{(k+1)b-bits},\ \underbrace{\underbrace{-10\dots0}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots0}_{b-bits}}_{(k+1)b-bits},\ \underbrace{\underbrace{00\dots1}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots0}_{b-bits}}_{(k+1)b-bits},$$
$$\underbrace{\underbrace{00\dots-1}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots0}_{b-bits}}_{(k+1)b-bits},\dots,\underbrace{\underbrace{00\dots0}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots1}_{b-bits}}_{(k+1)b-bits},\ \underbrace{\underbrace{00\dots0}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots-1}_{b-bits}}_{(k+1)b-bits}.$$

Since there are $2b(k+1)$ single symmetric errors in a codeword, the probability of erroneous decoding for the symmetric single errors will be $2b(k+1)\epsilon(1-\epsilon)^{(k+1)b-1}$. In case of double symmetric errors occurring randomly within a b-bit byte, the possible patterns are

$$\underbrace{\underbrace{11\dots0}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots0}_{b-bits}}_{(k+1)b-bits},\ \underbrace{\underbrace{-11\dots0}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots0}_{b-bits}}_{(k+1)b-bits},\ \underbrace{\underbrace{1-1\dots0}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots0}_{b-bits}}_{(k+1)b-bits},$$
$$\underbrace{\underbrace{-1-1\dots0}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots0}_{b-bits}}_{(k+1)b-bits},\dots,\underbrace{\underbrace{10\dots1}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots0}_{b-bits}}_{(k+1)b-bits},$$
$$\underbrace{\underbrace{-10\dots-1}_{b-bits}\underbrace{00\dots0}_{b-bits}\dots\underbrace{00\dots0}_{b-bits}}_{(k+1)b-bits},\dots,\underbrace{\underbrace{0\dots00}_{b-bits}\underbrace{0\dots00}_{b-bits}\dots\underbrace{0\dots-1-1}_{b-bits}}_{(k+1)b-bits}.$$

There are $(k+1)\binom{b}{2} = (k+1)\frac{b(b-1)}{2}$ positions in a codeword to have double errors randomly and there are 4 different patterns for each double symmetric error position. So the probability of erroneous decoding for double symmetric errors occurring within a b-bit byte randomly will be $(k+1)4\frac{b(b-1)}{2}\epsilon^2(1-\epsilon)^{(k+1)b-2} = 2(k+1)b(b-1)\epsilon^2(1-\epsilon)^{(k+1)b-2}$.

For symmetric triple adjacent errors within a b-bit byte, the possible patterns are

$$\underbrace{\underbrace{111\ldots0}_{b-bits}\underbrace{000\ldots0}_{b-bits}\ldots\underbrace{000\ldots0}_{b-bits}}_{(k+1)b-bits},\ \underbrace{\underbrace{-111\ldots0}_{b-bits}\underbrace{000\ldots0}_{b-bits}\ldots\underbrace{000\ldots0}_{b-bits}}_{(k+1)b-bits},$$

$$\ldots,\ \underbrace{\underbrace{-1-1-1\ldots0}_{b-bits}\underbrace{000\ldots0}_{b-bits}\ldots\underbrace{000\ldots0}_{b-bits}}_{(k+1)b-bits},\ \ldots,\ \underbrace{\underbrace{00\ldots0}_{b-bits}\underbrace{00\ldots0}_{b-bits}\ldots\underbrace{00\ldots-1-1-1}_{b-bits}}_{(k+1)b-bits}.$$

There are $(k+1)(b-2)$ positions for triple adjacent errors to occur in a codeword having $k+1$ b-bit bytes and there are 8 different patterns for each triple adjacent symmetric error position within a b-bit byte in the codeword. Thus the probability of erroneous decoding in this case will be $8(k+1)(b-2)\epsilon^3(1-\epsilon)^{(k+1)b-3}$.

Therefore, the total probability for the $((k+1)b, kb)$ integer $SBEC$ code will be

$$\begin{aligned}&(k+1)\left[2b\epsilon(1-\epsilon)^{(k+1)b-1}+2b(b-1)\epsilon^2(1-\epsilon)^{(k+1)b-2}+8(b-2)\epsilon^3(1-\epsilon)^{(k+1)b-3}\right]\\ =&2(k+1)\epsilon(1-\epsilon)^{(k+1)b-1}\left[b+b(b-1)\epsilon(1-\epsilon)^{(k+1)b-1}+4(b-2)\epsilon^2(1-\epsilon)^{(k+1)b-2}\right].\end{aligned}$$

□

Theorem 7.4. *The probability of erroneous decoding for a $((k+1)b, kb)$ integer SEC-$(B_lAEC)_b$ code is*

$$(k+1)\left[2b\epsilon(1-\epsilon)^{(k+1)b-1}+\epsilon^2(1-\epsilon)^{(k+1)b-2}\left\{\left(b+\frac{1-\epsilon}{\epsilon}\right)\left(\frac{1-\epsilon}{\epsilon}\right)\left(\left(\frac{1}{1-\epsilon}\right)^{l-1}-1\right)\right.\right.$$
$$\left.\left.-(l-1)\left(\frac{1}{1-\epsilon}\right)^{l-1}\left(\frac{1-\epsilon}{\epsilon}\right)\right\}\right],$$

where ϵ is the crossover probability in the BSC.

Proof. The code discussed here is capable of correcting single symmetric errors and asymmetric burst errors within a b-bit byte. As discussed in Theorem 7.3, the probability of erroneous decoding for single symmetric errors occurring within a b-bit byte is $2b\epsilon(1-\epsilon)^{(k+1)b-1}$. For asymmetric bursts of length 2, the pattern of errors within a b-bit byte are $\underbrace{110\ldots0}_{b-bits}$, $\underbrace{011\ldots0}_{b-bits}$, $\ldots$, $\underbrace{00\ldots11}_{b-bits}$. There are $b-1$ such errors, so the probability of erroneous decoding will be $(b-1)\epsilon^2(1-\epsilon)^{(k+1)b-2}$. Similarly, the pattern of errors for asymmetric bursts of length 3 are $\underbrace{111\ldots0}_{b-bits}$, $\underbrace{0111\ldots0}_{b-bits}$, $\ldots$, $\underbrace{00\ldots111}_{b-bits}$ and $\underbrace{101\ldots0}_{b-bits}$, $\underbrace{0101\ldots0}_{b-bits}$, $\ldots$, $\underbrace{00\ldots101}_{b-bits}$. Here the length considered is 3, whereas the number of non-zero components may be 2 or 3. Since the number of positions for such bursts is equal to $b-2$, the corresponding probability will be $(b-2)\left[\binom{1}{0}\epsilon^2(1-\epsilon)^{(k+1)b-2}+\binom{1}{1}\epsilon^3(1-\epsilon)^{(k+1)b-3}\right]$. Continuing this, the probability of erroneous decoding for asymmetric bursts of length l within a b-bit byte will be $(b-l+1)\sum_{i=0}^{l-2}\binom{l-2}{i}\epsilon^{i+2}(1-\epsilon)^{(k+1)b-i-2}$. Thus, by summing up these probabilities we obtain the probability of erroneous decoding within a b-bit byte as

$$
\begin{aligned}
&2b\Big\{\epsilon(1-\epsilon)^{(k+1)b-1}\Big\}+(b-1)\Big\{\epsilon^2(1-\epsilon)^{(k+1)b-2}\Big\}+(b-2)\Big\{\epsilon^2(1-\epsilon)^{(k+1)b-2}\\
&+\epsilon^3(1-\epsilon)^{(k+1)b-3}\Big\}+\ldots+(b-l+1)\Big\{\epsilon^2(1-\epsilon)^{(k+1)b-2}+\binom{l-2}{1}\epsilon^3(1-\epsilon)^{(k+1)b-3}\\
&+\binom{l-2}{2}\epsilon^4(1-\epsilon)^{(k+1)b-4}+\ldots+\binom{l-2}{l-2}\epsilon^l(1-\epsilon)^{(k+1)b-l}\Big\}\\
=&2b\epsilon(1-\epsilon)^{(k+1)b-1}+(b-1)\sum_{i=0}^{0}\binom{0}{i}\epsilon^{i+2}(1-\epsilon)^{(k+1)b-i-2}\\
&+(b-2)\sum_{i=0}^{1}\binom{1}{i}\epsilon^{i+2}(1-\epsilon)^{(k+1)b-i-2}+\ldots\\
&+(b-l+1)\sum_{i=0}^{l-2}\binom{l-2}{i}\epsilon^{i+2}(1-\epsilon)^{(k+1)b-i-2}\\
=&2b\epsilon(1-\epsilon)^{(k+1)b-1}+\sum_{j=1}^{l-1}\sum_{i=0}^{j-1}(b-j)\binom{j-1}{i}\epsilon^{i+2}(1-\epsilon)^{(k+1)b-i-2}\\
=&2b\epsilon(1-\epsilon)^{(k+1)b-1}+\epsilon^2(1-\epsilon)^{(k+1)b-2}\sum_{j=1}^{l-1}\Bigg[(b-j)\Bigg\{\left(\frac{\epsilon}{1-\epsilon}\right)^0+\binom{j-1}{1}\left(\frac{\epsilon}{1-\epsilon}\right)^1+\ldots\\
&\ldots+\binom{j-1}{j-1}\left(\frac{\epsilon}{1-\epsilon}\right)^{j-1}\Bigg\}\Bigg]\\
=&2b\epsilon(1-\epsilon)^{(k+1)b-1}+\epsilon^2(1-\epsilon)^{(k+1)b-2}\sum_{j=1}^{l-1}(b-j)\left(1+\frac{\epsilon}{1-\epsilon}\right)^{j-1}\\
=&2b\epsilon(1-\epsilon)^{(k+1)b-1}+\epsilon^2(1-\epsilon)^{(k+1)b-2}\sum_{j=1}^{l-1}(b-j)\left(\frac{1}{1-\epsilon}\right)^{j-1}\\
=&2b\epsilon(1-\epsilon)^{(k+1)b-1}+\epsilon^2(1-\epsilon)^{(k+1)b-2}\left[b\sum_{j=1}^{l-1}\left(\frac{1}{1-\epsilon}\right)^{j-1}-\sum_{j=1}^{l-1}j\left(\frac{1}{1-\epsilon}\right)^{j-1}\right]\\
=&2b\epsilon(1-\epsilon)^{(k+1)b-1}+\epsilon^2(1-\epsilon)^{(k+1)b-2}\left[b\left(\frac{\left(\frac{1}{1-\epsilon}\right)^{l-1}-1}{\left(\frac{1}{1-\epsilon}\right)-1}\right)\right.\\
&\left.-\left\{(l-1)\left(\frac{1}{1-\epsilon}\right)^{l-1}\left(\frac{1-\epsilon}{\epsilon}\right)-\left(\frac{1-\epsilon}{\epsilon}\right)^2\left\{\left(\frac{1}{1-\epsilon}\right)^{l-1}-1\right\}\right\}\right]\\
=&2b\epsilon(1-\epsilon)^{(k+1)b-1}+\epsilon^2(1-\epsilon)^{(k+1)b-2}\left[\frac{b(1-\epsilon)}{\epsilon}\left\{\left(\frac{1}{1-\epsilon}\right)^{l-1}-1\right\}\right.\\
&\left.-(l-1)\left(\frac{1}{1-\epsilon}\right)^{l-1}\left(\frac{1-\epsilon}{\epsilon}\right)+\left(\frac{1-\epsilon}{\epsilon}\right)^2\left\{\left(\frac{1}{1-\epsilon}\right)^{l-1}-1\right\}\right]
\end{aligned}
$$

$$=2b\epsilon(1-\epsilon)^{(k+1)b-1}+\epsilon^2(1-\epsilon)^{(k+1)b-2}\Bigg[\left(\frac{b(1-\epsilon)}{\epsilon}+\left(\frac{1-\epsilon}{\epsilon}\right)^2\right)\left\{\left(\frac{1}{1-\epsilon}\right)^{l-1}-1\right\}$$
$$-(l-1)\left(\frac{1}{1-\epsilon}\right)^{l-1}\left(\frac{1-\epsilon}{\epsilon}\right)\Bigg].$$

This code is capable of correcting the discussed errors only one at a time which occur within a b-bit byte and there are $k+1$ such b-bit bytes, so the probability will be

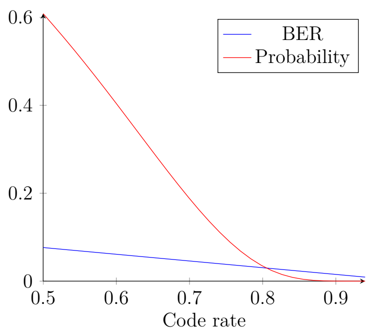
$$(k+1)\Bigg[2b\epsilon(1-\epsilon)^{(k+1)b-1}+\epsilon^2(1-\epsilon)^{(k+1)b-2}\Bigg\{\left(b+\tfrac{1-\epsilon}{\epsilon}\right)\left(\tfrac{1-\epsilon}{\epsilon}\right)\left(\left(\tfrac{1}{1-\epsilon}\right)^{l-1}-1\right)$$
$$-(l-1)\left(\tfrac{1}{1-\epsilon}\right)^{l-1}\left(\tfrac{1-\epsilon}{\epsilon}\right)\Bigg\}\Bigg]. \qquad \square$$

Bit Error Rate (BER) is the ratio between the number of corrupted bits and the number of bits transmitted. The number of corrupted bits in the case of codes correcting burst and CT-burst errors remains the same for the length specified. So, the BER for the integer $(B_lEC)_b$ and SEC-$(B_lAEC)_b$ codes discussed in this chapter is similar to the BER in Chapter 2 and Chapter 6, which is $\frac{1}{(k+1)bl}\left[\frac{l^2+5l-2}{4}\right]$. Since the number of corrupted bits in $((k+1)b, kb)$ integer $IDAEC$ codes varies between 1 and 2, we consider the BER equal to $\frac{1.5}{(k+1)b}$. Similarly, in $((k+1)b, kb)$ integer $SBEC$ codes, the number of corrupted bits is 1, 2 and 3. So, the BER will be $\frac{2}{(k+1)b}$.

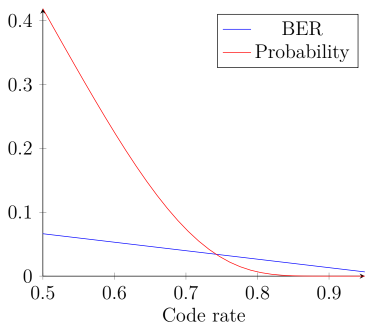
Table 7.1-7.2 present the probability of erroneous decoding and BER for the integer codes discussed in this chapter. The existence of the codes considered is given in their respective studies. By considering a few examples and $\epsilon = 0.1$, Figure 7.1-7.2 show the change in probability and BER with respect to different code rates for the discussed codes. In the graphs presented in Figure 7.1-7.2, it can be observed that the code rate and BER decrease with the increase in the code rate. In almost all of the cases, it can be seen that the rate of decrease in probability is faster compared to the BER. However, this depends on the value of ϵ. For instance, by considering $\epsilon = 0.0002$ in integer SEC-$(B_5AEC)_{32}$ code, the probability increases with the increase in code rate. In particular, for code rate $= 0.5, 0.67, 0.75, 0.8, 0.83, 0.85, \ldots, 0.9375$, the probability is $0.0252888, 0.0376911, 0.0499342, 0.0620195, 0.0739486, 0.0857229, \ldots, 0.18497$ respec-

Figure 7.1: **Change in probability and BER in $(B_lEC)_b$ and $IDAEC$ codes for different code rates**

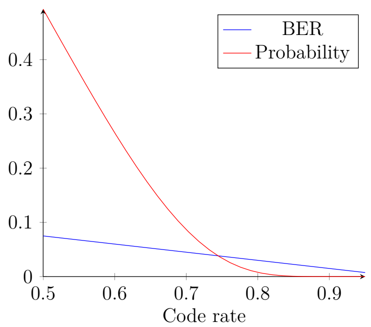
(a) **$(B_3EC)_{12}$ code**



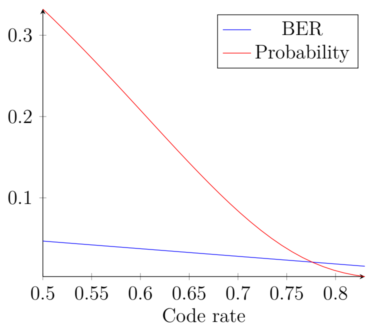
(b) **$(B_4EC)_{16}$ code**



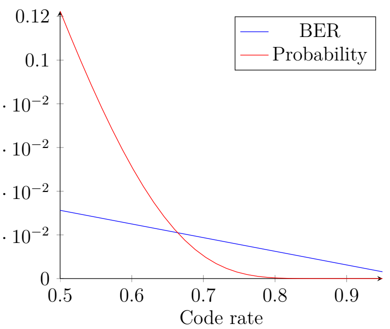
(c) **$(B_5EC)_{16}$ code**



(d) **$IDAEC$ code with $b = 16$**



(e) **$IDAEC$ code with $b = 24$**



(f) **$IDAEC$ code with $b = 32$**

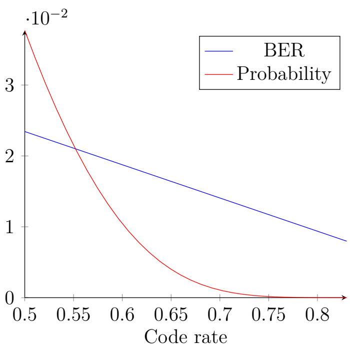


Figure 7.2: **Change in probability and BER in SEC-$(B_lAEC)_b$ and $SBEC$ codes for different code rates**

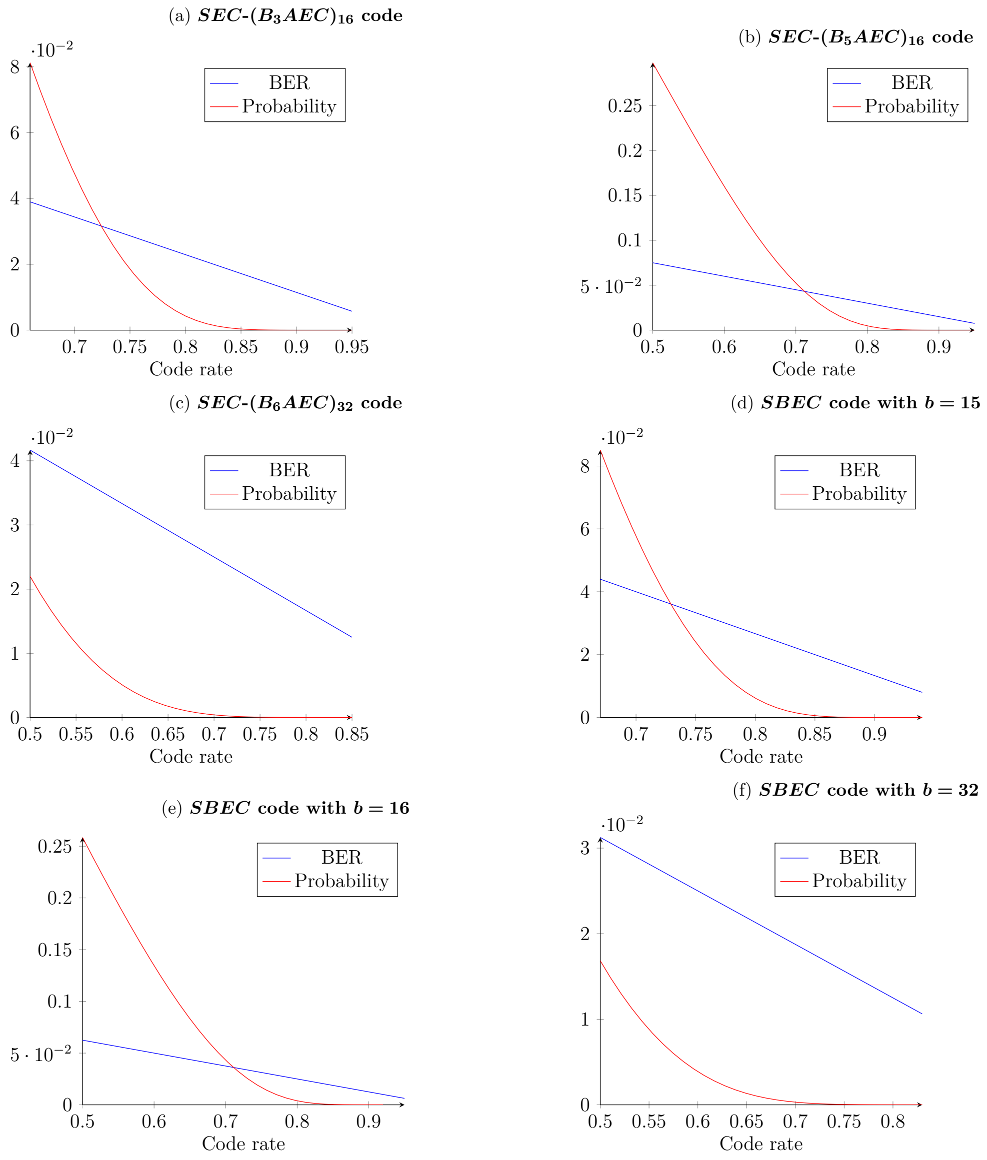


Table 7.1: **Probability of erroneous decoding ($\epsilon = 0.1$) and BER in $(B_l EC)_b$ and $IDAEC$ codes**

Codes	b	l	k	Probability	BER
$(B_l EC)_b$	12	3	3	0.097055	0.0381944
	12	3	6	0.00382636	0.0218254
	12	3	16	3.00081×10^{-8}	0.00898699
	16	4	30	3.81063×10^{-21}	0.00428427
	16	4	60	8.15398×10^{-43}	0.0021772541
	16	4	95	3.04871×10^{-68}	0.00138347
	16	5	5	0.00174458	0.025
	16	5	20	6.3674×10^{-14}	0.00714286
	16	5	38	7.84604×10^{-27}	0.00384616
$IDAEC$	16	NA	2	0.122541	0.03125
	16	NA	4	0.0104652	0.01875
	16	NA	5	0.00271091	0.015625
	24	NA	10	3.80906×10^{-10}	0.00568181
	24	NA	20	1.40598×10^{-20}	0.0029761
	24	NA	28	4.35482×10^{-29}	0.00215517
	32	NA	20	5.09378×10^{-28}	0.00223214
	32	NA	40	9.95672×10^{-57}	0.00114329
	32	NA	64	1.79708×10^{-91}	0.000721154

Table 7.2: **Probability of erroneous decoding ($\epsilon = 0.1$) and BER in SEC-$(B_lAEC)_b$ and $SBEC$ codes**

Codes	b	l	k	Probability	BER
SEC-$(B_lAEC)_b$	16	3	8	9.11725×10^{-6}	0.012731481
	16	3	16	2.39393×10^{-11}	0.0067401961
	16	3	32	8.97964×10^{-23}	0.00347222
	16	5	8	0.0000100477	0.016666667
	16	5	16	2.63824×10^{-11}	0.0088235294
	16	5	32	9.89606×10^{-23}	0.00454545
	32	8	8	6.2426×10^{-12}	0.0110677083
	32	8	16	2.27853×10^{-23}	0.005859375
	32	8	32	1.65152×10^{-46}	0.0030184659
$SBEC$	15	NA	5	0.00152373	0.0222222
	15	NA	15	5.56162×10^{-10}	0.0083333
	15	NA	25	1.23717×10^{-16}	0.00512821
	16	NA	10	3.45925×10^{-7}	0.011363636
	16	NA	20	3.15217×10^{-14}	0.005952381
	16	NA	30	2.22103×10^{-21}	0.004032259
	20	NA	8	2.3212×10^{-7}	0.01111111
	20	NA	16	$.09276 \times 10^{-14}$	0.005882353
	20	NA	24	1.46897×10^{-21}	0.004

tively. But in the case of BER, a decrease will always be observed with an increase in the code rate. This is because an increase in code rate leads to a gain in code length, which increases the denominator value in its expression. Also, BER is independent of the crossover probability. Similar graphs can be plotted in all of the cases by assuming different values of ϵ.

7.3 Conclusion

In this chapter, we have derived the probability of erroneous decoding for integer codes having symmetric and asymmetric natures of errors. This simplifies the process of analysing the codes to carry out research in different aspects of statistics used in coding theory. By replicating the approaches developed above, we can obtain the probabilities for any type of error in integer codes having symmetric and asymmetric patterns. By using similar approach, probability can also be determined for binary communication channels having different probabilities for $1 \to 0$ and $0 \to 1$.